# Union of two arithmetics sequences
# Basic calculation formula
# (1)

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**Abstract**. Let $A, B$ denote infinite arithmetic sequences in $\mathbb{N}$ with initial terms equal to 0. The union of $A, B$ is strictly increasing sequence $U$, containing only all elements of $A, B$ (without repetitions). We will derive the formula for the $n$-th element of the union $U(A,B)=(u_n)$ in the form: $u_n=f(n,a,b)$.

## 1 Purpose

**Definition 1.1** Let $A_1...A_n$ denote infinite, the same monotonic arithmetic sequences in $\mathbb{R}$. Union of $A_1...A_n$ is strictly monotonic sequence $U$, containing only all the terms of the sequences $A_1...A_n$ (without repetition).

In this paper we will deal with the simplest case - the union of two arithmetic sequences $A=(a_i), B=(b_j)$. We will consider only such $A, B$, that meet the following conditions:

**Conditions 1.1**

1) $A, B$ are composed of natural numbers: $\forall_{i,j\in\mathbb{N}}\ a_i, b_j\in\mathbb{N}$,

2) initial terms equal to 0: $a_0=0,\ b_0=0$,

3) common differences are: $a\geq2,\ b\geq2\ \ a,b\in\mathbb{N}$,

4) Greatest common divisor of common diffrences: $GCD(a,b)<Min(a,b)$.

5) $b>a^*$

Using condition 2): $a_i=ia,\ b_j=jb$. In 3) and 4) we immediately exclude trivial cases: for $a=0$ we have $U=B$, for $a=1$ $U=A$ and for $GCD(a,b)=a$ also $U=A$ (and *vice versa* the same for $b$). **(\*)** Condition 5) is not obligatory, but it segregates the properties and concepts used in this paper and will not matter in the end.

The goal is to derive the formula for the $n$-th union $U=(u_n)$ element in the form: $u_n=f(n,a,b)$ in accordance with the definition and conditions.

## 2 Notation and preliminary findings

### 2.1 Basic properties of the union

We will explain the terms we use in the example with a Table 1 (next page) containing the initial 34 elements of a union of sequences with common differences: $a=5, b=13$. The table was filled in so that all the terms of the sequence with a greater difference, ie $B$, were typed in the column

Table 1: Sample union

| $U(5,13)$ | | | $c$ | | | |
|---|---|---|---|---|---|---|
| $R$ | $g$ | $r$ | 0 | 1 | 2 | 3 |
| 0 | | 0 | 0 | 5 | 10 | |
| 1 | | 1 | 13 | 15 | 20 | 25 |
| 2 | 0 | 2 | 26 | 30 | 35 | |
| 3 | | 3 | 39 | 40 | 45 | 50 |
| 4 | | 4 | 52 | 55 | 60 | |
| 5 | | 1 | 65 | 70 | 75 | |
| 6 | | 1 | 78 | 80 | 85 | 90 |
| 7 | 1 | 2 | 91 | 95 | 100 | |
| 8 | | 3 | 104 | 105 | 110 | 115 |
| 9 | | 4 | 117 | 120 | 125 | |

$c$=0. In the columns $c$>0 terms of the sequence $A$, omitting the common terms, were written in rows, so that numbers in current row are less than the values in column 0 of the next row.

Terms of $A, B$ are different except for zero and all multiples of the least common multiple $a$ and $b$. Let us briefly $\omega=LCM(a,b)$ and $\Theta=GCD(a,b)$. The common terms of sequences $A, B$ will take the following values: $\omega_g=g\omega$ for $g\in\mathbb{N}$. For each $g$, all terms of union in the interval $\langle\omega_g,\omega_{g+1}-1\rangle$ are named group $G_g$. In particular, the terms of union of the interval $\langle\omega_0,\omega_1-1\rangle$ are group $G_0$. In the table the numbers of groups $g$ are in the second column.

In this paper we will use the notation of membership of set elements into a group: $X^G$ denote all and only elements of $X$ belonging to a certain group $G$.

## 2.2   Groups

For each group there are obvious relationships:

- Number of rows in a group: $|r^G|=\dfrac{\omega}{b}$. With the general characteristics of $LCM$ and $GCD$ we have $\omega=\dfrac{ab}{\Theta}$, so $|r^G|=\dfrac{a}{\Theta}$.

- Number of terms of sequence $B$ in each group: $|B^G|=|r^G|=\dfrac{a}{\Theta}$.

- Number of terms of sequence $A$ in each group: $|A^G|=\dfrac{\omega}{a}=\dfrac{b}{\Theta}$.

