Field Theory with Fourth-order Differential Equations

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Abstract

We introduce a new class of higgs type fields $\{U, U^\mu, U^{\mu\nu}\}$ with Feynman propagator $\sim 1/p^4$, and consider the matching to the traditional gauge fields with propagator $\sim 1/p^2$ in the viewpoint of effective potentials at tree level. With some particular restrictions on the convergence, there are a wealth of potential forms generated by the fields $\{U, U^\mu, U^{\mu\nu}\}$, such as: (1) in the case of U coupled to the intrinsic charges of matter fields, electromagnetic Coulomb potential with an extra linear potential and Newton's gravitation could be generated with the operators of different orders from the dynamics of U, respectively; (2) for the matter fields, with the multi-vacuum structure of a sine-Gordon type vector field A^μ induced from U, a seesaw mechanism for gauge symmetry and flavor symmetry of fermions could be generated, in which the heavy fermions could be produced; besides, by treating the fermion current as a field, a possible way for renormalizable gravity could be proposed; (3) the Coulomb potential in electromagnetism and gravitation could be generated by an anti-symmetric field strength of U^μ, when it's coupled to the intrinsic charge and momentum of matter fields, respectively; and, except for the Coulomb part in each case, there is a linear and a logarithmic part in the former case which might correspond to the confinement in strong QED, while there is a linear and a logarithmic part in the latter case which might correspond to the dark energy effects in the impulsive case and dark matter effects in the attractive case, respectively; besides, a symmetric field strength of U^μ could also generate the same gravitation form as the anti-symmetric case; (4) a nonlinear version Klein-Gordon equation, QED and the Einstein's general relativity, could be generated as a low energy approximation of the dynamics of U, U^μ and $U^{\mu\nu}$, respectively; moreover, in the weak field case, the gauge symmetry could superficially arise, and, a linear QED, linear gravitation and a 3rd-order tensor version QED could be generated by relating the field strength of U, U^μ and $U^{\mu\nu}$ to the corresponding gauge fields, respectively; (5) for the massive $\{U, U^\mu\}$, attractive potentials for particles with the same kind of charges could be generated, which might serve as candidate for interactions maintaining the s-wave pairing and d-wave pairing Cooper pairs in superconductors, with electric charge in the U case and magnetic moment in the U^μ case as interaction charge, respectively; etc.

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Contents

1 Introduction

As a very successful theory, the gauge field theory with the gauge invariance principle could be used to solve a huge part of questions for people. Certainly, there are some challenges to the gauge theory: one class about the extension for methods of application, such as the ones for non-perturbative problems; another class about the extension for new phenomenons, such as the ones for new particles or dark matter/dark energy effects; with an inevitable old topic about the unification and renormalization.

It's just the linear potential from the non-perturbative results in lattice gauge theory [1] that motivated us to consider a fourth order differential equations (D.E.). And, mathematically, a most straightforward way on the extensions for new particles could be related to the higher order D.E., generally with a trouble on dealing with the redundant unphysical/noncausual degrees of freedom (d.o.f) and an omittance/ignorance on the non-perturbative and unification problems. So, it would be significant to modify the higher order D.E. framework to cover the three sectors mentioned above, even with some man-made postulations or constraints. That is just what we have done in this paper.

In this paper, we have taken some postulations to construct our model within the 4th-order D.E. framework, mainly for the convergence(renormalization) and a reasonble performance on matching conditions of the model. For simplicity, we have concentrated our studies on the pro forma feasibility of the model in the view of effective potentials at tree level.

The remainder of this paper is organized as follows. In Sect. 2,3,4, we build the dynamics for the massless scalar, vector and tensor fields, respectively, in the 4th-order D.E. framework. In Sect. 5, we extend the models to a massive case and apply them to discuss the superconductor. The final section is reserved for our conclusions.

2 Field U

2.1 A Lagrangian for linear potential

2.1.1 Framework: effective potentials for tree-level

We can get the classic non-relativistic (NR) potential form from the amplitude of the tree-level "2→ 2" scattering process for a perturbative theory, within the Born-approximation framework, for instance, we can take [2]

$$(\text{vertex})_1 \ \otimes \ (\text{inner-line propagator}) \ \otimes (\text{vertex})_2 \Leftrightarrow \mathcal{V} \tag{1}$$

where the l.h.s is a part of the amplitude for a tree-level Feynman diagram, and the r.h.s is the classic potential. So, conversely, we can build theories for potentials with a definite form through the tree-level-correspondence, provided that the theories are perturbatively computable. For example, if there were neither momentums nor coordinates in the Feynman rules of vertices, we would extract different potentials with different inner-line propagators, such as:

$$\begin{aligned}\text{linear potential} &\leftrightarrow \frac{1}{p^4}\cdot\,,\\ \text{Coulomb potential} &\leftrightarrow \frac{1}{p^2}\,,\\ \text{van der Waals potential} &\leftrightarrow \frac{1}{p^\alpha}\,, \text{with } \infty < \alpha < 2.\end{aligned} \tag{2}$$

2.1.2 A Lagrangian for linear potential

We firstly write a Lagrangian, and then give the illustrations in following subsubsections.

We take $\{U, \psi\}$ as the physical particle degree of freedom(d.o.f), which have the transformation law under a $U(1)$ **global** group element V as

$$U \to VUV^\dagger\,,\ \psi \to V\psi\,, \tag{3}$$

and, with the method in Section 2.1.1, for a propagator $\sim \frac{1}{p^4}$, we take the Lagrangian with Lorentz symmetry and the $U(1)$ **global** symmetry as

$$\mathcal{L} = \mathcal{L}_U + \mathcal{L}_\psi + \mathcal{L}_I, \tag{4}$$

where the kinetic energy term

$$\begin{aligned}\mathcal{L}_U &= -\partial^\mu\partial^\nu U^\dagger \partial_\mu\partial_\nu U - \Lambda_U^4[(U+U^\dagger)+i(U-U^\dagger)] + m_U^4 U^\dagger U\,,\\ \text{with} \quad & U^\dagger U \le 1\,,\end{aligned} \tag{5}$$

is for free complex-valued field U;[1] the term

$$\mathcal{L}_\psi = \bar\psi(i\partial\!\!\!/ - m_\psi)\psi \tag{6}$$

is for free particle ψ; and,

$$\begin{aligned}\mathcal{L}_I = & -\alpha Q\Lambda\,\bar\psi[(U+U^\dagger)+i(U-U^\dagger)]\psi\\ & -\beta Q\,\bar\psi\partial\!\!\!/[(U+U^\dagger)+i(U-U^\dagger)]\psi\\ & -\rho Q\,\frac{1}{M}\partial_\mu[(U+U^\dagger)+i(U-U^\dagger)](\bar\psi i\overleftrightarrow{\partial}^\mu\psi)\\ & -\xi Q\,\frac{1}{M}\bar\psi\partial\!\!\!/\partial\!\!\!/[(U+U^\dagger)+i(U-U^\dagger)]\psi\\ & +\ldots(\text{higher order 3-field terms})\\ & -\kappa Q\,\frac{1}{M}\bar\psi\left\{\Lambda^2\cdot[U^\dagger U] + \Lambda\cdot[U^\dagger i\overleftrightarrow{\partial\!\!\!/}U] + \partial\!\!\!/[U^\dagger i\overleftrightarrow{\partial\!\!\!/}U]\right\}\psi\\ & +\ldots(\text{higher order multi-field terms})\end{aligned} \tag{7}$$

is for the gauge invariant interaction term under the condition of (3), where the "..." denotes terms for multi-field and higher order operators. Each of the coefficients $\{\alpha, \beta, \kappa\}$ takes a real number value for the sake of hermitian; and, Q is an operator corresponding to the generator of gauge group, with $\langle Q\rangle = \pm|Q|$ for particles and anti-particles respectively. Particularly, in the simplest case, for the sake of universality, we can take

$$\alpha = \beta = \rho = \xi = \kappa \equiv g_U\,. \tag{8}$$

[1]For a non-Abelian version of extension, the terms would be taken a trace of the "charge" indices.

For the parameters Λ and M, referring to Wilson's scheme for renormalization, we can propose the **postulation** as:

(i) each U(rather than ∂U) is tied with one small I.R. energy scale Λ, so the operators constituted with multi-U would be spontaneously depressed;

(ii) all the higher-dimensional($D > 4$) operators are depressed by the large U.V. energy scale M. [2]

And, the variable Λ and M for balancing the dimension is set to be

$$\Lambda = \mu_{IR} \xrightarrow{\text{(QED)}} 0, \;\; M = \mu_{UV} \xrightarrow{\text{(QED)}} \mu_{EW} \sim 246\, GeV \tag{9}$$

where μ_{IR} is the infrared boundary(but not the cutoff μ in next sections for dealing with I.R. divergences of loop level processes), and M the ultraviolet boundary for the theory. The first reason for taking a so small μ_{IR} is the fact that the linear potential hasn't been detected in the real QED sector. For the correspondence between μ_{IR} for U and μ_{IR}(or μ_{UV}) for QED, we won't consider in this work.

Since both $\hat{p} = i\partial$ for a complex field and $\hat{p} = \partial$ for a real field are hermite, the two interaction forms

$$\mathcal{L}_I = -\alpha\bar{\psi}(U + U^\dagger)\psi - \beta\bar{\psi}\partial(U + U^\dagger)\psi, \tag{10}$$

and

$$\mathcal{L}_I = -\alpha\bar{\psi}[i(U - U^\dagger)]\psi - \beta\bar{\psi}\partial[i(U - U^\dagger)]\psi, \tag{11}$$

are both hermite and right. Indeed, we can have

$$U \equiv U_1 - iU_2\,, \Rightarrow \frac{U + U^\dagger}{2} = U_1\,, \; \frac{i(U - U^\dagger)}{2} = U_2\,. \tag{12}$$

that means, both U_1 and U_2 include the effects from both U and $U^\dagger$. For future convenience, here we obviously write down the interaction Lagrangian for U_1, as

$$\mathcal{L}_I \;\; = \;\; -\alpha Q\Lambda\, \bar{\psi}U_1\psi - \beta Q\, \bar{\psi}\partial\!\!\!/ U_1\psi - \rho Q\, \frac{1}{M}\bar{\psi}\partial\!\!\!/\partial\!\!\!/ U_1\psi + \ldots\,, \tag{13}$$

and, contributions from U_2 should combine rather than cancel with that from U_1, otherwise, the introduction of U would be trivial.

We don't consider terms as

$$\mathcal{L}_I = -\alpha\Lambda\, \partial^\mu U_1\bar{\psi}[\sigma_{\mu\nu}(\gamma^\mu - i\partial^\mu)]\psi \tag{14}$$

in this work.

2.1.3 On the $\partial\partial U$ term (I): no ∂U, in kinetics term!

Each one of the $\{U, \partial U, \partial\partial U\}$ could be well-define and be taken as the block for constructing Lagrangian terms. However, there are two questions to answer:

1. why is ∂U absent in the kinetic energy term?

[2] If we offer a $1/M$ factor for every ∂ symbol in all the interaction terms, the linear potential would be a dominated part of the interaction since all other interaction terms are depressed, which isn't consistent with the real world.

2. which one is the canonic commutator, among $[U, \dot{U}]$, $[U, \partial\dot{U}]$ and $[\dot{U}, \partial\dot{U}]$?

Firstly, we should note, these are two isolated questions. In a word, for a block-variable, its appearance/absence in the kinetic energy term is irrelevant with its appearance/absence in canonic commutator. For instance, for a massless scalar field A, one of its canonic variable, the field variable A itself as a well-defined d.o.f., doesn't appear in the kinetic term, while for a Dirac field ψ, its canonic variables ψ and $\dot{\psi}$ both appear in the kinetic term.

Secondly, it's not incomprehensible for the kinetic energy term with four derivatives: now that the kinetic energy term of Dirac field could include only one derivative rather than two for the case of Klein-Gordon fields, the number of derivative in kinetic energy term could be possibly as many as it needed. So, we could say, which variables and how many ∂ operators appear in the kinetic energy term could be irrelevant with the ones in canonic commutator, and, it's allowed to construct the kinetic energy term with the $\partial\partial U$ variable; or, we could say, the E.O.M is the very core for a field, rather than the kinetic energy term which could be constructed according to the E.O.M. Besides, the stress tensor for a block of continuum material is not completely equivalent the acceleration of a particle, that is,

$$\partial_\mu \partial_\nu U \neq \partial^2 g_{\mu\nu} U \,. \tag{15}$$

Thirdly, for the Question-1, the term

$$(\partial U)^2 \text{ and } U^\dagger \partial\partial U$$

could not appear in our model, since it would give a term $\partial\partial U$ in the E.O.M so a propagator form $\sim 1/(p^4 - p^2)$. However, due to the singularity(pole) structure, we can't get the same results for the two propagators, $\sim 1/(p^4)$ and $\sim 1/(p^4 - p^2)$. Besides, if the E.O.M is not the form $\hat{p}^4 U = m^4 U$, that might break a generalized "charge" symmetry, see Sect. 2.1.5. So, we could only take $\partial\partial U$ to construct the kinetic term rather than ∂U.

Fourthly, for the Question-2, if we asked, "for variables at different level, such as a field U and its derivative $\{\partial U, \partial\partial U, \ldots\}$ or integration $\{\int U, \int\int U, \ldots\}$, which ones could be the well-defined blocks for constructing the canonic commutators, E.O.M and propagators, interaction terms for d.o.f?", then, at least we can say, for different successful frameworks, the matching among them should always be realized, and their results should be always equivalent in the matching region.

2.1.4 On the $\partial\partial U$ term (II): in the viewpoint of continuum medium

A particle is a field, and a field should be a field much more.

Based on the continuum mechanics theory, the stress tensor, a second order tensor, completely define the state of stress inside a material. So, if the velocity(momentum)p^μ are treated as canonic coordinates for a particle, the stress tensor(energy-momentum tensor) $T^{\mu\nu} \sim p^\mu p^\nu$ should be reasonably treated as canonic coordinates for a field. In other words, if the canonic momentum operators of particles are treated as the blocks for construction of the wave equations for particles, such as in relativistic quantum mechanics

$$\hat{p}\Psi = m\Psi,\ \hat{p}^2\Phi = m^2\Phi,\ \ldots.\text{(higher order D.E.)}, \tag{16}$$

where "D.E." denotes differential equations, then the canonic momentum operators of fields should be treated as the blocks for construction of the wave equations for fields, such as

$$\hat{T}\Psi = m^2\Psi,\ \hat{T}^2\Phi = (m^2)^2\Phi,\ ....(\text{higher order D.E.}). \tag{17}$$

The E.O.M is contained in the continuity equation for a **particle** current, as [2]

$$\begin{aligned} \partial_\mu J^\mu &= 0, \\ J^\mu &= \rho u^\mu = \Phi^\dagger i\partial^\mu\Phi - i\partial^\mu\Phi^\dagger\Phi \Rightarrow i\partial^\mu\Phi = p^\mu\Phi\,,\ p^\mu = mv^\mu & (18) \\ \xrightarrow{\text{(square)}} -\partial^2\Phi &= p^2\Phi = m^2\Phi \Rightarrow p^\mu = \pm mv^\mu\,, & (19) \end{aligned}$$

where we take directly a square for each operator to get the Klein-Gordon equation for a relativistic extension. So, under the correspondence, the continuity equation and E.O.M for a **field** should be as

$$\begin{aligned} \partial_\mu\partial_\nu T^{\mu\nu} &= 0, \\ T^{\mu\nu} &= (\rho+p)u^\mu u^\nu + pg^{\mu\nu} \quad \text{(symmetric tensor)} \\ &= \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_\alpha)}\partial^\mu\phi_\alpha - g^{\mu\nu}\mathcal{L} \\ &= (\partial_\mu\phi^\dagger\partial_\nu\phi + \partial_\nu\phi^\dagger\partial_\mu\phi) - g^{\mu\nu}(\partial_\alpha\phi^\dagger\partial^\alpha\phi - m^2\phi^\dagger\phi) \\ &= \phi^\dagger\left[\overleftarrow{\partial}_\mu\partial_\nu\phi + \overleftarrow{\partial}_\nu\partial_\mu - g^{\mu\nu}(\overleftarrow{\partial}_\alpha\partial^\alpha - m^2)\right]\phi \\ &= \phi^\dagger\left[(i\partial_\mu i\partial_\nu\phi + i\partial_\nu i\partial_\mu) - g^{\mu\nu}(i\partial_\alpha i\partial^\alpha - m^2)\right]\phi \\ \Rightarrow i\partial^\mu i\partial^\nu\phi &= p^\mu p^\nu\phi\,,\ p^2 - m^2 = 0 \\ \xrightarrow{\text{(square)}} \partial^4 U &= p^4 U = m_U^4 U \Rightarrow p^2 = \pm m_U^2\,, & (20) \end{aligned}$$

where we also take directly a square for each operator to get the relativistic extension.

2.1.5 On the $\partial\partial U$ term (III): in the viewpoint of anti-particle

Although there exist acausal solutions for differential equations with orders higher than 2, see Ref. [4], we can just omit them by treating them as non-physical (or, frozen) d.o.f, or, treat them as effects of hidden new degrees of freedom (existent but can't be directly measured for some reasons, such as being confined or spreading to the higher dimensions) beyond the standard model(SM) in particle physics; the latter one case is just what we want to propose, as discussed in Section 2.1.7.

Only on the viewpoint of mathematics, the general solution to (39) could be the form

$$U(x) = c_1 e^{ip\cdot x} + c_2 e^{-ip\cdot x} + c_3 e^{p\cdot x} + c_4 e^{-p\cdot x}\,, \tag{21}$$

which could be converted to a particular solution with a specific boundary condition. However, we can write the E.O.M in another form,

$$\hat{p}^4 U(x) = [\hat{p}^2\Phi(x)]^2 = [\hat{p}^2\widetilde{\Phi}(x)]\cdot[\hat{p}^2\Phi(x)]\,, \tag{22}$$

with the correspondence for $\widetilde{\Phi}$ to Φ here is just like a generalized version of the case that the anti-particles $\bar{\psi}$ associated with the particles ψ, which also arised from the treatment that the Dirac equation was formally from the square root of the Klein-Gordon equation. Besides, we can see, if the E.O.M is not the form $\hat{p}^4 U = m^4 U$, then that might break a generalized "charge" symmetry between Φ and $\widetilde{\Phi}$, as mentioned in Sect. 2.1.3. We can denote that as

$$\begin{aligned} & \Phi(x) \sim \langle\bar{\psi}\psi\rangle \\ \Rightarrow \quad & \text{K-G eq.} \ = [\text{ Dirac eq. }]^2 \,, \qquad (23) \\ & U(x) \sim \langle\widetilde{\Phi}\Phi\rangle \\ \Rightarrow \quad & \text{U-eq.} \ = [\text{ K-G eq. }]^2 \,. \qquad (24) \end{aligned}$$

Then we can have the new E.O.M

$$\hat{p}^2\Phi = m_U^2\Phi \Rightarrow \Phi = c_1 e^{ip\cdot x} + c_2 e^{-ip\cdot x} \qquad (25)$$

for the ordinary physical d.o.f, and

$$\begin{aligned} \hat{p}^2\widetilde{\Phi} &= -m_U^2\widetilde{\Phi}, \quad \text{(tachyon/higgs)} \qquad (26) \\ \not\Leftrightarrow -\hat{p}^2\widetilde{\Phi} &= m_U^2\widetilde{\Phi}, \qquad \text{(phantom)} \qquad (27) \\ \Rightarrow \widetilde{\Phi} &= c_3 e^{p\cdot x} + c_4 e^{-p\cdot x} \qquad (28) \end{aligned}$$

for the so-called unphysical d.o.f: the tachyons in (26), with an imaginary number valued mass [5], and the phantoms in (27), with a negative kinetic energy [6], respetively.

