Estimation of the Earth's "Unperturbed" Perihelion from Times of Solstices and Equinoxes

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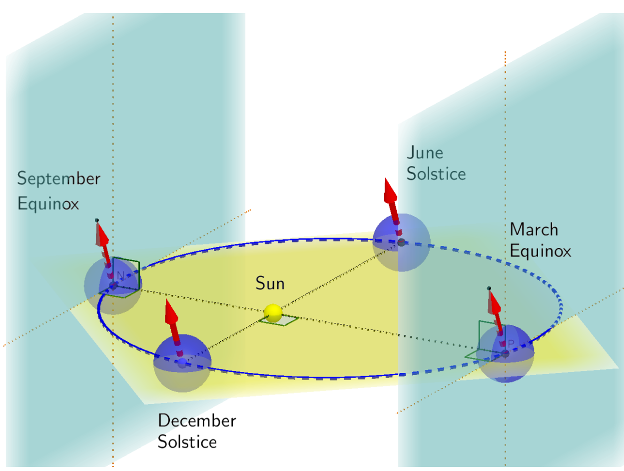
Abstract

Published times of the Earth's perihelions do not refer to the perihelions of the orbit that the Earth would follow if unaffected by other bodies such as the Moon. To estimate the timing of that "unperturbed" perihelion, we fit an unperturbed Kepler orbit to the timings of the year 2017's equinoxes and solstices. We find that the unperturbed 2017 perihelion, defined in that way, would occur 12.93 days after the December 2016 solstice. Using that result, calculated times of the year 2017's solstices and equinoxes differ from published values by less than five minutes. That degree of accuracy is sufficient for the intended use of the result.



"At the equinoxes, the Earth's axis of rotation lies within the plane that is perpendicular to the ecliptic and to the line connecting the centers of the Earth and Sun."

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1 Introduction

This document is prepared in support of a future one that will use Geometric Algebra (GA) to calculate the azimuth and elevation of the Sun as seen from any point on Earth, at any time. That calculation will make use of GA's convenient methods for rotating vectors and planes.

A key angle of rotation in the intended calculation is that that which is labeled θ_{DS} in Fig. 1. To know that angle, we must know the date and time of what is labeled (in that same figure) as the Earth's *perihelion*: the point in the Earth's orbit in which the Earth is closest to the Sun. If you're thinking that such an important and basic piece of information is available readily on line, you're right. For example, see Ref. [1]. However, the perihelions published on line are not the same sort as the perihelion shown in Fig. 1.

The difference is this: the perihelions published on line are for the Earth's real orbit, which is affected by the gravity of the Moon and other bodies. In contrast, the perihelion shown in Fig. 1—and which we need for our calculations—is for the orbit that the Earth would have if it were not affected by the gravity of other bodies.

How might we estimate the timing of that perihelion, for a given year? The approach taken here is to calculate the "unperturbed" orbit that best fits the timing of that year's equinoxes and solstices, and also the timing of the preceding year's December solstice. Details are presented in the sections that follow.

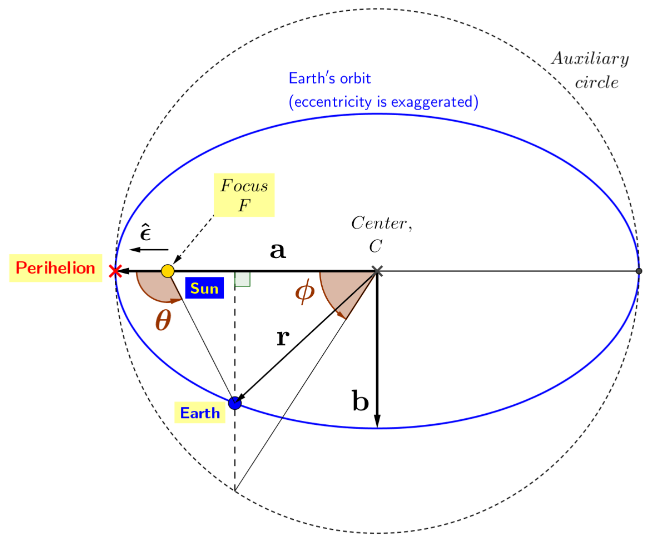

Figure 1: Features of the Earth's unperturbed Kepler orbit. Note the difference between the angles θ and ϕ.