- Number of terms of union $U$ in each group:

$$|U^G|=|B^G|+|A^G|-1=\frac{a}{\Theta}+\frac{b}{\Theta}-1=\frac{a+b-\Theta}{\Theta} \tag{1}$$

## 2.3   Rows

Rows will be marked with $r$ or $R$. When considering only one group (in particular the $G_0$ group), we will use relative rows numbers with lowercase letter $r$, where $r$ can only accept values from 0 to $\dfrac{a}{\Theta}-1$. The uppercase letter $R$ will be the absolute number of any row, referring to the entire table. The following identitys holds:

$$r=Mod\left(R,\frac{a}{\Theta}\right)=R-\frac{a}{\Theta}\left\lfloor\frac{R\Theta}{a}\right\rfloor \tag{2}$$

$$R = r + g\frac{a}{\Theta} \qquad (3)$$

## 2.4 Columns

Columns, marked with $c$, are numbered from 0. In the column 0 there are subsequent terms of sequence $B$: $b_j = b_R = Rb$. This column is always full. Number of columns $c > 0$ depends on the values of $b$ and $a$. Based on the Table 1, we conclude that the columns must be at least $\left\lfloor \frac{b}{a} \right\rfloor$. But, in some cases, they may be one more. This will happen when $0 < a_{min}^R < b - a\left\lfloor \frac{b}{a} \right\rfloor$, where $a_{min}^R$ is the smallest value $a_i$ in row $R$. It follows that all of the columns from 1 to $\left\lfloor \frac{b}{a} \right\rfloor$ are filled. The last column, always numbered $\left\lfloor \frac{b}{a} \right\rfloor + 1$, contains only the numbers of the $A$ sequence, which satisfy the condition $bR + \left\lfloor \frac{b}{a} \right\rfloor a < ai < b(R+1)$. This column is never completly full because at least the initial $(r=0)$ and the last row $\left( r = \frac{a}{\Theta} - 1 \right)$ of each group must be empty cells.

## 2.5 Indexes

As for the rows: when considering only one group, in particular the $G_0$ group, we will use the relative indices of the union indexes with lower case $n$, where $n$ can take values from 0 to $|U^G| - 1$ only (see (1)). The capital letter $N$ will be the absolute index of any union element, referring to the entire table.

## 2.6 Other properties

1. For the given coprime $a, b$ and any multiple of them, i.e. for $a' = \Theta a$, $b' = \Theta b$ the Table 1 layout for each group is identical, the number of rows in the group and the number of columns is the same, and in the last column the numbers always appear in the same positions as in the table for $a$ and $b$. Hence in the formulas there is $\Theta$.

2. Numbers in $G_g$ group are greater than the numbers in $G_0$ at the same positions by $g\omega$, ie for $R \in G_g$: $u(g, R, c) = u\left( 0, R - g\frac{a}{\Theta}, c \right) = u(r, c) + g\omega$. Therefore, knowing only the properties of the $G_0$, we can deduce all the properties of the whole union.

# 3 Deriving formula for the union

The final formula $u_N = f(N)$ will be derived in several steps:

1. We derive an indirect formula for union terms in group $G_0$, dependent on rows and columns numbers: $u(n, r, c) = f(n, r, c)$.

2. For each of the numbers $u(n, r, c)$ we will calculate the row number $r = r(n)$.

3. For any group this relative formula will be converted to an absolute row number: $R = R(N)$.

4. We specify the column number that determines the affiliation of the union element to sequence $A$ or $B$: $C = C(N)$.

5. We get the final result in the form: $u_N = f(N, R(N), C(N)) = f(N)$.

## 3.1  Step 1

We work within the $G_0$ group. All numbers in the column $c=0$ are multiples of $b$. The remaining columns contain multiples of $a$. For $c>0$ this multiple will be equal to the index $i$ of the term $a_i$. Because $n=r+i$, ie $i=n-r$, then:

$$u(n,r,c)=\begin{cases} br & \text{for: } c=0 \\ a(n-r) & \text{for: } c>0 \end{cases}$$

(4)

## 3.2  Step 2

For each $u_n$ the index $n$ is equal to number of union elements less then $u_n$. If $u_n$ is in column 0, then $n$ is equal to the number of all numbers in rows from 0 to $r(n)-1$. Among them are $r$ numbers from $B$ and $\left\lfloor \dfrac{b_r}{a} \right\rfloor = \left\lfloor \dfrac{rb}{a} \right\rfloor$ from $A$, which together give:

$$n=r+\left\lfloor \frac{rb}{a} \right\rfloor = \left\lfloor \frac{r(a+b)}{a} \right\rfloor \qquad \text{for } c=0$$