Now, we can understand how to deal with the divergence part $c_3 e^{p\cdot x} + c_4 e^{-p\cdot x}$ in (21): they could be limited as serving for the two particular cases of boundary conditions, that is, the tachyon/higgs/phantom/instanton solution,

$$\left[e^{p^0x^0}\cdot\theta(-x^0) + e^{-p^0x^0}\cdot\theta(x^0)\right]\cdot\left[e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} + e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\right], \qquad (29)$$

$$\left[e^{p^0x^0}\cdot\theta(-x^0) + e^{-p^0x^0}\cdot\theta(x^0)\right]\cdot C(\boldsymbol{x}), \qquad (30)$$

$$\text{or} \quad \left[e^{p^0x^0}\cdot\theta(-x^0) + e^{-p^0x^0}\cdot\theta(x^0)\right]\cdot\left[e^{\boldsymbol{p}\cdot\boldsymbol{x}}\cdot\theta(-\boldsymbol{x}) + e^{-\boldsymbol{p}\cdot\boldsymbol{x}}\cdot\theta(\boldsymbol{x})\right], \qquad (31)$$

with C a constant function.

Methodologically to say, wherever an infinity exist, it might be the place to discover new d.o.f. Or, we can take a generalization for the "pair" concept. For example:
a. the zero-temperature point , $T = 0$;
b. the light-speed point, $c = 1$;
b. the critical point for cosmological state parameter, $w = 1$, see Ref. [6] [7];
all of them could not be reached, but could be crossed by skipping it, with the introduction of a pair of some kind of conjugated "charges" for the two sides of the critical boundary, as:
a. the magnetic moment in a ferromagnetic system, leading to a generation for states of negative-temperature, with the critical boundary $T = 0^+$ and $T = 0^-$ still could not be reached;
b. the tachyon, leading to a generation for states faster than light, with the critical boundary $c = 1^+$ and $c = 1^-$ still could not be reached;
c. the phantom, leading to a generation for states of negative pressure, $w < 1$, with the critical boundary $w = 1^+$ and $c = w^-$ still could not be reached.

2.1.6 On the $\partial\partial U$ term (IV): hints from the lattice gauge theory

In fact, the term $\bar{\psi} U \psi$ in (7) has been formally introduced in the lattice gauge theory , though Eq.(305) by setting a finite minimal $\epsilon = a \nrightarrow 0$ for the space size, that is, [1]

$$n^\mu D_\mu \psi = \frac{1}{a}\left[\psi(x+\epsilon n) - \psi(x) + (1-U)]\psi(x)\right]. \tag{32}$$

And, there is a kind of orthogonal relations for the Wilson line U_{ij}, as the functional integrations below:

$$\int [dU]\, U_{ij} = 0\,, \int [dU]\, U_{ij} U^\dagger_{kl} = \frac{1}{N}\delta_{il}\delta_{jk}\,, \tag{33}$$

where the indices $ijkl$ denote the lattice grid points.

Besides, as what shown in the computation, it's just the employment of the Wilson loop U_P that ensured the availability of lattice gauge theory, while the usual concepts for the gauge field A in perturbative quantum field theory were almost unavailable and absent.

It is just these subtle hints that reminded us the importance of U_P, and inspired us to consider a field U, with a hidden correspondence of

$$U_P \to U\,, \tag{34}$$

rather than the gauge field A as a possible effective particle degree of freedom, which might even be a more general concept for all g-valued cases. And, since the field U is corresponding to the Wilson line $U_P(y,x)$, which could visually be seemed as a propagator of a particle in the lattice,

$$U \sim U_P(y,x) \sim \langle \widetilde{\Phi}\Phi \rangle\,, \tag{35}$$

so it's not difficult to understand that the E.O.M of U is the square of K-G equation, as shown in (24).

Some details in the lattice gauge theory

However, since U wasn't treated as a particle in Ref. [1], the term $\bar{\psi} U \psi$ in lattice gauge theory is essentially different with the one in (7), for instance, there would't be an apparent kinetics Lagrangian for a particle U (nevertheless, that surely doesn't matter with the employment of functional method, which is also available in the semi-classic framework). Indeed, in the derivation of the linear potential in Ref. [1], the field A (equivalently, the U) was treated as a classic field without free excitation modes, besides, with the gauge invariance of U_P, the Lagrangian were parameterized as $\mathcal{L}_{gluon} \sim \mathcal{L}_{gluon}(U_P)$ for gluons, and $S \sim U_P$ for the total action, which could also be said that, the Lagrangian of a "free particle" U was chosen to a nonstandard Lagrangian as

$$S \sim U_P \;\Rightarrow\; \mathcal{L}_U \sim \partial^4 (U + U^\dagger), \tag{36}$$

after a correspondence of $U_P \to U$.

Besides, from the parameterization $S \sim U_P$ in semi-classic sense in Ref. [1] we can know: if U_P was quantized, so would be the action S! However, we should note that, although U_P could be a particle degree of freedom, S couldn't be, since S would change its form after quantization, for instance, it would be constituted with multi-field coupled terms including the block U_P as a field!

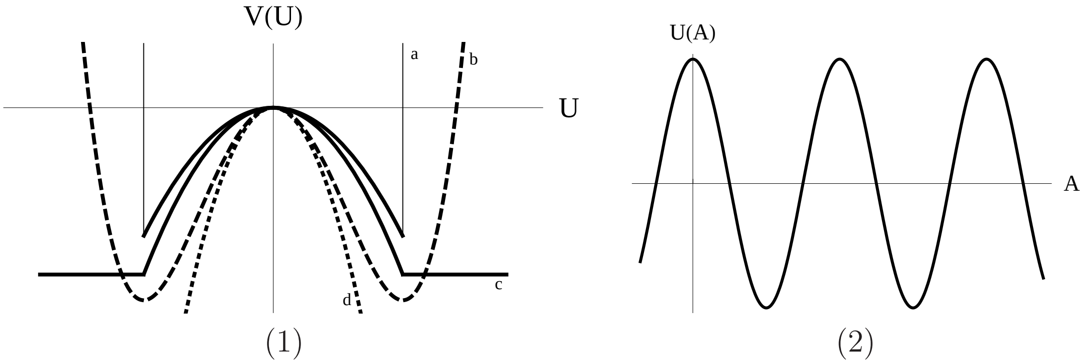

Figure 1: Self-interaction potentials for the field U and A.

2.1.7 On the mass term: more than higgs!

Why do we define $U^{\dagger}U < 1$?[3]

U is a special higgs field.

Firstly, U is a complex-valued field rather than a quaternion, so it has only two components. For a potential $V(U)$ with the form as the line-"a" in Fig.-1-(1), which is defined only for $|U| < 1$ rather than for all the U field configuration, we can treat U as a field including: a radial-direction component U_r, and a angular component U_θ as the conventional field(the Goldstone boson).

Secondly, for a higgs field U with a potential form as the line-"b" in Fig.-1-(1), we can generally decompose its radial-direction component to two fluctuation: a stable(physical) one based on the stable vacuum (minimum of the potential $V(U)$), and a unstable(unphysical) one based on the unstable vacuum (maximum of the potential $V(U)$); the former one could be seemed as the traditional excitation of "higgs particle", and the latter one would be "die out". Here, the most important point is, how to consider the U_r? For the case of line-"a", we can surely treat U_r as the former one, however, now we can also treat U_r as the latter one, since now U_r can keep exciting at the point of $U = 0$ without a "death"(which would happen in the case of line-"d" in Fig.-1-(1)), which could be more reasonable in the strong field case.

We will take the latter one choice, as discussed in Section 2.1.5. So, now, we needn't give too many query to the sign of the mass term in (5). We can say: yes, U is a kind of higgs-type field, and U does have a nonzero VEV, however, the U field (with $\hat{p}^4 U = m^4 U$) is not a traditional higgs field (or, tachyon, with $\hat{p}^2 U = -m^2 U$, see (26)). The choice for the sign of the mass term is very important and crucial for our following work.

2.2 The kinetics of U

2.2.1 The equation of motion of U

By the Euler-Lagrange equation [3]

$$\frac{\partial \mathcal{L}_U}{\partial U} - \partial_\mu \frac{\partial \mathcal{L}_U}{\partial(\partial_\mu U)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}_U}{\partial(\partial_\mu \partial_\nu U)} = 0\,, \tag{37}$$

[3]Indeed, we can define $U^{\dagger}U < v_0$, with v_0 a constant.

from (5) we can get the equation of motion(E.O.M) of U, see Appendix D,

$$\begin{aligned} -\partial^\mu\partial^\nu\partial_\mu\partial_\nu U &= -m_U^4 U + \Lambda_U^4 && (38)\\ \Leftrightarrow \qquad -\hat{p}^4 U &= -m_U^4 U + \Lambda_U^4\,,\ \hat{p}^\mu = i\partial^\mu\,, && (39)\end{aligned}$$

and the dynamical E.O.M for U, see (368) in Appendix D, as

$$-\,\partial^4 U = -m_U^4 U + \Lambda_U^4 + \alpha Q\Lambda\,\bar{\psi}\psi + \ldots\,. \tag{40}$$

So, that means, the media field U would only be influenced by the "scalar currents" of the matter field, but not the vector currents, which is fit with the common sense.

The appearance of term $(U+U^\dagger)$ must be in the combination with the term $U^\dagger U$, by the requirement for a stable vacuum, and, the role of term $(U+U^\dagger)$ is to provide a shift for the position of vacuum, as

$$V(U) = \Lambda_U^4(U+U^\dagger) - m_U^4 U^\dagger U = -m_U^4\left[\left(U-\frac{\Lambda_U^4}{m_U^4}\right)^\dagger\left(U-\frac{\Lambda_U^4}{m_U^4}\right) - (\frac{\Lambda_U^4}{m_U^4})^2\right]. \tag{41}$$

2.2.2 The canonic commutator and propagator

Please pay attention to the free propagator! There are two crucial problems about it:
1. whether it's reasonable for the application of the traditional canonic framework with only two canonic variables to a 4th-order D.E.?
2. how to construct the canonic commutator for U, especially, how to decide the "±" sign?

Firstly, if we **crudely** copy the tradition of the procedure for P-2 type field theory, then, according to the **custom** on the choice of "±" sign in classic Poisson bracket

$$[p_i, x_j] = -i\delta_{ij}\,, \tag{42}$$

and its quantum version for scalar field

$$[\dot{U}_i(\boldsymbol{x},t), U_i(\boldsymbol{y},t)] = -i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,, \tag{43}$$

we just need assign the canonic commutators below to quantize our model:
Postulation

$$\begin{aligned} \left[\partial^2\dot{U}_i(\boldsymbol{x},t), U_i(\boldsymbol{y},t)\right] &= -i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) && (44)\\ \xrightarrow{(\boldsymbol{x}=\boldsymbol{y})}\neq \left[\partial\dot{U}(\boldsymbol{x},t), \partial U(\boldsymbol{y},t)\right] &\sim \left[\dot{A}_\mu(\boldsymbol{x},t), A_\nu(\boldsymbol{y},t)\right] = ig_{\mu\nu}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,, && (45)\\ \text{others} &= 0\,. && (46)\end{aligned}$$

Formally, maybe we can understand (44) in another viewpoint (45), where $U \simeq 1 + i\epsilon n_\mu A^\mu$.

Or, maybe we can say, the Lagrangian should be originally constructed with the block $\partial\partial\partial U$, as

$$\begin{aligned}\mathcal{L} &= +\partial_\mu U\cdot\partial^2\partial^\mu U + m_U^4 UU = \partial_\mu U\partial_\nu\partial^\nu\partial^\mu U + m_U^4 UU\\ &= \partial_\nu\left(\partial_\mu U\partial^\nu\partial^\mu U\right) - \partial_\nu\partial_\mu U\partial^\nu\partial^\mu U + m_U^4 UU && (47)\end{aligned}$$

and then modified to the from in (5),

$$\mathcal{L} \ \rightarrow\ -\partial_\nu\partial_\mu U\partial^\nu\partial^\mu U + m_U^4 UU\,, \tag{48}$$

where the derivative term in (47) was dropped. And, with the corresponding Euler-Lagrangian equation

$$\frac{\partial\mathcal{L}_U}{\partial U}-\partial_\mu\frac{\partial\mathcal{L}_U}{\partial(\partial_\mu U)}+\partial_\mu\partial_\nu\frac{\partial\mathcal{L}_U}{\partial(\partial_\mu\partial_\nu U)}-\partial_\alpha\partial_\mu\partial_\nu\frac{\partial\mathcal{L}_U}{\partial(\partial_\alpha\partial_\mu\partial_\nu U)}=0\,, \tag{49}$$

we can get a definition for the "canonic momentum" as

$$\pi(\boldsymbol{x})=\frac{\partial\mathcal{L}_U}{\partial\dot{U}}=+\partial^2\partial^\mu U\,, \tag{50}$$

just the one in (44), with the Hamiltonian form

$$H\sim+\int d^3\boldsymbol{x}\pi(\boldsymbol{x})\dot{U}(\boldsymbol{x})-L_U\,. \tag{51}$$

Secondly, by inserting one of the definition ("physical version") of propagator ($U_{1,2}\equiv U$ for simplicity in this subsection)

$$\begin{aligned}D_F(x-y)&\equiv\langle 0|TU(x)U(y)|0\rangle\\&=\theta(x^0-y^0)\langle 0|U(x)U(y)|0\rangle+\theta(y^0-x^0)\langle 0|U(y)U(x)|0\rangle\end{aligned} \tag{52}$$

into the E.O.M, we can verify its correctness, as

$$\begin{aligned}&-(\partial^4-m^4)_x D_F(x-y)\equiv(\partial^4-m^4)_x\langle 0|TU(x)U(y)|0\rangle\\
=\;&-\{(\partial^4-m^4)\left[\theta(x^0-y^0)\langle 0|U(x)U(y)|0\rangle\right]\\
&+(\partial^4-m^4)\left[\theta(y^0-x^0)\langle 0|U(y)^\dagger U(x)|0\rangle\right]\}\\
=\;&-\{\partial^4\theta(x^0-y^0)\cdot\langle 0|U(x)U(y)|0\rangle+4\,\partial^3\theta(x^0-y^0)\cdot\partial\langle 0|U(x)U(y)|0\rangle\\
&+6\,\partial^2\theta(x^0-y^0)\cdot\partial^2\langle 0|U(x)U(y)|0\rangle+4\,\partial\theta(x^0-y^0)\cdot\partial^3\langle 0|U(x)U(y)|0\rangle\\
&+\theta(x^0-y^0)\cdot(\partial^4-m^4)\langle 0|U(x)U(y)|0\rangle\\
&+\partial^4\theta(y^0-x^0)\cdot\langle 0|U(y)U(x)|0\rangle+4\,\partial^3\theta(y^0-x^0)\cdot\partial\langle 0|U(y)U(x)|0\rangle\\
&+6\,\partial^2\theta(y^0-x^0)\cdot\partial^2\langle 0|U(y)U(x)|0\rangle+4\,\partial\theta(y^0-x^0)\cdot\partial^3\langle 0|U(y)U(x)|0\rangle\\
&+\theta(y^0-x^0)(\partial^4-m^4)\cdot\langle 0|U(y)U(x)|0\rangle\}\\
=\;&-\Big\{\delta'''(x^0-y^0)\cdot\langle 0|[U(x),U(y)]|0\rangle+4\,\delta''(x^0-y^0)\cdot\partial\langle 0|[U(x),U(y)]|0\rangle\\
&+6\,\delta'(x^0-y^0)\cdot\partial^2\langle 0|[U(x),U(y)]|0\rangle+4\,\delta(x^0-y^0)\cdot\partial^3\langle 0|[U(x),U(y)]|0\rangle\Big\}\\
=\;&-\{-\delta(x^0-y^0)\cdot\partial^3\langle 0|[U(x),U(y)]|0\rangle+4\,\delta(x^0-y^0)\cdot\partial^3\langle 0|[U(x),U(y)]|0\rangle\\
&-6\,\delta(x^0-y^0)\cdot\partial^3\langle 0|[U(x),U(y)]|0\rangle+4\,\delta(x^0-y^0)\cdot\partial^3\langle 0|[U(x),U(y)]|0\rangle\}\\
=\;&-\{\delta(x^0-y^0)\cdot\langle 0|[\partial^3U(x),U(y)]|0\rangle\}\\
=\;&+i\delta^{(4)}(x-y)\,,\end{aligned} \tag{53}$$

with the condition that $\delta'(x^0-y^0)$ and $\delta'''(x^0-y^0)$ are odd functions with respect to x_0, and the relations on δ-functions:

$$\int_{-\infty}^{\infty}\delta'(x)\varphi(x)\,dx=[\delta(x)\varphi(x)]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\delta(x)\varphi'(x)\,dx=-\int_{-\infty}^{\infty}\delta(x)\varphi'(x)\,dx\,, \tag{54}$$

$$\int_{-\infty}^{\infty}\delta''(x)\varphi(x)\,dx=-\int_{-\infty}^{\infty}\delta'(x)\varphi'(x)\,dx=\int_{-\infty}^{\infty}\delta(x)\varphi''(x)\,dx\,, \tag{55}$$

$$\int_{-\infty}^{\infty}\delta'''(x)\varphi(x)\,dx=-\int_{-\infty}^{\infty}\delta''(x)\varphi'(x)\,dx=\int_{-\infty}^{\infty}\delta'(x)\varphi''(x)\,dx=-\int_{-\infty}^{\infty}\delta(x)\varphi'''(x)\,dx\,. \tag{56}$$

That means, $D_F(x-y)$ is really the propagator of U. We can get the Feynman propagator $D_F^{\nu\rho}$ through its another definition("mathematical version"), that is, with an equation for $D_F^{\nu\rho}$ from the E.O.M in (39,40) or (368) in Appendix D, by setting $\Lambda_U = 0$, we have

$$-(\partial^4 - m_U^4) D_F(x-y) \;=\; i\delta^{(4)}(x-y)\,, \tag{57}$$

$$\text{or} \qquad -(p^4 - m_U^4)\widetilde{D}_F(p) \;=\; i\,, \tag{58}$$

with the solution in momentum space

$$\widetilde{D}_F(p) = \frac{-i}{p^4 - m_U^4 + i\epsilon} = \frac{-i}{(p^2 + m_U^2 - i\epsilon)(p^2 - m_U^2 + i\epsilon)} \quad (\text{when } \Lambda_U = 0)\,, \tag{59}$$

for $m_U \neq 0$, or

$$D_F(U) = \frac{-i}{p^4 + i\epsilon} \tag{60}$$

for $m_U = 0$.

So, the "$-$" factor in the E.O.M (39,40,53,58) is very crucial, which represents the sign of the mass term in Lagrangian, and, without the "$-$" factor, everything will be different! After all, the U here isn't the traditional scalar field.