2 Review of the Kepler Orbit, and of the Geometry of Solstices and Equinoxes

2.1 The Earth's Kepler Orbit

2.1.1 Mathematics of Kepler Orbit

Hestenes ([2], pp. 204-219) formulates and discusses the Kepler problem in GA terms, arriving at the well-known *Kepler equation* for planetary motion:

$$\frac{2\pi t}{T} = \phi - \epsilon \sin \phi. \tag{2.1}$$

where T is the planet's orbital period, t is the time elapsed since the planet was at its perihelion, and ϵ is the orbit's eccentricity. The angle ϕ (in radians) is as shown in Fig. 1.

Note the difference, in that figure, between the angles ϕ and θ. Because the angle that we will wish to identify in this document is a θ, and the angles that we must use in using Eq. (2.1) are ϕ's, we need to know how to convert between them. Hestenes ([2], p. 219) gives the required formula,

$$\tan \frac{\theta}{2} = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/2} \tan \frac{\phi}{2},$$

from which

$$\phi = 2 \tan^{-1} \left[\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/2} \tan \frac{\theta}{2} \right]. \tag{2.2}$$

Event	UTC Time and date ([1])	Days since Dec. 2016 solstice
Dec. 2016 solst	21/12/2016 10:44:00	N/A
Mar. 2017 equin	20/03/2017 10:29:00	88.99
June 2017 solst	21/06/2017 04:24:00	92.75
Sept. 2017 equin	22/09/2017 20:02:00	93.65
Dec. 2017 solst	21/12/2017 16:28:00	89.85

Table 1: Times and dates of solstices and equinoxes for the year 2017, and the number of days between each event and the December 2016 solstice.

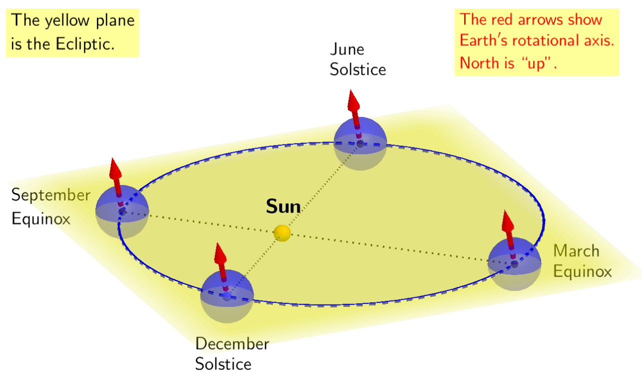

Figure 2: The relationship between the Sun, Earth, ecliptic, and the Earth's rotational axis. For our purposes, the orientation of the rotational axis is constant during a given year.

2.1.2 Data for Earth's orbit

For the Earth, $\epsilon = 0.01671022$, and T is the Tropical year, =365.242 days (Ref. [5]). Table 1 gives dates of solstices and equinoxes for the year 2017, and the time elapsed between each of those events and the December 2016 solstice.

One additional piece of important information: we know that the perihelion occurs within a few weeks of the December solstice, because that is the time of year when the Sun's apparent diameter is greatest. For example, see the data in Ref. [6].

2.2 The Geometry of Solstices and Equinoxes

For our purposes, the plane of the Kepler orbit is the same plane that is called the ecliptic, and the orientation of the Earth's axis of rotation is constant throughout any given year (Fig. 2)). Actually, of course, that axis precesses by 360° in approximately 26,000 years, or about one degree every 72 years (Ref. [3]).

Please note an important difference between the meaning of the word "solstice" in everyday language and in an astronomical context. In everyday

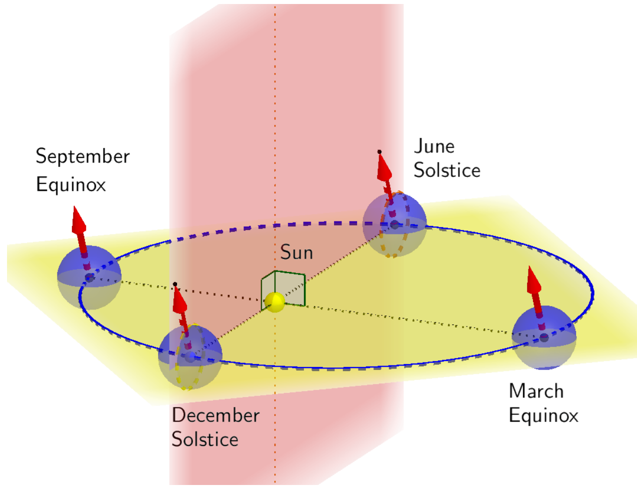

Figure 3: Geometry of the solstices: the Earth's axis of rotation lies within the plane that is perpendicular to the ecliptic, and which also contains the line that connects the centers of the Earth and Sun.

language, each "solstice" is a day, but in the astronomical context it is a precise instant in time, and a corresponding precise point on the Earth's orbit. Similar comments apply to the term "equinox".