We write the right floor by definition, taking into account that its internal fraction can not be integer (Condition 4), so both inequalities will be strict inequalities:

$$n<\frac{r(a+b)}{a}<n+1$$

$$\frac{na}{a+b}<r<\frac{(n+1)a}{a+b}$$

Since left inequality is strict and the difference between right and left expressions is less than 1, then:

$$r=\left\lfloor \frac{(n+1)a}{a+b} \right\rfloor$$

(5)

## 3.3  Step 3

The formula (5) is derived in $G_0$, but using only relative values of $r$ and $n$ is correct in each group. To determine the relation $R(N)$:

i) any index $N$ from a certain group $G_g$ we will convert to its equivalent $n$

ii) from (5) we will calculate $r(n)$

iii) next, we will recalculate the absolute row number $R$ from $G_g$, resulting in the desired dependence.

**i)**
For any $N$: $n=Mod\left(N,|U^G|\right)$. By substituting (1) we have:

$$n=Mod\left(N,\frac{a+b-\Theta}{\Theta}\right)$$

We write the $Mod$ from the definition:

$$n=N-\frac{a+b-\Theta}{\Theta}\left\lfloor \frac{N\Theta}{a+b-\Theta} \right\rfloor$$

**ii)**

We put the calculated $n$ to (5):

$$r = \left\lceil \frac{\left(N - \frac{a+b-\Theta}{\Theta}\left\lfloor \frac{N\Theta}{a+b-\Theta}\right\rfloor + 1\right)a}{a+b} \right\rceil \tag{6}$$

**iii)**

For further calculations we can use (3), but we will need a group number. We'll count it from the dependence:

$$g = \left\lfloor \frac{N}{|U^G|} \right\rfloor$$

We substitute (1):

$$g = \left\lfloor \frac{N\Theta}{a+b-\Theta} \right\rfloor \tag{7}$$

Now (6) and (7) we insert into (3):

$$R = \left\lceil \frac{\left(N - \frac{a+b-\Theta}{\Theta}\left\lfloor \frac{N\Theta}{a+b-\Theta}\right\rfloor + 1\right)a}{a+b} \right\rceil + \left\lfloor \frac{N\Theta}{a+b-\Theta} \right\rfloor \frac{a}{\Theta}$$

To shorten and simplify the formulas we temporarily use $g$ (from (7)):

$$R = \left\lceil \frac{\left(N - \frac{a+b-\Theta}{\Theta}g + 1\right)a}{a+b} \right\rceil + g\frac{a}{\Theta}$$

The right component is integer so we can put it to the floor:

$$R = \left\lceil \frac{\left(N - \frac{a+b-\Theta}{\Theta}g + 1\right)a}{a+b} + g\frac{a}{\Theta} \right\rceil$$

$$= \left\lceil \frac{a}{a+b}\left(N - \frac{1}{\Theta}(a+b-\Theta)g + 1\right) + g\frac{a}{\Theta} \right\rceil$$

$$= \left\lceil \frac{a}{a+b}(N+1) - \frac{a}{a+b}\frac{1}{\Theta}(a+b-\Theta)g + g\frac{a}{\Theta} \right\rceil$$

$$= \left\lceil \frac{a}{a+b}(N+1) + g\left(\frac{a}{\Theta} - \frac{a}{a+b}\frac{1}{\Theta}(a+b-\Theta)\right) \right\rceil$$

$$= \left\lceil \frac{a}{a+b}(N+1) + \frac{a}{a+b}g\left(\frac{a+b}{\Theta} - \frac{a+b-\Theta}{\Theta}\right) \right\rceil$$

$$= \left\lceil \frac{a}{a+b}(N+1) + \frac{a}{a+b}g\cdot(1) \right\rceil$$

$$= \left\lceil \frac{a}{a+b}(N+1+g) \right\rceil$$

$$= \left\lceil \frac{a}{a+b}\left\lfloor \frac{N\Theta}{a+b-\Theta} + N + 1 \right\rfloor \right\rceil \text{ after insertion of } g.$$

Ultimately:

$$R(N) = \left\lceil \frac{a}{a+b}\left\lfloor \frac{N(a+b)}{a+b-\Theta} + 1 \right\rfloor \right\rceil \tag{8}$$

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### 3.4   Step 4

We don't need to find a specific column number, but to distinguish whether the union element is in column 0 or outside. So the formula will be:

$$C(N)=\begin{cases}0 & \text{for: } c=0\\1 & \text{for: } c>0\end{cases}$$

We can construct it easily, noting that the union element of index $N>0$ occurs in column 0 only when, the row has changed at the same time.

$$C(N)=\begin{cases}0 & \text{for: } R(N)-R(N-1)=1\\1 & \text{for: } R(N)-R(N-1)=0\end{cases}\quad\text{for: } N>0$$