Besides, please pay attention to the poles in the propagator. The position and residue of a pole is crucial for the calculation results. For a general case, a contour integration in the p^0 complex plan would be equivalent to a complex integration $\int_{-\infty}^{+\infty} dp^0 + \int_{-i\infty}^{+i\infty} dp^0$, however, if we transfer the imaginary unit i in ip^0 to ix^0 through the product $p\cdot x$ in $e^{-ip\cdot x}$ and treat ix^0 as the temperature T, then, in a zero-temperature field theory, we can omit effects of the two poles $\{iE_U + \epsilon, -iE_U - \epsilon\}$, with $E_U = \sqrt{\boldsymbol{p}_U^2 + m_U^2}$ the energy of U. Otherwise, if we just rudely choose to detour the two poles, then our model would not give the results for superconductor in Section 5. Besides, for the $m_U = 0$ case, it's much more convenient for us since we can reduce the four simple-poles to two double-poles or just one quadruple-pole.

For convenience, we would call the model for U defined in the E.O.M $p^4 U = m_U^4 U$ as a "P-4" type, and the traditional model for U defined in Klein-Gordon equation $p^2 U = m_U^2 U$ as a "P-2" type.

2.3 Interaction I: coupled to intrinsic charges, QED?

A linear potential for strong QED?

At the beginning, we set the variables of the particles below:

$$p = (m, \boldsymbol{p}), \qquad k = (m, \boldsymbol{k}), \tag{61}$$

$$p' = (m, \boldsymbol{p}'), \qquad k' = (m, \boldsymbol{k}')\,. \tag{62}$$

For the non-relativistic approximation, we have the relations for kinetics variables as

$$q = p' - p \Rightarrow q^2 = (p'-p)^2 \xlongequal{\text{(NR limit)}} -|\boldsymbol{p}' - \boldsymbol{p}|^2 + \mathcal{O}(\boldsymbol{p}^4), \tag{63}$$

and

$$\bar{u}^{s'}(p')u^s(p) = 2m\delta^{ss'},\ \bar{u}^{s'}(p')\gamma^\mu u^s(p) \xlongequal{\text{(NR limit)}} v^\mu 2m\delta^{ss'}\,. \tag{64}$$

Suppose that the scattering is between a pair of particles with different kinds of charges, denoted as $Q_1 = -Q_2 \equiv 1$, that is, here the couplings $\alpha_{1,2} \equiv \alpha Q_{1,2}$, and $\beta_{1,2} = \beta Q_{1,2}$, then we have the complete amplitude for Fig. 4-(a), as[4]

$$\begin{aligned} i\mathcal{M}_a &= \bar{u}^{s'} i(\Lambda\alpha_1 + \beta_1 i\not{q})u^s \cdot \frac{-i}{q^4} \cdot \bar{u}^{r'} i(\Lambda\alpha_2 - \beta_2 i\not{q})u^r \\ &\simeq \frac{-i}{q^4} i(\Lambda\alpha_1 + \beta_1\gamma_\mu i q^\mu) i(\Lambda\alpha_2 - \beta_2\gamma_\nu i q^\nu) 2m\delta^{ss'} 2m\delta^{rr'} \\ &\simeq \frac{-i}{q^4}\left[-\Lambda^2\alpha_1\alpha_2 - i\Lambda(\alpha_2\beta_1 v_1 - \alpha_1\beta_2 v_2)\cdot q - \beta_1\beta_2\gamma_\mu\gamma_\nu q^\mu q^\nu\right] 2m\delta^{ss'} 2m\delta^{rr'} \\ &= -i\left[-\Lambda^2(\alpha_1\alpha_2)\frac{1}{|\boldsymbol{q}|^4} - i\Lambda\lambda\alpha_1\beta_2\frac{1}{|\boldsymbol{q}|^3} + \beta_1\beta_2 g^{00}\frac{1}{|\boldsymbol{q}|^2}\right] 2m\delta^{ss'} 2m\delta^{rr'} . \end{aligned} \tag{65}$$

where

$$(\alpha_2\beta_1 v_1 - \alpha_1\beta_2 v_2)\cdot q = \alpha_1\beta_2(v_1 - v_2)\cdot q \equiv \alpha_1\beta_2\lambda|\boldsymbol{q}|, \tag{66}$$

for the reason of $\alpha_2\beta_1 = \alpha Q_2\beta Q_1 = \alpha Q_1\beta Q_2 = \alpha_1\beta_2$, and we define

$$(v_1 - v_2)\cdot q \equiv v_{12}\cdot q \equiv \lambda|\boldsymbol{q}|\,,\ -\infty < \lambda < +\infty, \tag{67}$$

or

$$v_{1,2}\cdot q \equiv \lambda_{1,2}|\boldsymbol{q}|\,,\ \lambda = \lambda_1 - \lambda_2\,,\ -\infty < \lambda_{1,2} < +\infty, \tag{68}$$

and, particularly, for $v_{12} = 0$ case, $\lambda = 0$, and, for NR case, $\lambda \simeq 0$.

The amplitude $i\mathcal{M}$ should be compared with the Born approximation to the scattering amplitude in non-relativistic quantum mechanics, written in terms of the potential function $V(\boldsymbol{x})$: [2]

$$i\mathcal{M} \sim {}_{NR}\langle p'|iT|p\rangle_{NR} = -i\widetilde{V}(\boldsymbol{q})(2\pi)\delta(E_{\boldsymbol{p}'} - E_{\boldsymbol{p}}),\ (\boldsymbol{q} = \boldsymbol{p}' - \boldsymbol{p}). \tag{69}$$

By dealing with the kinetics factors as $2m\delta^{ss'} \to \delta^{ss'}$ and $(2\pi)\delta(E_{\boldsymbol{p}'} - E_{\boldsymbol{p}}) \to 1$, we can have

$$\widetilde{V}(\boldsymbol{q}) = -\Lambda^2(\alpha_1\alpha_2)\frac{1}{|\boldsymbol{q}|^4} + \beta_1\beta_2\frac{1}{|\boldsymbol{q}|^2}\,, \tag{70}$$

where the λ term in (65) was dropped according to the optical theorem, and the inverse Fourier transformation

$$V(\boldsymbol{x}) = \mathcal{F}^{-1}[\widetilde{V}(\boldsymbol{q})]\,. \tag{71}$$

With the formula in Appendix B, we can get the potential form for a pair of particles with different kinds of charge, as

$$V(r) = +\frac{\Lambda^2\alpha_1\alpha_2}{8\pi}r + \frac{\beta_1\beta_2}{4\pi r}\,. \tag{72}$$

There is a linear confined potential for the $\alpha_1\alpha_2 > 0$ case, which might be corresponding to the confinement for strong-coupled gauge theory.

[4]For simplicity, here we can only consider the contributions from U_1, and, for the contributions from U_2, the result just need a double.

2.4 Interaction II: coupled to momentum, gravity?

Could the term ∂U serve as gravity?

For the interaction term

$$\mathcal{L}_{(U+\partial\partial U)} = -\alpha Q^{-1}\Lambda\,\bar{\psi}U_1\psi - \rho Q\,\frac{1}{M}\bar{\psi}\partial\!\!\!/\partial\!\!\!/ U_1\psi, \tag{73}$$

which was extracted from the total Lagrangian (7), as the case for (101), we can write the corresponding part for the amplitude with the Feynman rules as

$$\begin{aligned}
i\mathcal{M}_{U+\partial\partial U} &= \bar{u}^{s'}(k')i(\alpha Q_1^{-1}\Lambda + \rho Q_1\frac{1}{M}iq\!\!\!/ iq\!\!\!/)u^s(k)\cdot\frac{-i}{q^4} \\
&\quad \cdot\bar{u}^{r'}(p')i(\alpha Q_2^{-1}\Lambda + \rho Q_2\frac{1}{M}i(-q\!\!\!/)i(-q\!\!\!/))u^r(p) \\
\xrightarrow{(Q_1=Q_2)} &= 2\bar{u}^{s'}(k')i(\alpha Q_1^{-1}\Lambda)u^s(k)\cdot\frac{i(q^2)}{q^4}\cdot\bar{u}^{r'}(p')i(\rho Q_2\frac{1}{M})u^r(p)+... \\
&= 2\frac{\Lambda}{M}\bar{u}^{s'}(k')i(\alpha Q_1^{-1})u^s(k)\cdot\frac{i}{q^2}\cdot\bar{u}^{r'}(p')i(\rho Q_2)u^r(p)+... \\
&= 2\frac{\Lambda}{M}\cdot[i\mathcal{M}_{Coulomb-like}]+...\,,
\end{aligned} \tag{74}$$

which was indeed a weak version of Coulomb-type potential (depressed by $\frac{\Lambda}{M}$) but attractive for particles with the same charge so could be a possible candidate for Newton's gravity. Maybe this could give an approach to unify the electromagnetic force and the gravity. For detail, if we set $\Lambda = \frac{1}{L} \simeq 10^{-41}GeV$ with $L \simeq 10^{11}l.y.$ corresponding to the size of universe, with the ratio of Newton's gravity force F_G and the Coulomb force F_C,

$$\frac{F_G}{F_C} = \left[G\left(\frac{m_e}{e}\right)^2\frac{e^2}{r^2}\right]/\left[k\frac{e^2}{r^2}\right] \simeq 10^{-43} \tag{75}$$

where m_e is the mass of electron and $k \simeq 9\times10^9(N\cdot m^2\cdot C^{-2})$ the Coulomb constant(in SI unit), then we have

$$M \simeq 10^2GeV\,, \tag{76}$$

by a lucky coincidence at the order of E.W. energy scale! See (9).

If this is true, we might say, the the smallness of gravitation constant G comes from a depression of the I.R. energy scale Λ or the size of universe. And, of course, directly to see, the two couplings in (172) and (203) would spontaneously become equal at a large enough energy scale, so a unification would be realized.

2.5 Comments on the potentials: Unification I

If we combine the potential terms in (72,74,), then, we can say, the field U could provide a wealth of interaction information, such as:

The list of potentials generated by U

1. there is a linear impulsive/confined potential (for the $\alpha_1\alpha_2 < 0$ and $\alpha_1\alpha_2 > 0$ case, respectively), which might be corresponding to the dark energy effects [6] [7] or the confinement for strong-coupled gauge theory; surely this term would be depressed or enhanced by

the energy scale Λ.

2. **(Unification I)** there are two Coulomb-type potentials, which might be corresponding to the ordinary Coulomb potential and Newton's gravitation; that the two kinds of forces appear in a single model with a relation on the coupling coefficients, might be seemed as a kind of unification;

Some notes for the potential

3. the special relativity (SR) effects are automatically served by the spinor basis $u^s(p)$. Since the coupling β is dimensionless, this theory would be a U.V. renormalizable one in the sense of superficial degree of divergence(or, in the dimensional regularization framework).

4. apparently, with different settings for the parameters, different part in the total potential would be the dominant part.

5. the linear potential would not influence the transmit of the free photons since the photon is a kind of source-free field, but the hyper-hyperfine structure of the optical spectrum of atoms would be influenced.

2.6 Renormalization: depression for higher-order processes

Postulation Amplitudes for all higher-order processes are depressed by I.R. renormalization, even couplings $\alpha, \beta \gg 1$!**(Renormalization II)**

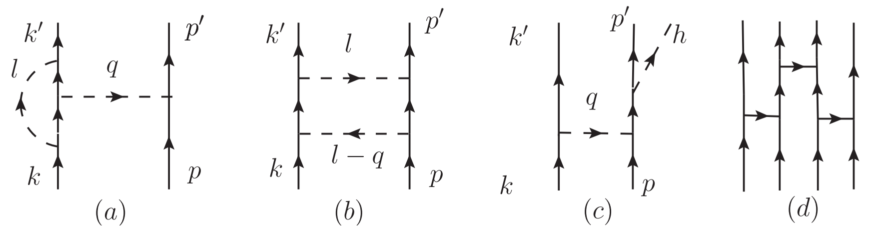

Figure 2: Some Feynman diagrams for higher order processes induced by the 3-particle vertices.

2.6.1 Ladder loop level: a infrared cutoff

When $g \to \infty$, rather than doing perturbative expansion in $1/g$ power series in the neighborhood region of $g = \infty$, we will show that in some particular cases we can directly do the expansion in g power series.

In the approximation $k \sim m_\psi \simeq 0$ and $k' \simeq k$, with the leading order (LO) interaction terms in (7)

$$\mathcal{L}_I = -\alpha Q \Lambda \, \bar{\psi} U \psi - \beta Q \, \bar{\psi} \partial\!\!\!/ U \psi \, , \tag{77}$$

the amplitude corresponding to the vertex correction part of the loop diagram in Fig. 2-(a) can be written as(for the loop integration formula, see Appendix C)

$$
\begin{aligned}
i\mathcal{M}_b &= \int \frac{d^4l}{(2\pi)^4}\frac{i}{l^4} i(\alpha_1\Lambda+\beta_1\not{l})\frac{i}{\not{k}'-\not{l}-m_\psi} i(\alpha_1\Lambda+\beta_1\not{q})\frac{i}{\not{k}-\not{l}-m_\psi} i(\alpha_1\Lambda-\beta_1\not{l}) \\
&= \int \frac{d^4l}{(2\pi)^4}\frac{i}{l^4} i(\alpha_1\Lambda+\beta_1\not{l})\frac{i(\not{k}'-\not{l}+m_\psi)}{(k'-l)^2-m_\psi^2} i(\alpha_1\Lambda+\beta_1\not{q})\frac{i(\not{k}-\not{l}+m_\psi)}{(k-l)^2-m_\psi^2} i(\alpha_1\Lambda-\beta_1\not{l}) \\
\xrightarrow[k'\simeq k]{k\sim m_\psi\simeq 0} &\simeq -\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^4}\frac{1}{(k-l)^4}(\alpha_1\Lambda+\beta_1\not{l})(-\not{l})(\alpha_1\Lambda+\beta_1\not{q})(-\not{l})(\alpha_1\Lambda-\beta_1\not{l}) \\
&= \left[-\int \frac{d^4l}{(2\pi)^4}\frac{a_4l^4+a_3l^3+a_2l^2}{l^4(l-k)^4}\right](\alpha_1\Lambda+\beta_1\not{q}) \\
&= \left[-\int \frac{d^4l}{(2\pi)^4}\frac{a_4}{(l-k)^4}-\int \frac{d^4l}{(2\pi)^4}\frac{a_3l^3+a_2l^2}{l^4(l-k)^4}\right](\alpha_1\Lambda+\beta_1\not{q}) \\
&= -i\left[\log\left(\frac{-k^2}{M^2}\right)+\frac{1}{k^2}\left(\frac{-k^2}{\mu^2}\right)+\frac{1}{k^4}\left(\frac{-k^2}{\mu^2}\right)\right](\alpha_1\Lambda+\beta_1\not{q})\,. \qquad (78)
\end{aligned}
$$

There are both ultraviolet divergences and infrared divergences in this integration, however, it's renormalizable, with a dimensional regularization for the ultraviolet part and a cutoff for the infrared part. Here, if we can renormalize the magnitude of the amplitude for the loop-level to be smaller than the result for the tree-level in Fig. 4-(a), and ensure the magnitude becomes smaller and smaller as the loops becoming more and more, then, the theory would be "perturbatively" computed "loop by loop"(attention: it is not equivalent to "order by order" in the power of $\{\alpha,\beta\}$ here).

For instance, reminding the second factor in (78) is just corresponding to a single vertex for the tree level, if we impose a very large μ (say, $\mu > \alpha\Lambda$) as the infrared cutoff, the amplitude in (78) could indeed become smaller than the one in (65) for the tree-level. We can take an example to show the reasonableness for this statement, in the viewpoint of the relaxation time: the stronger interaction, the short relaxation time, so the larger characteristic momentum scale.

The amplitude of the loop diagram in Fig. 2-(b) can be written as

$$
\begin{aligned}
i\mathcal{M}_b &= \int \frac{d^4l}{(2\pi)^4}\bar{u}^{s'} i(\alpha_1\Lambda+\beta_1\not{l})\frac{i}{\not{p}'-\not{l}-m_\psi} i(\alpha_1\Lambda+\beta_1(\not{q}-\not{l}))u^s \\
&\cdot\frac{i}{l^4}\cdot\frac{i}{(l-q)^4}\cdot\bar{u}^{r'} i(\alpha_2\Lambda-\beta_2\not{l})\frac{i}{\not{k}'+\not{l}-m_\psi} i(\alpha_1\Lambda-\beta_1(\not{q}-\not{l}))u^r \\
&= \int \frac{d^4l}{(2\pi)^4}\bar{u}^{s'} (\alpha_1\Lambda+\beta_1\not{l})\frac{\not{p}'-\not{l}+m_\psi}{(p'-l)^2-m_\psi^2}(\alpha_1\Lambda+\beta_1(\not{q}-\not{l}))u^s \\
&\cdot\frac{1}{l^4}\cdot\frac{1}{(l-q)^4}\cdot\bar{u}^{r'} (\alpha_2\Lambda-\beta_2\not{l})\frac{\not{k}'+\not{l}+m_\psi}{(k'+l)^2-m_\psi^2}(\alpha_1\Lambda-\beta_1(\not{q}-\not{l}))u^r \\
&\sim \left[\int \frac{d^4l}{(2\pi)^4}\frac{b_6l^6+b_5l^5+\ldots+b_1l+b_0}{a_0l^{12}+a_1l^{11}+\ldots+a_8l^4}\right]2m\delta^{ss'}2m\delta^{rr'}\,. \qquad (79)
\end{aligned}
$$

Apparently, it's hyper-renormalizable in the ultraviolet region and non-renormalizable in the infrared region for this integration, that is, there isn't ultraviolet divergences but only infrared divergences in this integration. Here if we impose a very large μ (say, $\mu > \alpha\Lambda$) as the infrared cutoff, the amplitude in (79) indeed becomes smaller than the one in (65) for the tree-level.

2.6.2 Higher order tree-level: kinetics/dynamics equivalence

The amplitude of the loop diagram in Fig. 2-(c) can be written as

$$i\mathcal{M}_c \sim i\mathcal{M}_a \cdot i(\alpha_1 \Lambda + \beta_1 \not{k}) \frac{i}{\not{p}' + \not{k} - m_\psi}. \tag{80}$$

The result can be used to describe two cases: (1) the amplitude indeed becomes larger for a many-body system, which should be in a bound state (or Bose - Einstein condensate state), with the same momentum for all the particls, like the case in Fig. 2-(d); (2)the amplitude could be "depressed" after the renormalization of the collinear divergences, for instance, one can introduce a cutoff for the phase-space parameters to avoid the collinear divergences, which would give an effective depressed factor for the physical cross section, and, the more particles, the more depressed factors. **(Renormalization III)**

Anyway, for some particular cases, in which the effective expansion factors could be "renormalized" to be smaller than 1 after the renormalization to the infrared divergences (through an introduction of a large energy scale) and the renormalization to the collinear divergences (through an introduction of a constraint on the kinetics phase space), the theory would be "perturbatively" computed "order by order" formally in the power series of the couplings $\{\alpha, \beta\}$, and, maybe we could say the theory is formally non-perturbative but practically perturbative.

♠ **the multi-particle vertices**

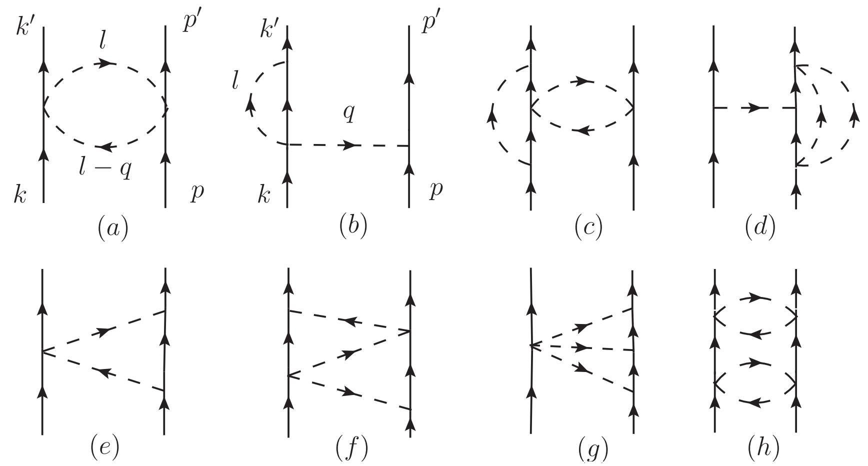

Figure 3: Some Feynman diagrams for loop-level processes induced by the multi-particle vertices.