At the solstices (Fig. 3), the Earth's axis of rotation lies within the plane that is perpendicular to the ecliptic, and which also contains the line that connects the centers of the Earth and Sun. In contrast, at the equinoxes that axis lies within the plane that is perpendicular to the ecliptic and to the line connecting the centers of the Earth and Sun (Fig. 4).

Adding this information to that which we saw in our review of the Earth's orbit (Section 2.1), we can deduce that θ_{ME}, the angle θ at the March equinox, is equal to $\theta_{DS} + \pi/2$ (Fig. 5). Similarly, θ_{JS} (the value at the June solstice) is $\theta_{DS} + \pi$, and θ_{SE} (the value at the September equinox) is $\theta_{DS} + 3\pi/2$.

2.3 Observations Derived from Our Review

In our review of the Kepler orbit (Section 2.1), we saw that although the differences in θ between successive solstices and equinoxes are always equal —to be specific, they're all equal to 90°—the elapsed times between any solstice and the two equinoxes which it separates are unequal. The same can be said of any equinox, and the two solstices which it separates. From that observation, and Kepler's Second Law (Ref. [4]) , we can deduce that

1. the December solstice does not occur when the Earth is at perihelion. Otherwise, the elapsed time between the September equinox and December solstice would be equal to that between the December solstice and March

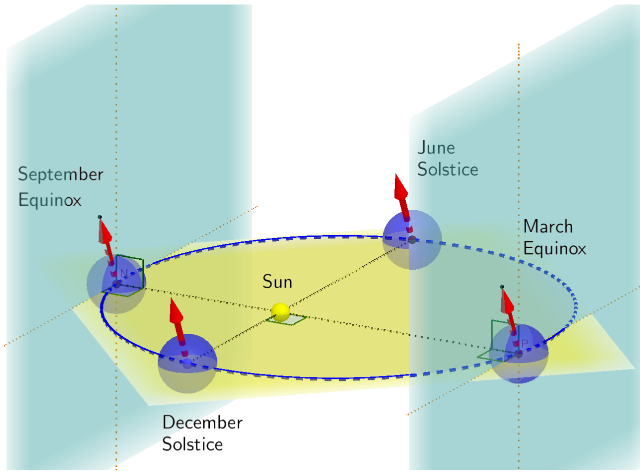

Figure 4: Geometry of the equinoxes: the Earth's axis of rotation lies within the plane that is perpendicular to the ecliptic and to the line connecting the centers of the Earth and Sun.

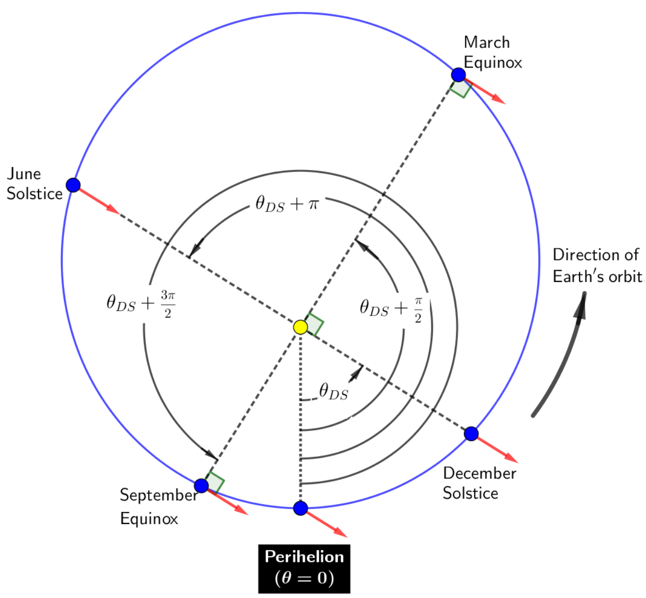

Figure 5: Angular relationships between the perihelion, solstices, and equinoxes, assuming that the angle θ_{DS} is positive. Cf. Fig. 7.

equinox); and

2. the time intervals between successive solstices and equinoxes must depend upon the value of θ_{DS}.