Let's say the same thing in unconditional form:

$$C(N)=1-(R(N)-R(N-1))=1+R(N-1)-R(N)$$

After substituting (8) we obtain:

$$C(N)=1+\left\lfloor\frac{a}{a+b}\left\lfloor\frac{(N-1)(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor-\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor \tag{9}$$

The above formula for $N=0$ gives:

$$C(0)=1+\left\lfloor\frac{a}{a+b}\left\lfloor\frac{-\Theta}{a+b-\Theta}+1\right\rfloor\right\rfloor-\left\lfloor\frac{a}{a+b}\right\rfloor=1+\left\lfloor\frac{a}{a+b}\cdot(-1)\right\rfloor-\left\lfloor\frac{a}{a+b}\right\rfloor=1-1-0=0$$

ie (9) is correct for $N>0$ and for $N=0$, so it is true for every $N$.

### 3.5   Step 5

The formula (4) for relative values will be transformed into formula for absolute values. The first member, for $c=0$ ($C=0$), will have the form: $bR$. For $c>0$ ($C=1$), the common element for sequences $A$ and $B$ from the column $c=0$ should be taken into account at the beginning of each group. This means adding a $G$-number to the index: $i=N-R+g$. After this correction we receive:

$$u(N,g,R,C)=\begin{cases}bR & \text{for: } C=0\\a(N-R+g) & \text{for: } C=1\end{cases} \tag{10}$$

We write the lower formula (for $C=1$):

$$u(N,g,R,1)=a(N-R+g)$$

We substitute (8) and (7) and continue for $C=1$:

$$u(N)=a\left(N-\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor+\left\lfloor\frac{N\Theta}{a+b-\Theta}\right\rfloor\right)$$

Place the first and last member into the middle floor by changing the sign:

$$u(N)=-a\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor-\left\lfloor\frac{N\Theta}{a+b-\Theta}\right\rfloor-N\right\rfloor$$

Place the final $N$ into the preceding floor:

$$u(N)=-a\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor-\left\lfloor\frac{N\Theta+N(a+b-\Theta)}{a+b-\Theta}\right\rfloor\right\rfloor$$

$$=-a\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor-\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor\right\rfloor$$

Exclude number 1 from the first floor:

$$u(N)=-a\left\lfloor\frac{a}{a+b}+\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor-\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor\right\rfloor$$

$$=-a\left\lfloor\frac{a}{a+b}+\left(\frac{a}{a+b}-1\right)\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor\right\rfloor$$

$$=-a\left\lfloor\frac{a}{a+b}-\frac{b}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor\right\rfloor$$

$$=-a\left\lfloor-\frac{b}{a+b}\left(\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor-\frac{a}{b}\right)\right\rfloor$$

Using the dependency $-\lfloor-x\rfloor=\lceil x\rceil$ we substitute the floor for the ceil:

$$u(N)=a\left\lceil\frac{b}{a+b}\left(\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor-\frac{a}{b}\right)\right\rceil \qquad\text{for}\quad C=1 \qquad\qquad (11)$$

Now, after substituting the formulas (8), (9) and (11) to (10), we obtain the final, complete formula for the $N$-th union element:

$$u_N=\begin{cases}b\left\lfloor\dfrac{a}{a+b}\left\lfloor\dfrac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor & \text{for}\quad C=0 \\[3ex] a\left\lceil\dfrac{b}{a+b}\left(\left\lfloor\dfrac{N(a+b)}{a+b-\Theta}\right\rfloor-\dfrac{a}{b}\right)\right\rceil & \text{for}\quad C=1\end{cases} \qquad (12)$$

$$\text{where}\quad C=1+\left\lfloor\frac{a}{a+b}\left\lfloor\frac{(N-1)(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor-\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor$$

To simplify formula (12) we substitute: $F(N)=b\left\lfloor\dfrac{a}{a+b}\left\lfloor\dfrac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor$ , $H(N)=\left\lceil\dfrac{b}{a+b}\left(\left\lfloor\dfrac{N(a+b)}{a+b-\Theta}\right\rfloor-\dfrac{a}{b}\right)\right\rceil$
Now we can write short formula in unconditional form:

$$u_N=F(1-C)+HC \qquad\qquad (13)$$

**Remark 1** If $b\leq a$, and all the terms of the sequence $B$ will be typed in the column $c=0$ of Table 1, then all the above reasoning can be repeated, obtaining the same result. Then, for $b<a$ the table will have only two columns and for $b=a$ formula (12) simplifies to $u_N=aN$.

**Remark 2** To be continued: In the following articles we will show more complex versions of formula (12), for $a_0, b_0 <> 0$ and for non-integer numbers. And something else...