For the amplitude in Fig. 3-(a), with the 4-particle coupled terms in (7), we have

$$i\mathcal{M}_a \sim \int \frac{d^4 l}{(2\pi)^4} \frac{a_4 l^4 + a_3 l^3 + a_2 l^2 + a_1 l + a_0}{l^4 (l-q)^4}. \tag{81}$$

Apparently, it's hyper-renormalizable in the ultraviolet region and non-renormalizable in the infrared region for this integration, as in the case of (79).

2.7 U out a nutshell: the generation of a nonlinear K-G equation

1. We can generate a Klein-Gordon equation for $\Phi \sim \sqrt{U}$.

Mathematically, it's allowed to reduce the one 4th-order D.E. (39) to two 2nd-order e-quations. Indeed, as mentioned in (22-24), we can decompose the E.O.M of U in (39) to two Klein-Gordon equations for a P-2 type conventional scalar field Φ and a P-2 type unconventional scalar field $\widetilde{\Phi}$ respectively, where $U \sim \widetilde{\Phi}\Phi$, for $\widetilde{\Phi}$, see (26-27).

2. We can't generate a Klein-Gordon equation for U.

By treating (39) as an inhomogeneous Laplace equation for $V = \partial^2 U$, we can get a solution

$$\partial^2 V = m_U^4 U - J \Rightarrow V \equiv (\underline{\partial^2 U = ...}) \tag{82}$$

as a formally 2nd-order D.E. for U, underlined in (82). However, that's not a real Klein-Gordon equation for U, even not a nonlinear one! For mathematical detail, we take the free E.O.M of U in (39) as an example:

a. if $m_U \neq 0$, (82)is truly an integro-differentia equation rather than a really 2nd-order D.E. for U, with the r.h.s a nonzero term $\int G(x, x')U(x')$, where $G(x, x')$ is the Green function of the homogeneous Laplace equation $\partial^2 V = \partial^2(\partial^2 U) = 0$;

b. if $m_U = 0$, with the boundary condition $[\partial^2 U]_B \neq 0$ for the nontrivial D.E. (39), we must get a solution $V = \partial^2 U = B \neq 0$, which is corresponding to a E.O.M for massive U or interactive U, rather than the original massless free field U, depending on the detail of the term B.

So, we can **not** say that the Klein-Gordon equation is based on the "solution" level of the 4th-order D.E. (39).

Nonlinear Klein-Gordon equation for U: out of a nutshell

Here we need the self-interaction term of U, which could be written as

$$\mathcal{L}_I = -g_U \Lambda_U^2 U \partial_\mu U \partial^\mu U + m_U^4 U^2 \,. \tag{83}$$

Since there is at least 3 U-field for the interaction term, there must be at least 0 or 2 ∂-symbols for constructing a Lorentz scalar Lagrangian; however, here we don't use the terms UUU, because, we'll see, that can't give us a qualified 2nd-order D.E..

In a word, for a pure U-field system, if $\partial\partial \ll \Lambda_U \Lambda$ (or, we can say, the system is "out of a nutshell" as an illustrative statement), then the kinetic energy term could be dropped, then we can get a E.O.M for U according to the Euler-Lagrangian equation, as

$$g_U \Lambda_U^2 (\partial U)^2 - 2 g_U \Lambda_U^2 U \partial^2 U = m_U^4 U \Rightarrow (\partial U)^2 - 2U\partial^2 U = \frac{m_U^4}{g_U \Lambda_U^2} U \,. \tag{84}$$

Apparently, that is a nonlinear 2nd-order D.E., so, we just call it "nonlinear Klein-Gordon equation". Particularly, for a special case, $\langle U \rangle \gg U - \langle U \rangle$ and $\langle U \rangle \gg \partial U$, we can get the "linear" Klein-Gordon equation

$$-\partial^2 U = \frac{m_U^4}{2 g_U \langle U \rangle \Lambda_U^2} U \,, \tag{85}$$

and there should be the relation $2 g_U \langle U \rangle \Lambda_U^2 = m_U^2$.

2.8 From U to QED: gauge symmetry arises

2.8.1 U as a group element in weak field case

For a complex-valued field U, we can decompose it as

$$U = \phi_1 \exp^{-i\phi_2} = \phi_1 \cos\phi_2 - i\phi_1 \sin\phi_2 \,, \; U^\dagger U \leq 1 \,, \tag{86}$$

then, in the weak field approximation, that is, $U \simeq \langle U \rangle$, there would be

$$\begin{aligned} U(x) \to \langle U \rangle e^{-ig\phi(x)} &= \langle U \rangle(\cos g\phi - i \sin g\phi) \simeq \langle U \rangle [1 - ig\phi(x)] \,, \\ g \simeq 0 &\Rightarrow U^\dagger U \simeq 1 \,. \end{aligned} \tag{87}$$

So we can see, within a nonlinear σ model framework, terms in (10) and (11) are now actually equivalent so that should be mutually exclusive in a definite Lagrangian of U, with the difference on the choice for the particle degree of freedom(d.o.f) between two real fields, $(U + U^\dagger)$ and $[i(U - U^\dagger)]$. For example, if a real vector field A^μ behaved as the fluctuation of U, then the imaginary part of U would be chosen for the proper "block" for constructing the Lagrangian.

Particularly, there is another approximation expansion of U,

$$\begin{aligned} U \simeq \langle U \rangle [1 - ig\epsilon n_\mu A^\mu(x)] &\to \langle U \rangle e^{-ig\epsilon n_\mu A^\mu(x)} \,, \\ g \simeq 0 &\Rightarrow U^\dagger U \simeq 1 \,. \end{aligned} \tag{88}$$

That means, **now U could be treated as a $U(1)$ group element with A^μ as its gauge particle d.o.f, and the superficial gauge symmetry of the Lagrangian arises!**

Hence, in the viewpoint of traditional gauge theory, the interaction term

$$\bar{\psi} U i \partial\!\!\!/ U^\dagger \psi \simeq \bar{\psi} i \partial\!\!\!/ (U - U^\dagger) \psi$$

is formally supplemented for the gauge invariance for the kinetics terms of the matter field ψ. Sine the former term above would be depressed as it's a multi-field term, the actual term for the gauge invariance is the latter one, $\bar{\psi} i \partial\!\!\!/ U^\dagger \psi$, and that might imply

$$\beta = e \,. \tag{89}$$

Effects of the nonzero VEV of U

U is a kind of higgs field, so it should exhibit its higgs-like property. From its definition

$$U = \exp[-ig\epsilon n^\mu A_\mu] \sim 1 - ig\epsilon n^\mu A_\mu + \ldots \,, \tag{90}$$

when $g \to 0$, there was $\langle U \rangle = 1$. And now, with the interaction term $\alpha\Lambda\bar{\psi}U\psi$ in (7), according to the higgs mechanism, the fermions will get a a mass correction

$$\triangle m \sim \alpha\Lambda\langle U \rangle \sim \alpha\Lambda \xrightarrow{\Lambda \simeq 0} 0 \,. \tag{91}$$

2.8.2 From A to U: U as a background and U as an excitation

We write the quantum mechanics amplitudes as

$$\langle f | e^{iS} | i \rangle \equiv e^{i\theta} = e^{i\bar{\theta}} e^{i\theta'} \equiv S(\bar{U}) e^{i\theta'} \,. \tag{92}$$

Traditionally, we separate $e^{i\theta}$ to a $\bar{\theta} = \infty$ part $e^{i\bar{\theta}}$ as classic background(BKG) effects and a $0 < \theta' \ll 1$ part $e^{i\theta'}$ as quantum fluctuation effects,

$$\begin{aligned} & \text{S-matrix} = (\text{BKG effects}) + (\text{fluctuation effects}) \\ \Rightarrow \quad & S(U) = S(\bar{U}) + S(A), \text{ or } \ U = \bar{U} + A \end{aligned} \tag{93}$$

that is, if $\theta = \infty$, the system could be described in a classic mechanics picture by taking a classic field $\bar{U}$ to serve the BKG effects, or, if $\theta \ll 1$, the system could be described in a quantum mechanics picture by taking a particle A(exactly the gauge field) to serve the fluctuation effects. But, in the case of $e^{iS} = \exp\{ig\hat{O}\}$ with $g \gg 1 \neq \infty$ so that $\bar{\theta} \gg 1 \neq \infty$, how do we calculate the amplitudes? The question itself is also to say, problems for strong coupling cases would have combined the classic effects and the quantum effects together under consideration.

From (93), at least one thing is definite, that is, no matter it's in the $g = \infty$ case or the $g \neq \infty$ case, all fluctuations A were defined on a definitely certain BKG. By taking a change on the form, we could rewrite (93) as

$$U - \bar{U} = A, \tag{94}$$

which is to say, when U was treated as a BKG, the particle A was not only a fluctuations, but also be the renormalized version of the background U(by a cancellation with the anti-BKG $\widetilde{U}$), or on the other hand, all the effects from A in fact include the background effects from U.

However, by sequentially taking a generalization on the meaning of U in (94), we could rewrite (93) as

$$(\text{BKG effects})_1 - (\text{BKG effects})_2 = (\text{fluctuation effects}), \tag{95}$$

then, it immediately comes to our mind that the situation for the cancelation effects of particle-antiparticle pairs, written as

$$(\text{BKG effects}) \ + (\text{anti-(BKG effects)}) \quad = \quad (\text{fluctuation effects}) \tag{96}$$

$$\Rightarrow U + \widetilde{U} \quad = \quad A \neq 0, \tag{97}$$

which would motivate us that, if U itself became a new effective degree of freedom, then A needn't to be isolated out and defined as a new degree of freedom any more, since whose effects had been innately included in U.

As we all know, it is relative and not absolute to treat a particle as BKG effects or fluctuation effects, or, in other words to say, the criteria for particles and quasi-particles are relativistic. The best example is the electron: in the relativistic limit case, say, in decays of nuclei, the electron was a quantized field degree of freedom, with its partner, the positrons; in the NR limit case, the energy spectrum of atoms, wer determined by the energy of the NR motion of electrons, which was seemed as the fluctuation of a BKG, the mass of the rest electrons; in the case of considering the fine structures of the energy spectrum, the relativistic effects of electrons would appear; and so on.

2.8.3 Where is the Coulomb potential? $\partial U \sim A$!

When $g \to 0$, the effects of U should be matched to the effects of A_μ by the matching between their respective Lagrangian terms, or their respective predictions for the physical amplitudes.

Renormalization I

As discussed in Section 2.8.1, when $g \to 0$, with the definition of U in (256), $U^\dagger U \simeq 1$, so now $\mathcal{L}_U$ is a non-linear σ model for U. In the nonlinear σ model frame, when $g = 0$, the radial-direction component of U was stable with taking the value $U = \langle U \rangle$, so, the "propagating effects" of the vacuum(actually the radial-direction component) has been reduced originally, that means, for the propagator

$$\begin{aligned} \langle 0|TU(y)U(x)|0\rangle &= \langle |T[1 - i\phi(y) + ..][1 - i\phi(x) + ..]|\rangle \\ &= 1 - \langle 0|[i\phi(y) + i\phi(x)]|0\rangle + \langle 0|T[i\phi(y)i\phi(x)]|0\rangle + ... \\ &\to 0 - \langle 0|[i\phi(y) + i\phi(x)]|0\rangle + \langle 0|T[i\phi(y)i\phi(x)]|0\rangle + ... , \end{aligned} \tag{98}$$

the infinity from the VEV $\int d^4x \cdot 1$ has been renormalized, with $\langle 0|\phi(x)|0\rangle = 0$. Besides, the VEV of U don't influence the interactions induced by ∂U.

The amplitude

Remind that, when U is a group element, in the weak coupling case the gauge interaction term

$$\begin{aligned} g\bar{\psi}\not{A}\psi = \bar{\psi}(Ui\not{\partial}U^\dagger)\psi &= \bar{\psi}((1 - i\mathcal{O}(g\epsilon))i\not{\partial}((1 + i\mathcal{O}(g\epsilon)))\psi \\ &\simeq \bar{\psi}\not{\partial}[i(U - U^\dagger)]\psi, \end{aligned} \tag{99}$$

that means, the two terms,$\bar{\psi}(U\not{\partial}U^\dagger)\psi$ and $\bar{\psi}\not{\partial}U\psi$ both appears in (7), are now numerically approximately equal. However, are these two terms still give the equivalent results when U is a particle in (7), and, what are the difference between them?

In the viewpoint of S-matrix, the answer is positive. For detail, in the weak fluctuation case, there is $U \simeq 1 - i\mathcal{O}(g \cdot \epsilon)$, so that, $U^\dagger U = UU^\dagger \simeq 1$ could be inserted into the S-matrix element, as

$$\begin{aligned} \mathcal{S}_{2\to 2} = \exp[i\int d^4x\mathcal{L}] &= 1 + : \frac{1}{2}(i\mathcal{L})^2 : + ... \\ &= 1 + \frac{1}{2}i\beta_1 i\beta_2 : \bar{\psi}i\not{\partial}U^\dagger\psi \cdot \bar{\psi}i\not{\partial}U\psi : + ... \\ &= 1 + \frac{1}{2}i\beta_1 i\beta_2 : \bar{\psi}i\not{\partial}U^\dagger\psi \cdot (U^\dagger U) \cdot \bar{\psi}i\not{\partial}U\psi : + ... \\ &= 1 + \frac{1}{2}i\beta_1 i\beta_2 : \bar{\psi}Ui\not{\partial}U^\dagger\psi \cdot \bar{\psi}U^\dagger i\not{\partial}U\psi : + ... , \end{aligned} \tag{100}$$

that means, now the term $\bar{\psi}i\not{\partial}U^\dagger\psi$ and $\bar{\psi}Ui\not{\partial}U^\dagger\psi$ are equivalent.

To answer which term is corresponding to the gauge interaction, or, how to understand each term in the Lagrangian (7), let's firstly consider the contributions of the term $\bar{\psi}\not{\partial}U\psi$ to a scattering amplitudes, see Fig. 4-(a).

We can extract the corresponding term for the amplitude straightforward from the Feyn-

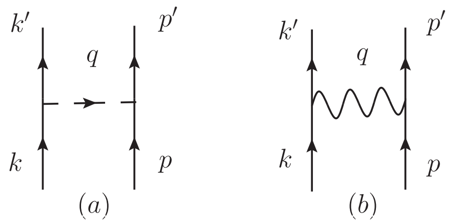

Figure 4: The Feynman diagrams for the leading order tree level processes and for Coulomb potential in QED.

man rules as

$$
\begin{aligned}
i\mathcal{M}_{\partial U} &= \bar{u}^{s'}(k')i(-\beta Q_1 i\not{q})u^s(k)\cdot\frac{-i}{q^4}\cdot\bar{u}^{r'}(p')i(\beta Q_2 i\not{q})u^r(p)\\
&= \bar{u}^{s'}(k')i(\beta Q_1\gamma^\mu)u^s(k)\cdot\frac{-iq_\mu q_\nu}{q^4}\cdot\bar{u}^{r'}(p')i(\beta Q_2\gamma^\nu)u^r(p)\\
&\overset{\hat{L}}{=\!=} \bar{u}^{\hat{L}s'}(\hat{L}k')i(\beta Q_1\gamma^\mu)u^{\hat{L}s}(\hat{L}k)\cdot\frac{-i\hat{L}q_\mu\hat{L}q_\nu}{(\hat{L}q)^4}\cdot\bar{u}^{\hat{L}r'}(\hat{L}p')i(\beta Q_2\gamma^\nu)u^{\hat{L}r}(\hat{L}p)\\
&= \bar{u}^{\hat{L}s'}(\hat{L}k')i(\beta Q_1\gamma^\mu)u^{\hat{L}s}(\hat{L}k)\cdot\frac{-iq^2 g_{\mu\nu}}{q^4}\cdot\bar{u}^{\hat{L}r'}(\hat{L}p')i(\beta Q_2\gamma^\nu)u^{\hat{L}r}(\hat{L}p)\\
&= \bar{u}^{\hat{L}s'}(\hat{L}k')i(\beta Q_1\gamma^\mu)u^{\hat{L}s}(\hat{L}k)\cdot\frac{-ig_{\mu\nu}}{q^2}\cdot\bar{u}^{\hat{L}r'}(\hat{L}p')i(\beta Q_2\gamma^\nu)u^{\hat{L}r}(\hat{L}p)\\
&\overset{\hat{L}^{-1}}{=\!=\!=} \bar{u}^{s'}(k')i(\beta Q_1\gamma^\mu)u^s(k)\cdot\frac{-ig_{\mu\nu}}{q^2}\cdot\bar{u}^{r'}(p')i(\beta Q_2\gamma^\nu)u^r(p)\\
&= i\mathcal{M}_{Coulomb},
\end{aligned}
\tag{101}
$$

where $\hat{L}$ denotes a Lorentz rotation which changed the tensor $q_\nu q_\nu$ to

$$q_\nu q_\nu \to \hat{L}q_\mu\hat{L}q_\nu = (\hat{L}q)^2 g_{\mu\nu} = q^2 g_{\mu\nu}, \tag{102}$$

with the $i\mathcal{M}_{\partial U}$ and q^2 invariant for the reason of the Lorentz invariance of the Lagrangian term $\bar{\psi}\not{\partial}U\psi$. And we can find that (101) is just the amplitude of a scattering process corresponding to the Coulomb potential in QED, as in Fig.4-(b).

So, now we can say, when U is a particle, it's the term $\bar{\psi}\not{\partial}U\psi$ serving for the effects corresponding to the gauge interaction in QED. And, definitely, with the result from (101), we can say, the term $\bar{\psi}(U\not{\partial}U^\dagger)\psi$ in (7) isn't the leading order contribution corresponding to the gauge interaction, since the term $\bar{\psi}(U\not{\partial}U^\dagger)\psi$ is serving for a high order contribution.

It is easy to understand that through the physical picture, followed as the meaning of Eq.(97):

1.The gauge particle field A, which was originally defined as a d.o.f with $U\partial U^\dagger$ to denote the fluctuation effects from the cancelation of BKG U and anti-BKG $\widetilde{U}$, could be a good degree of freedom only when U was a frozen BKG(or, a classic field, rather than a particle). But, when U was excited, A wouldn't be a good degree of freedom, and the isolation of A from U in Eq.(113) was not available any more.

2.However, the effects corresponding to the "old" A would be still included in the one particle U, which should be surely determined by term $\bar{\psi}\not{\partial}U\psi$ rather than terms $\bar{\psi}(U\not{\partial}U^\dagger)\psi$, which was corresponding to the mixing effects of two particles U and $U^\dagger$(see (81) in next sections). Or, in other words, when U behaved as a BKG field(or a VEV), the gauge interaction absorbed into an effective field A is the all effects of U, but, when U was excited, it is just a part of the whole effects of U. Anyway, it's affirmed again that the gauge particles could be seemed as fluctuation effects of U, either a particle U in a new perspective for $g \gg 1$ or a classic field U in the old perspective for $g \to 0$.