Putting all of this information together, we might conjecture that a given year's θ_{DS} can be identified by using trial and error to find the value thereof that gives the best "fit" to the timings of the following year's solstices and equinoxes. In that process of trial and error, we will be helped by knowing that although the perihelion and the December solstice do not coincide, they occur within a few weeks of each other (Section 2.1.2) .

3 Estimating θ_{DS}

3.1 Definition of Purpose; Strategy; and Key Assumptions

The purpose of this document is to provide an estimate of θ_{DS} for use in a future document that will use GA to calculate azimuths and elevations of the Sun. At the end of Section 2.3, we observed that in principle, the value of θ_{DS} can be estimated through trial and error, by finding the value thereof that gives the best "fit" to the actual timings of a given year's solstices and equinoxes. We will use that strategy here, using a least-squares definition of "best fit", as explained in greater detail in the sections that follow.

In our calculations, we will assume that during any given year, the eccentricity vector $\boldsymbol{\epsilon}$ is constant, as is the orientation of the Earth's rotational axis (Section 2.2).

3.2 Implementation

Let's continue our use of the subscripts ME, JS, SE, and DS to denote (respectively) the March equinox, June solstice, September equinox, and December solstice. Our strategy, then, is as follows, using the time period between the December solstices of 2016 and 2017 as an example:

Step	Operation
1	Guess a value of θ_{DS} for 2016.
2	From the value guessed in Step 1, calculate the following for the year 2017: θ_{ME}, θ_{JS}, θ_{SE}, and θ_{DS}
3	Using Eq. (2.1) and the θ's calculated in Step 2, calculate t_{ME}, t_{JS}, t_{SE}, and t_{DS} for the year 2017.
4	Using the values calculated in Step 3, and the published date of the December 2016 solstice, find the differences a $Calculated\, t_{ME}\,(of\ 2017) - t_{DS}\,(of\ 2016)$, b $Calculated\, t_{JS}\,(of\ 2017) - t_{DS}\,(of\ 2016)$, c $Calculated\, t_{SE}\,(of\ 2017) - t_{DS}\,(of\ 2016)$, and d $Calculated\, t_{DS}\,(of\ 2017) - t_{DS}\,(of\ 2016)$. Call these results 4a, 4b, 4c, and 4d, respectively.
5	Using published times of the 2017 solstices and equinoxes, find the differences a $Published\, t_{ME}\,(of\ 2017) - t_{DS}\,(of\ 2016)$, b $Published\, t_{JS}\,(of\ 2017) - t_{DS}\,(of\ 2016)$, c $Published\, t_{SE}\,(of\ 2017) - t_{DS}\,(of\ 2016)$, and d $Published\, t_{DS}\,(of\ 2017) - t_{DS}\,(of\ 2016)$. Call these results 5a, 5b, 5c, and 5d, respectively.
6	Calculate $(5a - 4a)^2 + (5b - 4b)^2 + (5c - 4c)^2 + (5d - 4d)^2$.
7	Repeat Steps 1-6, to find the value of θ_{DS} (with the necessary precision) that minimizes the sum calculated in Step 6.

The strategy is implemented in the spreadsheet shown in Fig. 6. The Solver tool in Microsoft Excel is convenient for Step 7.

3.3 Results and Discussion

As shown in Fig. 6, the best-fit estimate of θ_{DS} for 2016 is -0.23009863 radians, meaning that the perihelion occurred after the December 2016 solstice (Fig. 7) rather than before it, as was assumed in constructing Fig. 5. The value of t for that solstice is -12.93 days, meaning that the “unperturbed” perihelion that interests us occurred on 3 January 2017. This result is consistent with dates of ”true” perihelions given in Ref. [1], all of which fall between the second and fifth of January in their respective years.

The maximum difference between the published times of 2017’s solstices

Determination of the Times of Perihelia from Times of Solstices and Equinoxes

Purpose: The yearly perihelia for which times are published are, truly, the times at which the Earth is closest to the Sun. Because the Earth's trajectory is affected by the Moon's gravity, instead of by the sun's alone, each year's perihelion can differ from the perihelion to which the Kepler equation refers. This worksheet attempts to calculate value of θ for the Kepler-equation perihelion from times of Solstices and Equinoxes.