2.8.4 The generation of a linear QED

To generate a QED from $\mathcal{L}_U$ in (5), the crucial point is a good d.o.f or "block" for constructing Lagrangian.

1. The weak field case

In this case, the gauge connection field A_μ is a good d.o.f, and $U^\dagger U \to 1$. We will see, the good block is not $\partial_\mu U \to A_\mu$, but $\partial\partial U \leftrightarrow F_{\mu\nu}$.

The variable $i\partial_\mu U$ is a gradient field, so it does serve as the gauge transformation $A_\mu \to A_\mu + i\partial_\mu U$ for a gauge field A. However, just because it's only a gradient field, $i\partial_\mu U$ could **never** serve as a gauge connection field, even as a longitude component for a gauge field. The "field strength" of a gradient field would be zero, which could be confirmed in another way here, as

$$F_{\mu\nu} = \partial_\mu(\partial_\nu U) - \partial_\nu(\partial_\mu U) = 0\,. \tag{103}$$

So, how would we realize the correspondence for $\partial_\mu U \to A_\mu$? Here are two methods:

a. we modify $\partial_\mu U$ to a qualified vector field. As in (120,99), there is $A_\mu \equiv U\partial_\mu U^\dagger \simeq \partial_\mu U^\dagger$, so we can take the Maurer-Cartan 1-form $A_\mu = U\partial_\mu U^\dagger$ as gauge field; essentially, that is an operation in the way of extending the number of d.o.f, either $\partial_\mu U^\dagger \to U\partial_\mu U^\dagger$, or

$$A_\mu \sim \partial_\mu U^\dagger \to \phi(x)\partial_\mu U \tag{104}$$

with ϕ an arbitrary scalar field, and, particularly, in the case of $\phi(x)^\dagger\phi(x) = 1$, we can just treat $\phi \simeq U$. Then, we can rebuild the kinetic energy term for $A_{\alpha\beta} \sim A_\mu$, as

$$\begin{aligned} & \partial_\mu\partial_\nu U^\dagger\partial^\mu\partial^\nu U = \partial_\mu\partial_\nu U^\dagger \cdot \phi^\dagger\phi \cdot \partial^\mu\partial^\nu U \\ = \;& \left[\partial_\mu(\phi^\dagger\partial_\nu U^\dagger) - \partial_\mu\phi^\dagger \cdot \partial_\nu U^\dagger\right] \cdot \left[\partial^\mu(\phi\partial^\nu U) - \partial^\mu\phi \cdot \partial^\nu U\right] \\ = \;& \left[\partial_\mu(\phi^\dagger\partial_\nu U^\dagger) - \partial_\mu\phi^\dagger \cdot \partial_\nu U^\dagger\right] \cdot \left[\partial^\mu(\phi\partial^\nu U) - \partial^\mu\phi \cdot \partial^\nu U\right] \\ = \;& \partial_\mu A_\nu \partial^\mu A^\nu + (\text{multi-field terms}) \end{aligned} \tag{105}$$

For dealing with the lack of a term $\partial_\mu A_\nu \partial^\nu A^\mu$ in the full kinetic energy term, indeed, for an Abelian case, we can take a anti-symmetric tensor (complex-valued) $A_{\alpha\beta}$ as the spin-1 field, which is equivalent to the vector A_μ, as

$$A^\mu \equiv \frac{(\epsilon^{\alpha\beta\mu} - \epsilon^{\beta\alpha\mu})}{2} A_{\alpha\beta} = \phi(x)\partial^\mu U\,. \tag{106}$$

b. we use the correspondence between $\partial_\mu\partial_\nu U$ and $F_{\mu\nu}$. As in (298,300), we have defined U as the Wilson loop,

$$\begin{aligned} U(x) \equiv U_P(x,x) \;&\equiv\; \exp\left[-ig\oint_P dz^\mu A_\mu(x)\right] \\ &=\; \exp\left[-i\frac{g}{2}\int_\Sigma d\sigma^{\mu\nu}F_{\mu\nu}\right] \\ &=\; 1 - i\epsilon^2 gF_{12} + \mathcal{O}(\epsilon^3)\,, \end{aligned} \tag{107}$$

with the field strength defined as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \simeq \frac{iU(x)}{\epsilon^2} \leftrightarrow \partial\partial U\,. \tag{108}$$

Note, it's not true for $\partial\partial U = F_{\mu\nu}$ since $\partial\partial U$ is a symmetric tensor but $F_{\mu\nu}$ an anti-symmetric one, but just for a correspondence in the sense of "block" for constructing Lagrangian.

For the interaction terms in the full Lagrangian (7), we can directly set $U \to 1$, $\partial_\mu U_1 \to A_\mu$, and U_2 could be seemed as frozen. Or, we can directly omit terms including

$$U(x) = \int_\infty^x dz_\mu \partial_z^\mu U(z) \sim \int_\infty^x dz_\mu \frac{1}{\phi(z)} A^\mu(z)\,, \tag{109}$$

since there is a depression for these terms from the small Λ.

2. The strong field case

In this case, $g \cdot \epsilon \simeq 1$, we can't reduce U to a single gauge field A^μ, for the reason, we can say, as in (131), now the instanton ϕ would be excited, so A^μ should be combined with ϕ to serve for the full interaction. However, it's just the significance that we construct a P-4 type model for U, otherwise, our P-4 type model would be trivial.

2.8.5 Matching I: the degree of freedom

It is the crucial point for the construction of (4), that, for $g \to \infty$, U is treated as a particle degree of freedom, while for $g \to 0$, U is treated as a classic field function. For the detail, when g is small, from (292,306,308,297,300) in Appendix A, we have

$$\begin{aligned} U(x+\epsilon n, x) &= 1 - ig\epsilon n^\mu A_\mu(x) + \mathcal{O}((g\epsilon)^2), & (110)\\ U_{P_{ij}}(x,x) &= 1 - i\epsilon^2 g F_{ij} + \mathcal{O}(\epsilon^3), & (111)\\ B_\mu(x) &= U(x)\partial_\mu U^\dagger(x) = -igA_\mu + \mathcal{O}(g^2), & (112)\\ D_\mu &= \partial_\mu + B_\mu = \partial_\mu - igA_\mu + \mathcal{O}(g^2), & (113) \end{aligned}$$

where in (111) the subscript ij meant a chosen path P_{ij} was in the $i-j$ plane for instance. We give a list for variables in (110-113) in Table 1.

Then, the Lagrangian of a $U(1)$ gauge theory, for instance, the QED, was expressed in the form

$$\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \tag{114}$$

that is, the physical particle degrees of freedoms are ψ and A, and the theory could be calculated in the perturbative scheme.

However, for the $g \to \infty$ case, that is,

$$g \cdot \epsilon \geq 1, \tag{115}$$

the expression of (113) and (114)is unavailable. So, we construc a theory in which U is treated as a particle degree of freedom for the case $g \to \infty$ as in (4), as mentioned in the beginning of this section.

P-4 type		P-2 type	
$\{U, U^{\mu}, U^{\mu\nu}\}$	field		
$\partial \cdot U^{\mu} = 0$	gauge fixed condition		
$\partial^{\mu}U^{\nu}$ $\partial^{\mu}U\,, U^{\dagger}\partial^{\mu}U$	field strength of U fluctuation of U	A^{μ}	field
$\partial \cdot \partial U = 0$		$\partial \cdot A^{\mu} = 0$	gauge fixed condition
$\partial^{\mu}\partial^{\nu}U$	stress tensor	$\partial^{\mu}A^{\nu}$	field strength of A
		$\partial \cdot \partial A^{\mu} = 0$	E.O.M
$\partial\partial \cdot \partial\partial U = 0$	E.O.M		

Table 1: Correspondence between variables in (110-113) .

2.8.6 Matching II: the canonic commutator

If $g \to 0$, U could be treated as classic field, U and A could be the equivalent d.o.f, and there could be $A_{\mu} \sim \partial_{\mu}U^{\dagger}(x) \sim U(x)\partial_{\mu}U^{\dagger}(x)$; on the other hand, if $g \to \infty$, U should be treated as particle, U and A could **not** be good d.o.f at the same time, and in in fact there is $A_{\mu} \sim \partial_{\mu}U^{\dagger}(x) \nrightarrow U(x)\partial_{\mu}U^{\dagger}(x)$ for $g \to \infty$.

Let's show the two cases in the viewpoint of the relation $A_{\mu} \sim \partial_{\mu}U^{\dagger}(x)$.

If we just want to get a $\mathcal{L}_{QED}(g \to \infty)$ form which could restore or match the $\mathcal{L}_{QED}(g \to 0)$ in (114) as g becomes small, there are many different ways to realize that goal. For example, we can just take B instead of A in (113) and (114),

$$\mathcal{L}_{QED}(g \to \infty) = (\partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu})^2 + \bar{\psi}(i\not{\partial} + g_B\not{B} - m)\psi, \tag{116}$$

that is, the physical particle degrees of freedom are ψ and B, but the theory couldn't be calculated in the perturbative frame since the coupling constant $g_B = 1$.

Although we can still formally treat A as a physical degree of freedom with whose all the complicate effects included in B, we should note that, A and B couldn **not** be physical degrees of freedom at the same time, since the two canonical quantized conditions couldn't be true at the same time, as below

$$\begin{aligned}
\left[B(x), \dot{B}(y)\right] &= i\delta^{(4)}(x-y) \\
&= \left[-igA(x) - \frac{g^2}{2}A(x)^2 + \ldots, -ig\dot{A}(y) - \frac{g^2}{2}\dot{A}(y)^2 + \ldots\right] \\
&= -g^2\left[A(x), \dot{A}(y)\right] + i\frac{g^3}{2}\left[A(x)^2, \dot{A}(y)\right] + \ldots .
\end{aligned}$$

In another viewpoint, for an Abelian gauge field case, when g is small, we have

$$\begin{aligned}
\left[A(x)_{\mu}, \dot{A}_{\nu}(y)\right] &= i\delta^{(3)}(x-y)\delta_{\mu\nu} \\
&\simeq \left[\frac{i}{g}U\partial_{\mu}U^{\dagger}(x), \frac{\partial}{\partial t}\left[\frac{i}{g}U\partial_{\nu}U^{\dagger}\right](y)\right] \\
&= -\frac{1}{g^2}\left\{\left[U\partial_{\mu}U^{\dagger}(x), \dot{U}\partial_{\nu}U^{\dagger}(y)\right] + \left[U\partial_{\mu}U^{\dagger}(x), U\partial_{\nu}\dot{U}^{\dagger}(y)\right]\right\} \\
&= -\frac{1}{g^2}\left\{U(x)\left[\partial_{\mu}U^{\dagger}(x), \dot{U}\partial_{\nu}U^{\dagger}(y)\right] + \left[U(x), \dot{U}\partial_{\nu}U^{\dagger}(y)\right]\partial_{\mu}U^{\dagger}(x)\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \left[U(x), U\partial_\nu \dot{U}^\dagger(y)\right]\partial_\mu U^\dagger(x) + \left[U(x), U\partial_\nu \dot{U}^\dagger(y)\right]\partial_\mu U^\dagger(x)\Big\} \\
&\sim \quad \left[U, \dot{U}\right]\epsilon \cdot A + \left[U, \partial\dot{U}\right](\epsilon \cdot A)^2 + ..., \qquad (117)
\end{aligned}
$$

and

$$
\left[U, \dot{U}\right] \simeq \left[1 + g\epsilon \cdot A, gA \cdot \dot{A}\right] = g^2(\epsilon A)^{\mu\nu} \cdot \left[A_\mu, \dot{A}_\nu\right], \qquad (118)
$$

which showed that A and U are equivalent degrees of freedom. Contrarily, when g is large, A and U couldn't be good degrees of freedom at the same time. In another point of view, even $\left[A_\mu, \dot{A}_\nu\right] \to \varepsilon \sim 0$, $\left[U, \dot{U}\right]$ could still be $g^2\epsilon\varepsilon \to \infty \simeq \delta^{(3)}(0)$, which is an indication for the quantization canonic commutator.

Besides, the kinetics energy term $\mathcal{L}_U$ in (5) could not be constructed through the way of directly inserting $A \sim U\partial U^\dagger$ into the $\mathcal{L}_A$ in QED. For instance, there is

$$
\begin{aligned}
& (\partial^\mu A^\nu)(\partial_\nu A_\mu) \\
\to \; & \mathrm{Tr}\left([\partial^\mu(\partial^\nu U U^\dagger)][\partial_\nu(U\partial_\mu U^\dagger)]\right) \\
= \; & \mathrm{Tr}\left([\partial^\mu U\partial^\nu U^\dagger + \partial^\mu\partial^\nu U U^\dagger][\partial_\nu U\partial_\mu U^\dagger + U\partial_\nu\partial_\mu U^\dagger]\right) \\
= \; & \mathrm{Tr}\left(\partial^\mu U\partial^\nu U^\dagger \cdot \partial_\nu U\partial_\mu U^\dagger + \partial^\mu\partial^\nu U\partial_\nu U^\dagger U\partial_\mu U^\dagger\right) \\
& + \mathrm{Tr}\left(\partial^\nu U\partial^\mu U^\dagger U\partial_\nu\partial_\mu U^\dagger + \partial^\mu\partial^\nu U \cdot \underline{U^\dagger U} \cdot \partial_\nu\partial_\mu U^\dagger\right), \qquad (119)
\end{aligned}
$$

which could only give the self-interaction terms of U for a general case of $U^\dagger U \neq 1$. Or, in the weak fluctuation case, $U^\dagger U \simeq 1$, there is a trivial result,

$$
\begin{aligned}
F^{\mu\nu} &\equiv \partial^\mu A^\nu - \partial^\nu A^\mu \\
&\simeq \frac{i}{g}\left(\partial^\mu B^\nu - \partial^\nu B^\mu\right) \\
&= \frac{i}{g}\left[\partial^\mu(U\partial^\nu U^\dagger) - \partial^\nu(U\partial^\mu U^\dagger)\right] \\
&\simeq \frac{i}{g}[\partial^\mu(1 + ig\epsilon n_\alpha A^\alpha) \cdot \partial^\nu(1 - ig\epsilon n_\alpha A^\alpha) \\
&\quad -\partial^\nu(1 - ig\epsilon n_\alpha A^\alpha) \cdot \partial^\mu(1 - ig\epsilon n_\alpha A^\alpha)] \\
&= \frac{i}{g}[igA^\mu(-ig)A^\nu - igA^\nu(-ig)A^\mu] = 0 \\
\text{or} \quad &\simeq \frac{i}{g}\left(\partial^\mu \cdot 1 \cdot \partial^\nu \cdot 1 - \partial^\nu \cdot 1 \cdot \partial^\mu \cdot 1\right) = \frac{0}{g} \xrightarrow{g \to 0} \neq 0\,. \qquad (120)
\end{aligned}
$$

In a word, we can't generate $\mathcal{L}_U$ from $\mathcal{L}_A$!

2.8.7 $U \sim \epsilon \cdot A$, so, is U a string?

According to the analysis above, and the definition

$$
U(x + \epsilon n, x) = 1 - ig\epsilon n^\mu A_\mu(x) + ... \qquad (121)
$$

we could find that the product $\epsilon \cdot A$ as an entire variable might be concentrated on.

When $g \to 0$, A is a good d.o.f. of particle, then the relation of action

$$\begin{aligned} S &= \int d^4x \partial_\mu A_\alpha \partial_\nu A_\beta \eta^{\mu\nu}\eta^{\alpha\beta} \\ &= \int d^2x \left| \frac{J(dx_1, dx_2)}{J(\epsilon_1, \epsilon_2)} \right| \partial_\mu(\epsilon_1 A_\alpha)\partial_\nu(\epsilon_2 A_\beta)\eta^{\mu\nu}\eta^{\alpha\beta} \\ &\simeq \int d^2x \left| \frac{J(dx_1, dx_2)}{J(\epsilon_1, \epsilon_2)} \right| \partial_\mu U \partial_\nu U \eta^{\mu\nu} \end{aligned} \tag{122}$$

might indicate that U is a string if there was

$$\left| \frac{J(dx_1, dx_2)}{J(\epsilon_1, \epsilon_2)} \right| = Const.. \tag{123}$$

Apparently, it's equivalent to what here we have done, that is, treat U is a "field" for $g \to \infty$, with a different form of $(\partial\partial U)^2$ as a kinetic energy term, by contrast with the particles A with $(\partial A)^2$ as kinetic energy term; as well as for the case of $g \to 0$. On the other hand, we might say, the interaction of matter fields ψ coupled to U is a kind of "point-string" interaction.

The reduction of $U \to \epsilon \cdot A$ might be seemed as a kind of realization of holographic principle (take the ADS/CFT duality as an example), that is, both of them worked through taking a correspondence between different d.o.f or operators with different dimensions. Or, we might say, the reduction of "string→ point" could be realized by the approximation in Section 2.7, instead of the compactification.

2.8.8 Multi-vacuum structure for higgs vector A^μ

1. Multi-vacuum structure for A^μ

One thing should be noted. If we write

$$U(x) = \exp[-ig\epsilon n^\mu A_\mu(x)] = \cos[g\epsilon n^\mu A_\mu(x)] - i\sin[g\epsilon n^\mu A_\mu(x)], \tag{124}$$

then the potential term

$$V(A) \sim U(A) + U^\dagger(A) = \cos[(g\epsilon)A], \tag{125}$$

would mean that the dynamics for the field A^μ is of a sine-Gordon type (or, a kind of higgs vector), see Fig. 1-(2), in which there might be many excitations for A at different vacuums (or, VEVs), with heavy masses in the large g cases($g\epsilon \simeq 1$) and small masses in the small g cases.

Similarly, we can write

$$U(x) = \exp[-ig\epsilon\phi(x)] = \cos[g\epsilon\phi(x)] - i\sin[g\epsilon\phi(x)], \tag{126}$$

for a scalar P-2 type field ϕ, and give ϕ the similar results as the ones below for A^μ.

2. Fermion mass spectrum

Like the mass correction in (91) from U, with the term $\bar{\psi}\not{A}\psi$, the fermions can get a mass correction from A^μ,

$$\Delta m \sim \alpha\Lambda\langle A\rangle \sim \alpha\Lambda\frac{(2n+1)\pi}{g\epsilon}\ ,\ n = 0, 1, 2, \dots . \qquad (127)$$

For instance,

a. if Δm is the mass differences between the current quarks and the constituent quarks, then, by setting

$$g \sim \frac{(2n+1)\alpha\Lambda}{\Delta m \cdot \epsilon} \xrightarrow{\mathcal{O}(\Lambda)\sim\mathcal{O}(\epsilon)} \frac{(2n+1)\alpha}{\Delta m} \sim 1\,, \qquad (128)$$

with $\Delta m \sim 1GeV$ and $n = 0$, we have $\alpha \sim 1$.

b. if $g \sim 0.01$ for the E.W. interaction, then, $\Delta m \sim 100GeV$, corresponding to the possible heavy fermions.

3. Instanton d.o.f is excited

If U is the full effects of gauge symmetry, then it will include the instanton effects. One of the instanton solution for an non-Abelian gauge field could be written as [8]

$$A_\mu(x) = \frac{i}{g}\phi(r)U(x)\partial_\mu U^{-1}(x)\,, \qquad (129)$$

$$\phi(r) = \frac{r^2}{r^2+\lambda^2}\,. \qquad (130)$$

So, inversely, we have

$$U(x)i\partial_\mu U^{-1}(x) = g\frac{1}{f(r)}A_\mu(x) \equiv \phi(r)A_\mu(x)\,, \qquad (131)$$

that means, if one still choose $A_\mu(x)$ as d.o.f in the $g \gg 1$ case, then, it must be combined with a $\phi(r)$ field to represent the full effects of the $U(x)\partial_\mu U^{-1}(x)$ term.

On the other hand, when $g \to 0$, we can treat the instanton d.o.f is very heavy and frozen.

4. A seesaw mechanism for gauge symmetry and flavor symmetry

See Fig. 1-(2), with (125), for a vacuum at $A = \langle A\rangle_i$, the potential could be written as

$$V(A \simeq A_i) \simeq -1 + (g\epsilon)^2(A - A_i) + \dots, \qquad (132)$$

which means the mass of the excitation $A' = A - A_i$ is of order $\sim m = g\epsilon$. So, we can get the conclusions below:

I. when $g \to 0$,

a. A'_μ is nearly massless, so the gauge symmetry is restored;

b. the instanton could be treated as very heavy and frozen, as discussed for (131);

c. the VEV $\langle A\rangle_i$ are of very different magnitudes, so, through (127), the fermion masses would be also of very different magnitudes, including very heavy fermions; this is a kind of flavor symmetry breaking for fermions;

II. when $g \to \infty$,

a. A'_μ is massive, with the diagonal elements in its mass matrix being large, so the gauge symmetry is broken;

b. since the instanton in (131) was excited now, the tunnelling(oscillating) effect would become strong, so the off-diagonal elements in the mass matrix of A'_i become large, too; or, in another viewpoint, now it's A'_μ that was frozen, and the instanton was the real d.o.f for mediating interactions; we can treat the instanton massless or nearly massless according to the absence of heavy bosons in a hadron;

c. the VEV $\langle A\rangle_i$ in the neighbour minimum are nearly equal, so, there would be a degenerate for the fermion mass, or, we can say, the flavor symmetry for fermions would be restored; besides, it's now allowed for very small fermion masses through (127), which might be an underlying reason for the feasibility of the "large N_c" or "large N_f" hypothesis for a real hadron, and for the possible neutrino-Dark Matter oscillation.

So, maybe this is a new kind of dynamical symmetry breaking/restoring mechanism, with a seesaw for gauge symmetry and flavor symmetry.

2.9 Matter fields and current

2.9.1 New type matter fields

The matter field ϕ could also be treated as

$$\Phi(x+\epsilon n, x) = 1 - ig\epsilon\phi(x) + ... \tag{133}$$

with kinetic energy term $(\partial\partial\Phi)^2$ transforming to $(\partial\phi)^2$ for $g \to 0$ case, and, for an interaction term

$$\alpha\partial\Phi^\dagger U\partial\Phi \to \alpha'\phi^\dagger U\phi \tag{134}$$

the transition of couplings $\alpha \to \alpha'$ would be canceled with the redefinition of wave function $\Phi \sim g\phi(x) \to \phi(x)$, so indeed we would have

$$\alpha' = \alpha\,. \tag{135}$$

For fermion field ψ, there is

$$\Psi(x+\epsilon n, x) = 1 - ig\epsilon\psi(x) + ... \tag{136}$$

with kinetic energy term $\bar\Psi\partial\partial\partial\Psi$ transforming to $\bar\psi\partial\psi$ for $g \to 0$ case, and, for interaction term there would be

$$\alpha\partial U\cdot\partial\bar\Psi\partial\Psi \to \alpha'\partial U\cdot\bar\psi\partial\psi\,,\ \alpha = \alpha'\,. \tag{137}$$

However, what we want to present here is for the current,[5]

$$\text{(current)}\quad J^\mu(x) \;=\; \phi^\dagger i\partial^\mu\phi(x) \to \Phi^\dagger i\partial^\mu\Phi(x) \equiv J^\mu(x) \quad \text{(field)}\,, \tag{138}$$

Let's illustrate our motivation for the conversion in (138) with an example, for instance, the interaction term in(7)

$$\begin{aligned}\mathcal{L}_I \;&=\; -\rho Q\,\frac{1}{M}\partial_\mu U_1(\bar\psi i\overleftrightarrow{\partial}^\mu\psi)\\ &\quad -\rho Q\,\frac{1}{M}\partial_\mu U_1[\bar\psi i\partial^\mu\psi - (i\partial^\mu\bar\psi)\cdot\psi]\,,\end{aligned} \tag{139}$$

[5]The expansion could also be taken for a non-Abelian case or tensor current case.

which is a non-renormalized one because of the vertex, with a momentum included in. So, if we want to convert this interaction to a renormalized one, what should we do to deal with the $\partial U \cdot \bar{\psi}\partial\psi$ term for $\partial \gg 1$ case? If we only take a correspondence of $\phi \to \Phi$ hence $\alpha\phi^\dagger U\partial\phi \to \partial\Phi^\dagger U \cdot (\Phi - 1)$, it would give trivial substitution for the "coupling": $\partial_\phi \to \partial_\Phi$, with the non-renormalizable property (with momentum included in vertex) remained.

But, now, within the P-4 framework, after an extension $\phi \to \Phi$, it's allowed to further treat a current J^μ to a field J^μ, then the term $\partial U \cdot J^\mu$(current) would turn to $\partial U \cdot J^\mu$(field), and, there would be other terms to represent the interactions, such as $\partial\partial U \cdot J$(field)$\cdot J$(field).

Moreover, there is the expansion,

$$\Theta(x + \epsilon n, x) = 1 - ig\epsilon n_\mu \frac{1}{M} J^\mu(x) + ... , \tag{140}$$

where the charge should be $\left[g\frac{1}{M}\right]$, so, that means, when $\left[g\epsilon\frac{1}{M}\right] \sim 1$, the "field" J^μ would turn back to the P-4 type field Θ, with the conversion $\partial U \cdot J^\mu$(field) $\to \partial U \cdot i\partial(\Theta - \Theta^\dagger)$, and, there would be other terms to represent the interactions, such as $\partial_\mu U \cdot (\Theta^\dagger \hat{O}^\mu \Theta)$, with $\hat{O}^\mu$ an intrinsic operator of the field Θ to contract with the tensor indices in $\partial_\mu U$.

2.9.2 Current = Field? A possible way for renormalizable gravity.

Is it feasible for (138)?

Firstly, what's the difference between a current and a vector field? A field has a E.O.M, while a current hasn't; for other things, they could be treated as the same. So, it's more or less reasonable for (138).

Secondly, reminding the Maxwell equation,

$$\partial^2 A^\mu \sim J^\mu , \tag{141}$$

which would be generalized to a new equaiton

$$\partial^2 J^\mu = 0 \Rightarrow \partial^4 A^\mu = 0 \tag{142}$$

we can say, if J^μ became a P-2 type field, then, A^μ would become a P-4 type field (which would be studied in Section 3). So, it is feasible for (138), based on the foundation for the dynamics of a P-4 type A^μ.

2.10 The van der Waals-potential: the introduction of $\int dxA$ is trivial!

It's not necessary to construct van der Waals-potential by introducing new media particles Φ(bosons or fermions) with new type propagators such as $\sim 1/p^N$ with $N < 0$(which would be always converted to an $N > 0$ case by redefining another new field with an equivalent dynamics), but only need to set restrictions on the interaction terms.

For detail, firstly, let's take the symbol $\mathcal{L}_{(LO)}$ to be the Lagragian interaction term

$$\mathcal{L}_{(LO)} \sim \bar{\psi} \otimes \Phi \otimes \ldots \otimes \partial\Phi \otimes \ldots \otimes \psi$$

with the least number of $\partial\Phi$ terms, which we call it as "leading order(LO) interaction term". Then, for $\mathcal{L}_{(LO)}$ includes $N \geq 1$ $\partial\Phi$-symbols, we get van der Waals-potential $V(r) \sim 1/r^N$ (with $N > 1$) through setting the propagator $\sim 1/p^{4-\alpha}$ with $\alpha > 0$ for Φ , for instance, $\sim 1/p^3$, $\sim 1/p^2$, $\sim 1/p$,and so on, since the amplitude of "$2 \to 2$" scattering process would be

$$\begin{aligned} & \mathcal{M} \sim p \otimes \frac{1}{p^{4-\alpha}} \otimes p \sim \frac{1}{p^{2-\alpha}} \sim -\mathcal{V}(p) \\ \Rightarrow \quad & V(r) \sim \int d^3p\mathcal{V}(p) \sim p^{1+\alpha} \sim \frac{1}{r^{1+\alpha}}. \end{aligned} \tag{143}$$

Particularly, for a field Φ with a propagator $\sim p$, we can treat Φ as an extraordinary field with an integral equation type EOM

$$\frac{1}{\hat{p}}\Phi(x) = \frac{1}{m_\Phi}\Phi(x), \text{ with } \frac{1}{\hat{p}} = \int_\infty^x dy \tag{144}$$

It's allowed to use an integral equations to be the EOM of field Φ in the case of $\partial\Phi$ is ill-defined, for instance, Φ or $\partial\Phi$ was a singular function.

However, since our quantization framework is canonic commutator, the EOM should be always constructed through derivative qeuations rather than integral equations. On the other hand, we could always define a new effective field and its E.O.M as

$$U \equiv \int_\infty^x dy\Phi(y) \Rightarrow \hat{p}U = m_U U \tag{145}$$

to be the new d.o.f, and the corresponding Lagrangian terms, which could give equivalent results to the original Φ-terms for all orders(by corresponding $\mathcal{L}_{(LO)} \sim \bar{\psi}\partial U\psi$ including one ∂-symbols for U to the equivalent original $\mathcal{L}_{(LO)} \sim \bar{\psi}\Phi\psi$ for Φ), so that, it's only formally meaningful but practically trivial to construct fields with integral equation type E.O.M.

3 Field U^μ

3.1 Lagrangian for U^μ

Now we take a generalization of $U \to U^\mu$, with the transform property of U^μ under the $U(1)$ **global** group element V as

$$U^\mu \to VU^\mu V^\dagger\,, \tag{146}$$

where the indices μ means the transform is for each component of U^μ.

With the gauge fixed condition, for a general complex-valued vector field

$$U = \text{Re}[U] + i\,\text{Im}[U] \equiv U_1 - i\,U_2\,, \tag{147}$$

whose motion obeying the P-4 type **Klein-Gordon** equation as for a scalar U field in (40), the Lagrangian could be written as

$$\mathcal{L}_U = +\partial_\alpha\partial_\beta U_\mu^\dagger\partial^\alpha\partial^\beta U^\mu - m_U^4 U_\mu^\dagger U^\mu\,,\; U_\mu^\dagger U^\mu < 1\,, \tag{148}$$

or

$$\mathcal{L}_U = +\partial_\alpha(aF_{\beta\mu} + b\overline{F}_{\beta\mu})^\dagger\partial^\alpha(aF^{\beta\mu} + b\overline{F}^{\beta\mu}) - m_U^4 U_\mu^\dagger U^\mu \,,\; U_\mu^\dagger U^\mu < 1 \,, \tag{149}$$

with

$$a^2 + b^2 = 1 \,, \tag{150}$$

where

$$F^{\beta\mu} \equiv F^{\beta\mu}(U) \equiv \partial^\beta U^\mu - \partial^\mu U^\beta \,,\; \overline{F}_{\beta\mu} \equiv \overline{F}_{\beta\mu}(U) \equiv \partial^\beta U^\mu + \partial^\mu U^\beta \,. \tag{151}$$

Particularly, in the case of $b = 0$ and $m_U = 0$, to get a **Maxwell** equation, for a specific irreducible representation of the tensors, we can choose a Lagrangian for $F^{\beta\mu}$ as

$$\begin{aligned}
\mathcal{L}_U &= +\frac{1}{2}(\partial_\alpha F_{\beta\mu} - \partial_\beta F_{\alpha\mu})^\dagger(\partial^\alpha F^{\beta\mu} - \partial^\beta F^{\alpha\mu}) - m_U^4 U_\mu^\dagger U^\mu \\
&= +\left[\partial_\alpha F_{\beta\mu}^\dagger \partial^\alpha F^{\beta\mu} - \partial_\alpha F_{\beta\mu}^\dagger \partial^\beta F^{\alpha\mu}\right] - m_U^4 U_\mu^\dagger U^\mu \\
&= +\left[\partial_\alpha(\partial_\beta U_\mu - \partial_\mu U_\beta)^\dagger \partial^\alpha(\partial_\beta U_\mu - \partial_\mu U_\beta) - \partial_\alpha(\partial_\beta U_\mu - \partial_\mu U_\beta)^\dagger \partial^\beta(\partial^\alpha U^\mu - \partial^\mu U^\alpha)\right] \\
&\quad - m_U^4 U_\mu^\dagger U^\mu \\
&= +\Big[\partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\beta U^\mu - \partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\mu U^\beta - \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\alpha\partial^\beta U^\mu + \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\alpha\partial^\mu U^\beta \\
&\quad - \partial_\alpha\partial_\beta U_\mu^\dagger \partial^\beta\partial^\alpha U^\mu + \partial_\alpha\partial_\beta U_\mu^\dagger \partial^\beta\partial^\mu U^\alpha + \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\beta\partial^\alpha U^\mu - \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\beta\partial^\mu U^\alpha\Big] \\
&\quad - m_U^4 U_\mu^\dagger U^\mu \\
&= +\Big[(\partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\beta U^\mu + \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\alpha\partial^\mu U^\beta - \partial_\alpha\partial_\beta U_\mu^\dagger \partial^\beta\partial^\alpha U^\mu) \\
&\quad + (-\partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\mu U^\beta - \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\alpha\partial^\beta U^\mu + \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\alpha\partial^\beta U^\mu + \partial_\beta\partial_\alpha U_\mu^\dagger \partial^\beta\partial^\mu U^\alpha \\
&\quad - \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\beta \underline{\partial^\mu U^\alpha})\Big] - m_U^4 U_\mu^\dagger U^\mu \\
&= +\Big[\partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\beta U^\mu - \partial_\alpha \underline{\partial_\mu U_\beta^\dagger} \underline{\partial^\beta} \partial^\alpha \underline{U^\mu}\Big] - m_U^4 U_\mu^\dagger U^\mu \\
&= +\left[\partial_\alpha\partial_\beta U_\mu^\dagger(\partial^\alpha\partial^\beta U^\mu - \underline{\partial^\mu\partial^\alpha} U^\beta)\right] - m_U^4 U_\mu^\dagger U^\mu \\
&= +\left[\partial_\alpha\partial_\beta U_\mu^\dagger(\partial^\alpha\partial^\beta U^\mu - \partial^\alpha\partial^\mu U^\beta)\right] - m_U^4 U_\mu^\dagger U^\mu \\
&= +\frac{1}{2}\left\{\left[\partial_\alpha\partial_\beta U_\mu^\dagger(\partial^\alpha\partial^\beta U^\mu - \partial^\alpha\partial^\mu U^\beta)\right] + (\beta \leftrightarrow \mu)\right\} - m_U^4 U_\mu^\dagger U^\mu \\
&= +\frac{1}{2}\Big\{\left[\partial_\alpha\partial_\beta U_\mu^\dagger(\partial^\alpha\partial^\beta U^\mu - \partial^\alpha\partial^\mu U^\beta)\right] + \Big[\partial_\alpha\partial_\mu U_\beta^\dagger(\partial^\alpha\partial^\mu U^\beta - \partial^\alpha\partial^\beta U^\mu)\Big]\Big\} - m_U^4 U_\mu^\dagger U^\mu \\
&= +\frac{1}{2}\left(\partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha F^{\beta\mu} + \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\alpha F^{\mu\beta}\right) - m_U^4 U_\mu^\dagger U^\mu \\
&= +\frac{1}{2}\left(\partial_\alpha\partial_\beta U_\mu^\dagger - \partial_\alpha\partial_\mu U_\beta^\dagger\right)\partial^\alpha F^{\beta\mu} - m_U^4 U_\mu^\dagger U^\mu \\
&= +\frac{1}{2}\partial_\alpha F_{\beta\mu}^\dagger \partial_\alpha F^{\beta\mu} - m_U^4 U_\mu^\dagger U^\mu \,,\; U_\mu^\dagger U^\mu < 1 \,,
\end{aligned} \tag{152}$$

where the underlined indices would be interchanged, and then, we can get the P-4 type Maxwell equation

$$(\partial^4 g_{\mu\nu} - \partial^2\partial_\mu\partial_\nu)A^\nu = 0 \tag{153}$$

with an extra minus sign by contrast with (40), see Appendix D.

Note that, for a normal vector field, the sign for mass term in (148) is "+" so that it's "−" for each space component U_i, however, here the U_μ is a kind of higgs field, so that the sign for mass term should be "−". Then, the gauge fixed condition and corresponding free E.O.M would be

$$\hat{p}\cdot U = 0 \;\Rightarrow\; +\hat{p}^4 U^\mu - m_U^4 U^\mu = 0,\; \hat{p} = i\partial, \tag{154}$$

so, the propagator in momentum space would be

$$D_F^{\mu\nu}(U) = \frac{+ig^{\mu\nu}}{p^4 - m_U^4 + i\epsilon}\,, \tag{155}$$

rather than $D_F^{\mu\nu}(U) = \frac{-ig^{\mu\nu}}{p^4 - m_U^4 + i\epsilon}$, with the same reason as for (53,59).

The interaction term of U^μ with the matter could be written as

$$\begin{aligned}
\mathcal{L}_I \;=\;& -\alpha\,\Lambda\bar{\psi}[(\not{U}+\not{U}^\dagger) + i(\not{U}-\not{U}^\dagger)]\psi - \alpha\,\frac{\Lambda}{M}[(U+U^\dagger)+i(U-U^\dagger)]_\mu \bar{\psi} i\partial^\mu\psi \\
& -\beta\,[F_{\mu\nu}^{(U+U^\dagger)} + iF_{\mu\nu}^{(U-U^\dagger)}]\cdot\bar{\psi}[(\varepsilon^{\mu\nu}+\sigma^{\mu\nu}) + \frac{1}{M}(\gamma^\mu i\partial^\nu - \gamma^\nu i\partial^\mu]\psi \\
& -\beta\,[\overline{F}_{\mu\nu}^{(U+U^\dagger)} + i\overline{F}_{\mu\nu}^{(U-U^\dagger)}]\cdot\bar{\psi}[g^{\mu\nu} + \frac{1}{M}(\gamma^\mu i\partial^\nu + \gamma^\nu i\partial^\mu)]\psi \\
& -\kappa\,\frac{1}{M}\bar{\psi}\left[\Lambda^2(\not{U}^\dagger\not{U})\right]\psi \\
& +(\text{higher-order operators})\,,
\end{aligned} \tag{156}$$

where $\varepsilon^{\mu\nu}$ is an antisymmetric real-valued constant tensor, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$, so that for each $\mu\nu$ component, there are $(\varepsilon^{\mu\nu})^\dagger = \varepsilon^{\mu\nu}$ and $(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$.

The self-interaction term of U^μ might be written as

$$\begin{aligned}
\mathcal{L}_I^{self} \;=\;& -\Lambda_U\left[(\epsilon_{\rho\alpha}+g_{\rho\alpha})(U+U^\dagger)^\rho F_{\beta\mu}^{(U+U^\dagger)} F^{\alpha\beta\mu}_{(U+U^\dagger)}\right] \\
& -\Lambda_U\left[\text{terms with } i(U-U^\dagger)\;\right]\,.
\end{aligned} \tag{157}$$

where the $\epsilon^{\mu\nu}$ in the Λ_U term is chosen to be antisymmetric. Since there is at least 3 U-field for the interaction term, there must be at least 1 or 3 ∂-symbols for constructing a scalar; however, here we don't write the terms with only one ∂-symbol, as $U_\mu U_\nu \partial^\mu U^\nu$ or $U_\mu U^\mu \partial_\nu U^\nu$, because, we'll see in Section 3.4, that they can't give us a qualified 2nd-order D.E..