Data for Earth's orbit:		
Eccentricity	0.01671022	
Period	365.242	days

Key to date codes: **DS**: December Solstice; **JS**: June Solstice; **ME**: March Equinox; **SE**: September Equinox

Year: December 2016 Solstice to December 2017 Solstice

Select an initial value of θ_{DS} 2016, then adjust it to minimize the maxiumum difference between calculated and published elapsed times

θ_{DS} 2016: -0.23009863 radians

Event	Date and time of event (dd/mm/yyyy hh:mm, UTC)	Time elapsed between DS 2016 and event (days)	Values calculated from θ_{SD} 2016: θ (radians)	ϕ (radians)	t (days)	Calculated elapsed time between DS 2016 and event (days)	Square of difference between calculated and published elapsed times (days^2)
DS 2016	21/12/2016 10:44	0	-0.23009863	-0.226317967	-12.93791342	0	0
ME 2017	20/03/2017 10:29	88.98958333	1.340697697	1.324458063	76.04879703	88.98671045	8.25348E-06
JS 2017	21/06/2017 04:24	181.7361111	2.911494024	2.907651341	168.7967996	181.734713	1.95467E-06
SE 2017	22/09/2017 20:02	275.3875	4.48229035	4.498591996	262.4527169	275.3906303	9.79902E-06
DS 2017	21/12/2017 16:28	365.2388889	6.053086677	6.056867341	352.3040866	365.242	9.67901E-06

The largest squared difference between calculated and published dates of events is 0.979902E-06 days^2 (yellow square), at the September Solstice. Therefore, the largest diffence between calculated and published dates is the square root of that number, = 0.00313 days, or 4.5 minutes.

Sum of squares of differences (days^2)	2.969E-05

Figure 6: Screen shot of the Excel spreadsheet (Ref. [7]) used to identify the best-fit value of θ_{DS} for the December 2016 solstice.

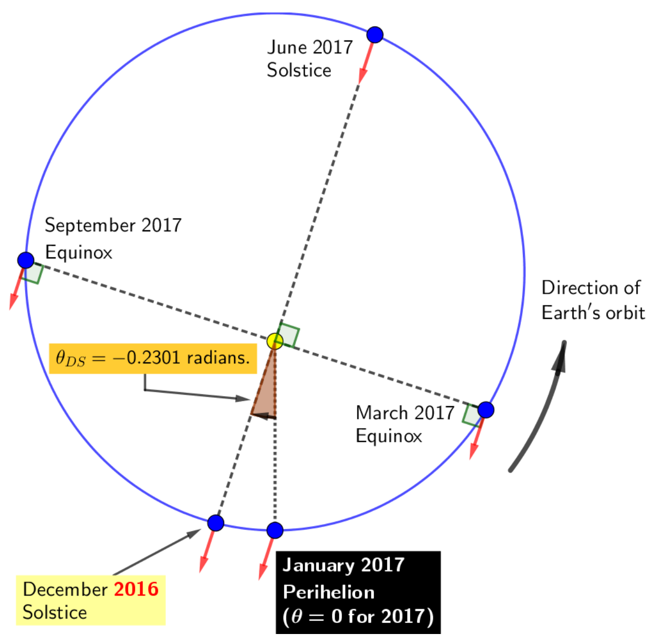

Figure 7: The position of the December 2016 solstice in relation to the 2017 perihelion, according to the best-fit value of θ_{DS} obtained in this document. Note that θ_{DS} is negative rather than (as was assume in Fig. 5) positive.

Event	Days after Dec. 2016 solst.		
	Published*	Calculated**	Calculated-Published (min.)
Dec. 2016 solst	N/A	N/A	N/A
Mar. 2017 equin	88.99	88.99	-4.137
June 2017 solst	92.75	92.75	-2.013
Sept. 2017 equin	93.65	93.65	4.508
Dec. 2017 solst	89.85	89.85	4.480

*See Table 1. **For the best-fit value of $\theta_{DS}, = -0.23009863$ radians.

Table 2: Comparison between published times of 2017 solstices and equinoxes and those calculated from the best-fit value (=-0.23009863 radians) of θ_{DS} found in this study.

and equinoxes, and those calculated using the best-fit estimate of θ_{DS}, is 4.5 minutes (Table 2). That degree of accuracy is sufficient for the future document in which this best-fit value of θ_{DS} will be used.

4 Conclusions

We found that for the December 2016 solstice, θ_{DS} is -0.23009863 radians. Using that result, calculated times of the year 2017's solstices and equinoxes differ from published values by less than five minutes. That degree of accuracy is sufficient for the calculations in which the best-fit θ_{DS} will be used.

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