Actually, like the definition of the field strength for a non-Abelian gauge field A_α^i as

$$F_{\alpha\beta}^i t^i = \partial_\alpha A_\beta^i t^i - \partial_\beta A_\alpha^i t^i - ig[A_\alpha^i t^i, A_\beta^j t^j]\,, \tag{158}$$

we might "fabricate" a formally definition of the "field strength $F_{\alpha\beta\mu}$ of field strength of $F_{\beta\mu}$ of U^μ" as the form

$$F_{\alpha\beta\mu}T^\beta = \left(\partial_\alpha F_{\beta\mu}T^\beta - \partial_\mu F_{\beta\alpha}T^\beta\right) + \left[F_{\nu\alpha}T^\nu, F_{\beta\mu}T^\beta\right]\,, \tag{159}$$

with T^μ a class of group generator-like operators. However, here we would ignore this fictive construction of (159) and (157), and that would not influence the results we concerned in this paper (that is, the kinetic energy term of of $U^{\mu\nu}$ and the interaction term of $U^{\mu\nu}$ and matters in the Lagrangian). For detail, the first term for the r.h.s of (157)

$$\begin{aligned}
& \partial_\alpha F_{\beta\mu} - \partial_\mu F_{\beta\alpha} \\
=\;& \partial_\alpha(\partial_\beta U_\mu - \partial_\mu U_\beta) - \partial_\mu(\partial_\beta U_\alpha - \partial_\alpha U_\beta) \\
=\;& \partial_\beta(\partial_\alpha U_\mu - \partial_\mu U_\alpha) = \partial_\beta F_{\alpha\mu}
\end{aligned} \tag{160}$$

would be a "trivial" construction(for the meaning of "trivial", it means, $\partial_\alpha F_{\beta\mu} - \partial_\mu F_{\beta\alpha}$ has only one term in practice, which has been shown in (152)), although the second term would include $\partial U \partial U$ term would give a nontrivial self-interaction term with multi-derivative for the U_μ fields, like the term $U_\alpha^\dagger U_\beta^\dagger U_\mu^\dagger \partial^\alpha F^{\beta\mu}$ in (165) a 4-particle coupled term.

For future convenience, as for the scalar U field, for each μ component, we can have the decomposition

$$U^\mu = U_1^\mu - iU_2^\mu \,, \tag{161}$$

and the self-interaction terms of U^μ could be written as

$$\begin{aligned}\mathcal{L}_I^{self} &= -\Lambda_U U_{1\alpha}\partial_\beta U_{1\mu}\partial^\alpha\partial^\beta U_1^\mu - \Lambda_U^2 U_{1\alpha}U_{1\beta}U_{1\mu}\partial^\alpha\partial^\beta U_1^\mu \\ &\quad +(U_2 \text{ terms}) + (U_1 \cdot U_2 \text{ mixed terms})\,. \end{aligned}\tag{162}$$

We don't consider terms as

$$\mathcal{L}_I = -\alpha\Lambda\, U_1^\mu \bar{\psi}[\sigma_{\mu\nu}(i\partial^\mu + \gamma^\mu)]\psi \tag{163}$$

in this work.

3.2 Version I: anti-symmetric field strength $F_{\mu\nu}$

3.2.1 Version I.1: the interaction coupled to intrinsic charges

1. Lagrangian

The antisymmetric tensor $F_{\mu\nu}$ has 6 independent components, so, it might serve as the contribution from two vector field, the strength $\{\boldsymbol{E}, \boldsymbol{B}\}$ of an off-shell photon. Note, now the E.O.M for $F_{\mu\nu}$ wouldn't be true since the U_μ was off-shell, that is,

$$\partial^2\partial_\mu F^{\mu\nu} = \partial^2\partial_\nu F^{\mu\nu} \neq 0\,. \tag{164}$$

However, if U_μ was on-shell, then $F_{\mu\nu}$ has 6-4=2 independent component, which might now again serve as the on-shell state of a photon. Let's check whether $F_{\mu\nu}$ could serve for the photon in the sense of the effective interaction form.

The $F_{\mu\nu}$ part of Lagrangian for free particle U^μ could be written as

$$\begin{aligned}\mathcal{L}_U &= +\frac{1}{2}\partial_\alpha F_{\beta\mu}^\dagger \partial^\alpha F^{\beta\mu} - m_U^4 U_\mu^\dagger U^\mu \\ &= +\frac{1}{2}\left[\partial_\alpha(\partial_\beta U_\mu^\dagger - \partial_\mu U_\beta^\dagger)\right]\left[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)\right] - m_U^4 U_\mu^\dagger U^\mu\,, \end{aligned}\tag{165}$$

and, the interaction term could be written as

$$\begin{aligned}\mathcal{L}_I &= -\alpha\,\Lambda\bar{\psi}[(\not{U}+\not{U}^\dagger) + i(\not{U}-\not{U}^\dagger)]\psi \\ &\quad -\beta\,\bar{\psi}(\varepsilon^{\mu\nu}+\sigma^{\mu\nu})[F_{\mu\nu}^{(U+U^\dagger)} + iF_{\mu\nu}^{(U-U^\dagger)}]\psi \\ &\quad +(\text{higher-order operators}), \end{aligned}\tag{166}$$

For future convenience, we write the interaction terms for U_1^μ obviously, as

$$\begin{aligned}\mathcal{L}_I &= -\alpha\,\Lambda\bar{\psi}\not{U}_1\psi - \beta\,\bar{\psi}(\varepsilon^{\mu\nu}+\sigma^{\mu\nu})F_{\mu\nu}^{(U_1)}\psi \\ &\quad +(\text{higher-order operators})\,. \end{aligned}\tag{167}$$

2. The E.O.M for tensor theory for Version I.1

See (376-377) in Appendix D, we can get the dynamical equation for the field U^μ as

$$\partial^2 \left[\partial^2 g_{\sigma\tau} - \partial_\sigma \partial_\tau\right] U^\tau \xlongequal{(\partial\cdot U=0)} \partial^4 U_\sigma = (i\partial)^4 U_\sigma = +m_U^4 U^\sigma + J^\sigma \tag{168}$$

with the gauge fixed condition $\partial_\sigma U^\sigma = 0$, and

$$J^\sigma = \alpha\, \Lambda \bar\psi \gamma^\sigma \psi - 2\beta\, \partial_\tau \left[\bar\psi(\varepsilon^{\tau\sigma} + \sigma^{\tau\sigma})\psi\right] . \tag{169}$$

3. Quantization and the effective potential for Version I.1

Now we can write an effective potential mediated by U^μ. By omitting the $\epsilon^{\mu\nu}$ term in Lagragian (165), the amplitude of the process in Fig. 4-(a) could be written as[6]

$$\begin{aligned}
i\mathcal{M}_a &= \bar u^{s'} i[\alpha_1 \Lambda \gamma^\mu g_{\mu\rho} + \beta_1 \sigma_{\alpha\mu}(iq^\alpha g^\mu_\rho - iq^\mu g^\alpha_\rho)]u^s \cdot \frac{+ig^{\rho\lambda}}{q^4} \\
&\quad \cdot \bar u^{r'} i[\alpha_2 \Lambda \gamma^\nu g_{\nu\lambda} - \beta_2 \sigma_{\beta\nu}(iq^\beta g^\nu_\lambda - iq^\nu g^\beta_\lambda)]u^r \\
&= \bar u^{s'} i[\alpha_1 \Lambda \gamma_\rho + \beta_1 (i\sigma_{\alpha\mu} q^\alpha g^\mu_\rho - i\sigma_{\alpha\mu} q^\mu g^\alpha_\rho)]u^s \cdot \frac{+ig^{\rho\lambda}}{q^4} \\
&\quad \cdot \bar u^{r'} i[\alpha_2 \Lambda \gamma_\lambda - \beta_2 (i\sigma_{\beta\nu} q^\beta g^\nu_\lambda - i\sigma_{\beta\nu} q^\nu g^\beta_\lambda)]u^r \\
&= \bar u^{s'} i[\alpha_1 \Lambda \gamma^\rho + \beta_1 (q_\mu g^\mu_\rho + q_\alpha g^\alpha_\rho)]u^s \cdot \frac{+ig^{\rho\lambda}}{q^4} \\
&\quad \cdot \bar u^{r'} i[\alpha_2 \Lambda \gamma^\lambda - \beta_2 (q_\nu g^\nu_\lambda + q_\beta g^\beta_\lambda)]u^r \\
&= \bar u^{s'} i(\alpha_1 \Lambda \gamma^\rho + 2\beta_1 q_\rho)u^s \cdot \frac{+ig^{\rho\lambda}}{q^4} \cdot \bar u^{r'} i(\alpha_2 \Lambda \gamma^\lambda - 2\beta_2 q_\lambda)u^r \\
&\simeq -i \cdot \left\{ 4\Lambda^2 \alpha_1 \alpha_2 \frac{1}{|\boldsymbol{q}|^4} - 2\Lambda \alpha_1 \beta_2 (v_1 - v_2) \cdot q \frac{1}{|\boldsymbol{q}|^4} + 4\beta_1 \beta_2 \frac{1}{|\boldsymbol{q}|^2} \right\} \\
&\quad \cdot 2m\delta^{ss'} 2m\delta^{rr'}
\end{aligned} \tag{170}$$

with the approximate relations below(Gordon's identity)

$$\begin{aligned}
\bar u^{p'} \gamma^\mu u^p &= \frac{1}{2m} \bar u^{p'} \left[(p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\nu\right] u^p ,\ (p' \equiv p+q) \\
\Rightarrow \bar u^{p'} \left[i\sigma^{\mu\nu} q_\nu\right] u^p &= \bar u^{p'} \left[2m\gamma^\mu - (p+q+p)^\mu\right] u^p \\
&= \bar u^{p'} \left[2m\gamma^\mu - (2p+q)^\mu\right] u^p \\
\Rightarrow \qquad \left[i\sigma^{\mu\alpha} q_\alpha\right] &\to \left[2m\gamma^\mu - (2p+q)^\mu\right] \simeq -q^\mu \\
-\left[i\sigma^{\nu\beta} q_\beta\right] &\to \left[2m\gamma^\nu - (2k-q)^\nu\right] \simeq q^\nu ,\ (k' \equiv k-q) .
\end{aligned} \tag{171}$$

Then, by comparing with the Born approximation to the scattering amplitude in non-relativistic quantum mechanics, see (69), we can get the effective potential as

$$V(r) = -\frac{4\Lambda^2 \alpha_1 \alpha_2}{8\pi} r + \frac{2\lambda \Lambda \alpha_1 \beta_2}{2\pi^2} \log \frac{r}{r_0} + \frac{4\beta_1 \beta_2}{4\pi} \frac{1}{r} - \frac{2\lambda \Lambda \alpha_1 \beta_2}{2\pi^2}(1 - \gamma_E) , \tag{172}$$

where $-\infty < \lambda < +\infty$ was defined in (67), and, particularly, for NR case, $\lambda \simeq 0$. The interpretation for (172) would be like (72).

---

[6] As in (65), for simplicity, here we can only consider the contributions from U_1, and, for the contributions from U_2, the result just need a double.

There is a combination of a linear and a logarithmic potential for the $\alpha_1 \cdot \alpha_2 < 0$ case, which might be corresponding to the confinement for strong-coupled gauge theory, or the dark matter effects.

Note that, as we only concentrate the non-relativistic case, we have chosen the definite spinor basis for the outer-line particles as $\bar{u}^{s'}\gamma^\mu u^s \to \delta^{ss'}$, that means, the spin orientation hasn't changed. That's very important! On one hand, spin changing would be a kind of relativistic effects, on the other hand, the "charge" $\beta\sigma^{\mu\nu}$corresponding to the term $\bar{\psi}F^{\mu\nu}\psi$ was a kind of magnetic moment, hence the change of spin orientation would influence the sign of the coefficient β in (166).

3.2.2 Matching for the d.o.f: generation of a linear QED

1. Can $F(U) \sim \partial U$ serve for the photon?

According to the square form of the kinetic energy term $(\partial F)^2$ and the Coulomb-type interaction arising from the term $\bar{\psi}F(U)\psi$ in (166), we can say, if U^μ is a field with propagator $\sim 1/p^4$, then its field strength

$$F(U) \sim \partial U \tag{173}$$

could be treated as a particle d.o.f with propagator $\sim 1/p^2$ serving for the E.M. force, that means, $F(U)$ could be the photon.

Formally, there is a correspondence on the order of derivative:

$$U^\mu \xrightarrow{\text{Field Strength}} A^\mu \xrightarrow{\text{Field Strength}} \{\boldsymbol{E}, \boldsymbol{B}\}\,.$$

2. An interaction from magnet moment?

Why not? It's not important that whether an interactions generated by the field strength would be interpreted as a magnet moment interaction or not; and, the important thing is the form of the interaction!

we can formally parameterize $F_{\mu\nu}$ to two vector, as

$$F_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu\,. \tag{174}$$

So, the effects of $F_{\mu\nu}$ could be corresponding to the process in Fig. 3-(a), that is, an interaction mediated by two particles. The difference is that, for example, when A_μ and B_ν are P-2 type field, $F_{\mu\nu}$ gives an interaction of Van der Waals form, which corresponds to the magnet moment interaction, however, when A_μ and B_ν are P-4 type field, $F_{\mu\nu}$ gives an interaction of Coulomb form, see Appendix C, which would correspond to a P-2 type field propagator.

On another viewpoint, mathematically, as for (106), we can indeed have a correspondence between $F_{\mu\nu}$ and a P-2 type field A_λ, as

$$F_{\mu\nu}\Gamma^{[\mu\nu]}_{\lambda} = A_\lambda\,,\; \Gamma_{[\mu\nu]\lambda}\Gamma^{[\mu\nu]\lambda} = 1\,, \tag{175}$$

where $\Gamma^{\lambda}_{[\mu\nu]}$ is a 3rd-order constant-valued tensor, anti-symmetric for the $\mu\nu$ indices.[7] If we insert (175) into the Lagrangian, it truly generates a kinetic terms for Lagrangian of a P-2 type field A_λ, and, for the interaction terms with $F_{\mu\nu}$, the charge/current would get a change, as

$$\bar{\psi}\sigma^{\mu\nu}\psi \to \bar{\psi}\sigma^{\mu\nu}\Gamma^{\lambda}_{[\mu\nu]}\psi \to \bar{\psi}\gamma^{\lambda}\psi\,, \tag{176}$$

which is just the E.M. current in QED!

Actually, the charge for a classic potential is purely put by hand, which might not be the exact charge at quantum level, but just a correspondence.

Besides, as in (177), for the interaction terms in the full Lagrangian (166), we can directly set $U_\mu \to 1_\mu$ (unit vector), $F(U_1)_{\mu\nu} \to A_\mu$, and $U_{2\mu}$ could be seemed as frozen. Or, we can directly omit terms with

$$U_\mu(x) = \int_\infty^x dz_\nu \partial_z^\nu U_\mu\,, \tag{177}$$

since there is a depression for these terms from the small Λ.

3.2.3 Version I.2: the interaction coupled to momentum

The interaction term could be written as

$$\begin{aligned}\mathcal{L}_I &= -\alpha\frac{\Lambda}{M}\,[(U+U^\dagger)_\mu + i(U-U^\dagger)_\mu]\bar{\psi}i\partial^\mu\psi \\ &\quad -\beta\frac{1}{M}\,[\overline{F}^{(U+U^\dagger)}_{\mu\nu} + i\overline{F}^{(U-U^\dagger)}_{\mu\nu}]\bar{\psi}(\gamma^\nu i\partial^\mu - \gamma^\mu i\partial^\nu)\psi \\ &\quad +(\text{higher-order operators})\,.\end{aligned} \tag{178}$$

For future convenience, we write the interaction terms for U_1^μ obviously, as

$$\mathcal{L}_I = -\alpha\frac{\Lambda}{M}\,U_{1\mu}\bar{\psi}i\partial^\mu\psi - \beta\frac{1}{M}\,\overline{F}^{(U_1)}_{\mu\nu}\bar{\psi}(\gamma^\nu i\partial^\mu - \gamma^\mu i\partial^\nu)\psi + \ldots..\,. \tag{179}$$

We don't consider terms as

$$\mathcal{L}_I = -\alpha\frac{\Lambda}{M}\,U_1^\mu\bar{\psi}\sigma_{\mu\nu}i\partial^\mu\psi \tag{180}$$

in this work.

2. The E.O.M for tensor theory of Version I.2: coupled to momentum

See (388) in Appendix D, we can get the dynamical equation for the field U^μ as

$$\partial^4 U_\sigma = m_U^4 U^\sigma + J^\sigma\,, \tag{181}$$

with the gauge fixed condition $\partial_\sigma U^\sigma = 0$, and

$$\begin{aligned}J^\sigma &= +\alpha\frac{\Lambda}{M}\,\bar{\psi}\bar{\psi}i\partial^\sigma\psi \\ &\quad + -2\beta\frac{1}{M}\,\partial_\tau[\bar{\psi}(\gamma^\sigma i\partial^\tau - \gamma^\tau i\partial^\sigma)\psi] + \ldots\,.\end{aligned} \tag{182}$$

[7] Maybe we can treat $\Gamma^{\lambda}_{[\mu\nu]}$ as a nonzero VEV of the torsion tensor of our universe.

3. Quantization and the effective potential for Version I.2: gravity?

Now we can write an effective potential mediated by U^μ. The amplitude of the process in Fig. 4-(a) could be written out(with the underlined terms are generated by the underlined term in (178)), as[8]

$$
\begin{aligned}
i\mathcal{M}_a &= \bar{u}^{s'} i\{\alpha_1 \frac{\Lambda}{M} iik_\rho + \beta_1 \frac{1}{M}(iq_\sigma g_{\kappa\rho} + iq_\kappa g_{\sigma\rho})[\gamma^\kappa iik^\sigma - \gamma^\sigma iik^\kappa]\}u^s \cdot \frac{+ig^{\rho\rho'}}{q^4} \\
&\cdot \bar{u}^{r'} i\{\alpha_2 \frac{\Lambda}{M} iip_{\rho'} - \beta_2 \frac{1}{M}(iq_{\sigma'} g_{\kappa'\rho'} + iq_{\kappa'} g_{\sigma'\rho'})[\gamma^{\kappa'} iip^{\sigma'} - \gamma^{\sigma'} iip^{\kappa'}]\}u^r \\
&= -\bar{u}^{s'}\{-\alpha_1 \frac{\Lambda}{M} k_\rho - i\beta_1 \frac{1}{M} 2[q\cdot k\gamma_\rho - q_\sigma \gamma^\sigma k_\rho]\}u^s \cdot \frac{+ig^{\rho\rho'}}{q^4} \\
&\cdot \bar{u}^{r'}\{-\alpha_2 \frac{\Lambda}{M} p_{\rho'} + i\beta_2 \frac{1}{M} 2[q\cdot p\gamma_{\rho'} - q_{\sigma'}\gamma^{\sigma'} p_{\rho'}]\}u^r \\
&= \frac{-i}{q^4}\cdot\{\frac{\Lambda^2\alpha_1\alpha_2 k\cdot p}{M^2} \\
&+ i\frac{2\Lambda\alpha_1\beta_2}{M^2}\{-[q\cdot p\gamma_{\rho'} k^\rho - q_{\sigma'}\gamma^{\sigma'} p\cdot k] + [q\cdot k\gamma_\rho p^{\rho'} - q_\sigma\gamma^\sigma p\cdot k]\} \\
&+ \frac{4\beta_1\beta_2}{M^2}[+q\cdot k q\cdot p - q\cdot k \not{p}\not{q} - q\cdot p\not{q}\not{k} + q^2 k\cdot p]\} \\
&\cdot 2m\delta^{ss'} 2m\delta^{rr'} \\
&= \frac{-i}{q^4}\cdot\{\frac{\Lambda^2\alpha_1\alpha_2 k\cdot p}{M^2} + i\frac{2\Lambda\alpha_1\beta_2 p\cdot k}{M^2}(+q_{\sigma'}\gamma^{\sigma'} - q_\sigma\gamma^\sigma) \\
&+ \frac{4\beta_1\beta_2}{M^2}(k\cdot p)q^2\}\cdot 2m\delta^{ss'} 2m\delta^{rr'} \\
&= -i\cdot\{\frac{\Lambda^2\alpha_1\alpha_2 k\cdot p}{M^2}\frac{1}{q^4} + i\frac{2\Lambda\alpha_1\beta_2 k\cdot p}{M^2}(+q_{\sigma'}\gamma^{\sigma'} - q_\sigma\gamma^\sigma)\frac{1}{q^4} \\
&+ \frac{4\beta_1\beta_2}{M^2}(k\cdot p)\frac{1}{q^2}\}\cdot 2m\delta^{ss'} 2m\delta^{rr'} \\
&\simeq -i\cdot\{\frac{\Lambda^2\alpha_1\alpha_2 k^0 p^0}{M^2}\frac{1}{|\boldsymbol{q}|^4} + i\frac{\lambda 2\Lambda\alpha_1\beta_2 k^0 p^0}{M^2}\frac{1}{|\boldsymbol{q}|^3} \\
&- \frac{4\beta_1\beta_2 k^0 p^0}{M^2}\frac{1}{|\boldsymbol{q}|^2}\}\cdot 2m\delta^{ss'} 2m\delta^{rr'}
\end{aligned} \tag{183}
$$

with $\alpha_1\beta_2 = \alpha_2\beta_1$, and the N.R. approximation $q\cdot k \simeq q\cdot p \simeq 0$, and the definition for λ as in (67)

$$
+q_{\sigma'}\gamma^{\sigma'} - q_\sigma\gamma^\sigma \simeq (v_1 - v_2)\cdot q \equiv \lambda|\boldsymbol{q}| \,, -\infty < \lambda < +\infty \,, \tag{184}
$$

particularly, for NR case, $\lambda \simeq 0$.

Then, by comparing with the Born approximation to the scattering amplitude in non-relativistic quantum mechanics, see (69), by omitting the λ term according to the optical theorem, we can get the effective potential as

$$
V(r) = -\frac{\Lambda^2\alpha_1\alpha_2 k^0 p^0}{8\pi M^2} r - \frac{4\beta_1\beta_2 k^0 p^0}{4\pi M^2}\frac{1}{r}\,. \tag{185}
$$

In Ref. [11], a gravity was generated by the antisymmetric $F^{\mu\nu}$ of an ordinary vector field B^μ with a symmetric energy-momentum tensor, which seemed superficially a little non-

[8] As in (170), for simplicity, here we can only consider the contributions from U_1, and, for the contributions from U_2, the result just need a double.

uniform.

3.2.4 Version I.3: mixing Version I.1 and I.2 for dark matter effect

With the Lagrangian in (166) and (178),

$$
\begin{aligned}
\mathcal{L}_{I(\alpha\beta)} &= -\alpha\,\Lambda\bar{\psi}[(\not{U}+\not{U}^{\dagger})+i(\not{U}-\not{U}^{\dagger})]\psi \\
&\quad -\beta\,\bar{\psi}(\varepsilon^{\mu\nu}+\sigma^{\mu\nu})[F_{\mu\nu}^{(U+U^{\dagger})}+iF_{\mu\nu}^{(U-U^{\dagger})}]\psi \\
&\quad +(\text{higher-order operators})\,, \qquad (166)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{I(\xi\rho)} &= -\xi\frac{\Lambda}{M_{Planck}}[(U+U^{\dagger})_{\mu}+i(U-U^{\dagger})_{\mu}]\bar{\psi}i\partial^{\mu}\psi \\
&\quad -\rho\frac{1}{M_{Planck}}[\overline{F}_{\mu\nu}^{(U+U^{\dagger})}+i\overline{F}_{\mu\nu}^{(U-U^{\dagger})}]\bar{\psi}(\gamma^{\nu}i\partial^{\mu}-\gamma^{\mu}i\partial^{\nu})\psi \\
&\quad +(\text{higher-order operators})\,, \qquad (178)
\end{aligned}
$$

where we use the coupling constants as subscripts to denote them, respectively.

We would show that we can list the different kinds of possible potentials generated by the field U^{μ} in a table, see Table-(2). In the table, we take m as the mass of fermions, and the word "imaginary" means this term would be absent according to the optical theorem; for potentials including only $\{\alpha,\beta\}$, see (172), and, for potentials including only $\{\xi,\rho\}$, see (185). Now we will show the calculations about the mixing effects from both $\{\alpha,\beta\}$ and $\{\xi,\rho\}$.

	$\alpha\Lambda$	β	$\xi m\Lambda/M_{Planck}$	$\rho m/M_{Planck}$
$\alpha\Lambda$	$-\alpha_1\alpha_2\Lambda^2\cdot r$ (confine)	$\pm\alpha_1\beta_2\Lambda\cdot\log r$ (confine)	$+\frac{m}{M_{Plank}}\alpha_1\xi_2\Lambda^2\cdot r$ (confine)	$\times$ (imaginary)
β		$+\beta_1\beta_2\cdot\frac{1}{r}$ (E.M.)	$\pm\frac{m}{M_{Plank}}\beta_1\xi_2\Lambda\cdot\log r$ (dark matter)	$\times$ (imaginary)
$\xi m\Lambda/M_{Planck}$			$-\frac{m^2}{M_{Plank}^2}\xi_1\xi_2\Lambda^2\cdot r$ (dark energy)	$\times$ (imaginary)
$\rho m/M_{Planck}$				$-\frac{m^2}{M_{Plank}^2}\rho_1\rho_2\cdot\frac{1}{r}$ (gravity)

Table 2: Possible potentials generated by field U^{μ}.

From the amplitude

$$
\begin{aligned}
i\mathcal{M}_a &= \bar{u}^{s'}i\{\alpha_1\Lambda\gamma^{\mu}g_{\mu\rho}+\beta_1\sigma_{\alpha\mu}(iq^{\alpha}g^{\mu}_{\rho}-iq^{\mu}g^{\alpha}_{\rho}) \\
&\quad +\xi_1\frac{\Lambda}{M}iik_{\rho}+\rho_1\frac{1}{M}(iq_{\sigma}g_{\kappa\rho}+iq_{\kappa}g_{\sigma\rho})[\gamma^{\kappa}iik^{\sigma}-\gamma^{\sigma}iik^{\kappa}]\}u^{s}\cdot\frac{+ig^{\rho\rho'}}{q^4} \\
&\quad \cdot\bar{u}^{r'}i\{\alpha_2\Lambda\gamma^{\nu}g_{\nu\rho'}-\beta_2\sigma_{\beta\nu}(iq^{\beta}g^{\nu}_{\rho'}-iq^{\nu}g^{\beta}_{\rho'}) \\
&\quad +\xi_2\frac{\Lambda}{M}iip_{\rho'}-\rho_2\frac{1}{M}(iq_{\sigma'}g_{\kappa'\rho'}+iq_{\kappa'}g_{\sigma'\rho'})[\gamma^{\kappa'}iip^{\sigma'}-\gamma^{\sigma'}iip^{\kappa'}]\}u^{r}\,, \qquad (186)
\end{aligned}
$$

it's apparent to see the $\{\alpha\rho, \beta\rho\}$ terms would give an imaginary-valued effective potential. For the $\alpha\xi$ term, we have

$$\begin{aligned} i\mathcal{M}_a &= \bar{u}^{s'} i\{\alpha_1 \Lambda \gamma^\mu g_{\mu\rho} + \xi_1 \frac{\Lambda}{M} iik_\rho\} u^s \cdot \frac{+ig^{\rho\rho'}}{q^4} \cdot \bar{u}^{r'} i\{\alpha_2 \Lambda \gamma^\nu g_{\nu\rho'} + \xi_2 \frac{\Lambda}{M} iip_{\rho'}]\} u^r \\ &= -i\frac{g^{\rho\rho'}}{q^4}(\alpha_1 \Lambda \gamma^\mu g_{\mu\rho} \xi_2 \frac{\Lambda}{M} iip_{\rho'} + \xi_1 \frac{\Lambda}{M} iik_\rho \alpha_2 \Lambda \gamma^\nu g_{\nu\rho'}) \cdot 2m\delta^{ss'} 2m\delta^{rr'} \\ &= -i\frac{1}{q^4}\left[-\frac{\Lambda^2}{M}\alpha_1 \xi_2 (p^0 + k^0)\right] \cdot 2m\delta^{ss'} 2m\delta^{rr'}, \end{aligned} \tag{187}$$

and the effective potential

$$V(r) = +\frac{\Lambda^2 \alpha_1 \xi_2 (p^0 + k^0)}{8\pi M} r\,. \tag{188}$$

And, for the $\beta\xi$ term, we have

$$\begin{aligned} i\mathcal{M}_a &= \bar{u}^{s'} i\{\beta_1 \sigma_{\alpha\mu}(iq^\alpha g^\mu_\rho - iq^\mu g^\alpha_\rho) + \xi_1 \frac{\Lambda}{M} iik_\rho\} u^s \cdot \\ &\quad \frac{+ig^{\rho\rho'}}{q^4} \cdot \bar{u}^{r'} i\{-\beta_2 \sigma_{\beta\nu}(iq^\beta g^\nu_{\rho'} - iq^\nu g^\beta_{\rho'}) + \xi_2 \frac{\Lambda}{M} iip_{\rho'}\} u^r \\ &= -i\frac{g^{\rho\rho'}}{q^4}\{\beta_1 2q_\rho \xi_2 \frac{\Lambda}{M} iip_{\rho'} - \xi_1 \frac{\Lambda}{M} iik_\rho \beta_2 2q_{\rho'}\} \cdot 2m\delta^{ss'} 2m\delta^{rr'} \\ {\scriptstyle (\beta_1\xi_2 = \beta_2\xi_1)} &\simeq -i\left[\frac{2\Lambda\beta_1\xi_2}{M}(-p+k)\cdot q\frac{1}{q^4}\right] \cdot 2m\delta^{ss'} 2m\delta^{rr'} \\ &\simeq -i\left[\frac{2\Lambda\lambda\beta_1\xi_2}{M}\frac{|\boldsymbol{q}|}{|\boldsymbol{q}|^4}\right] \cdot 2m\delta^{ss'} 2m\delta^{rr'} \end{aligned} \tag{189}$$

with the definition

$$(-p+k)\cdot q \equiv \lambda|\boldsymbol{q}|\,, \tag{190}$$

and then the effective potential

$$V(r) = \frac{2\Lambda\lambda\beta_1\xi_2}{M_{Planck}}\left[-\frac{1}{2\pi^2}\log(\frac{r}{r_0}) + (1-\gamma_E))\right]. \tag{191}$$

This term is a logarithmic-type potential (with r_0 put by hand to balance the dimension), attractive only for $\lambda < 0$), which might be corresponding to the dark matter effects(the velocity dispersions for elliptical galaxies required an attractive force $\boldsymbol{F} = -\nabla V(r) = v^2/r \simeq C_0/r$ required a potential $V(r) \sim \log(r/r_0)$, see Ref. [9]). For the sign of $\lambda < 0$, we don't know much about it, but, we can know, if the clockwise rotated galaxies are corresponding to $\lambda < 0$, then anti-clockwise rotated galaxies must be corresponding to $\lambda > 0$, and vice versa. So, if the universe are rotated, the impulsive version for the potential in (191) might also serve as the dark energy effects. Besides, it's allowed to treat r_0 as an adjustable parameter, and, a large r_0 of order $\mathcal{O}(1/M_{Planck})$ would be needed to generate a strong enough attractive force with the comparable magnitude with Newton's gravity.

3.2.5 Unification II: comments on potentials generated by $F^{\mu\nu}$

If we combine the potential terms in (172,185,188,191), then, we can say, the field U^μ with field strength $F^{\mu\nu}$ could provide a wealth of interaction information, as shown in Table 2.

The list of potentials generated by U^μ with field strength $F^{\mu\nu}$

1. for the $\{\alpha\alpha, \alpha\beta, \alpha\xi\}$ terms, there is a linear potential of order $\mathcal{O}(\alpha^2\Lambda^2)$, a logarithmic potential of order $\mathcal{O}(\alpha\beta\Lambda)$, and a linear potential of order $\mathcal{O}(\alpha\xi m\frac{\Lambda^2}{M_{Planck}})$, which might be corresponding to the confinement for strong-coupled gauge theory for the attractive case; surely these terms would be depressed or enhanced by the energy scale Λ, and, their effects would only be apparent at long distance range with respect to a Coulomb potential.

2. for the $\{\beta\beta\}$ term, there is a Coulomb-type potential of order $\mathcal{O}(\beta^2)$ which might be corresponding to the ordinary Coulomb potential.

3. for the $\{\beta\xi\}$ term, there is a logarithmic potential (with a r_0 put by hand to balance the dimension) of order $\mathcal{O}(\beta\xi m\frac{\Lambda}{M_{Planck}})$ which might contribute to the dark matter effects in the attractive case (see Ref. [9], the velocity dispersions for elliptical galaxies required an attractive force $\boldsymbol{F} = -\nabla V(r) = v^2/r \simeq C_0/r$ required a potential $V(r) \sim \log(r/r_0)$) and the dark energy effects in the impulsive case; except for Λ, this term would be depressed or enhanced by the parameter λ and a size parameter r_0.

4. for the $\{\xi\xi\}$ term, there is a linear potential of order $\mathcal{O}(\xi^2 m^2\frac{\Lambda^2}{M^2_{Planck}})$ which might contribute to the dark energy effects. [6] [7]

5. for the $\{\rho\rho\}$ term, there is a Coulomb-type potential of order $\mathcal{O}(\rho^2 m^2\frac{1}{M^2_{Planck}})$ which might be corresponding to Newton's gravitation potential.

Some notes for the potential

6. the special relativity effects are automatically served by the spinor basis $u^s(p)$. Since the coupling β is dimensionless, the β term in Lagrangian (166) would be a U.V. renormalizable one in the sense of superficial degree of divergence (or, in the dimensional regularization framework).

7. apparently, with different settings for the parameters, different part in the total potential would be the dominant part.

8. the logarithmic potential is determined by both the charge and the velocity of each particle, so, is this a new way to combine the E.M. and gravity?

9. both the logarithmic and linear potential would not influence the transmit of the free photons since the E.M. field is a kind of source-free field, but the hyper-hyperfine structure of the optical spectrum of atoms would be influenced.

10. the logarithmic would give corrections to the impulsive/confined effects generated from the linear potential part, so that would lead to a nonlinear red-shift, which might give an approach to understand some cosmological experiment data, such as, an indications of a spatial variation of the electromagnetic fine structure constant [10].

Unification II

By comparing (172) and (185), not like the method in (72,74), where the unity between E.M. force and gravity are realized with only two energy scale: Λ and M, here, we can unify this two kinds of forces by defining a new different energy scale, which has been proposed in Table (2):

[“each ∂-symbol for the matter field ψ in (195) should be tied with an energy scale M”] ,

with

$$M = M_{Plank} \quad (\text{for } \partial\psi \text{ or } \partial\bar{\psi}), \tag{192}$$

as a supplement to (9) for the **postulation**. Of course, directly to see, the two couplings in (172) and (185) would spontaneously become equal at a large enough energy scale, as the case in (74).

3.3 Version II: symmetric field strength $\overline{F}_{\mu\nu}$ for SR Gravity

Although there is a potential form for Newton's gravity generated in (183), the current in the corresponding Lagrangian is not the traditional energy-momentum tensor, since the former one is an antisymmetric tensor while the latter one is a symmetric one. Now we'll check whether the symmetric $\overline{F}_{\mu\nu}$ corresponding a symmetric current could serve as a Special Relativity(SR) gravity or not.

1. The interaction Lagrangian

The symmetric tensor $\overline{F}_{\mu\nu}$ has 10 independent components, so, it might serve as the contribution from the field strength of an off-shell gravitons. Note, as for $F_{\mu\nu}$ in (164), the E.O.M for $\overline{F}_{\mu\nu}$ wouldn't be true since the U_μ was off-shell, that is,

$$\partial^2\partial_\mu\overline{F}^{\mu\nu} = \partial^2\partial_\nu\overline{F}^{\mu\nu} \neq 0\,. \tag{193}$$

However, if U_μ was on-shell, then, combining the traceless condition, $\overline{F}_{\mu\nu}$ would have only 10-4-1=5 independent component, which might now again serve as the on-shell state of a graviton. Let's check whether $\overline{F}_{\mu\nu}$ could serve for the graviton, in the sense of the effective interaction form.

With the Lorentz gauge fixed condition, see Appendix D, the $\overline{F}_{\mu\nu}$ part of Lagrangian for free particle U^μ in an irreducible complete-symmetric tensor representation could be written as

$$\begin{aligned}
\mathcal{L}_U &= +\frac{1}{2}\partial_\alpha\overline{F}^\dagger_{\beta\mu}\partial^\alpha\overline{F}^{\beta\mu} - m_U^4 U_\mu^\dagger U^\mu \\
&\quad -\Lambda_U \cdot \left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho\overline{F}^{(U+U^\dagger)}_{\beta\mu}\partial^\alpha\overline{F}^{\beta\mu}_{(U+U^\dagger)}\right] \\
&= +\frac{1}{2}\left[\partial_\alpha(\partial_\beta U_\mu^\dagger + \partial_\mu U_\beta^\dagger)\right]\left[\partial^\alpha(\partial^\beta U^\mu + \partial^\mu U^\beta)\right] - m_U^4 U_\mu^\dagger U^\mu \\
&\quad -\Lambda_U \cdot [...] \,,
\end{aligned} \tag{194}$$

and, the interaction term could be written as

$$\begin{aligned}\mathcal{L}_I &= -\alpha\frac{\Lambda}{M}[(U+U^\dagger)_\mu + i(U-U^\dagger)_\mu]\bar{\psi}i\partial^\mu\psi \\ &\quad -\beta\frac{1}{M}[\overline{F}^{(U+U^\dagger)}_{\mu\nu} + i\overline{F}^{(U-U^\dagger)}_{\mu\nu}]\bar{\psi}(\underline{-2g^{\mu\nu}m} + \gamma^\nu i\partial^\mu + \gamma^\mu i\partial^\nu)\psi \\ &\quad +(\text{higher-order operators}),\end{aligned} \tag{195}$$

where $\overline{F}^{(U)}_{\mu\nu} \equiv \overline{F}_{\mu\nu}(U)$ is the field strength.[9] Note, the term

$$\mathcal{L}_\psi = g_{\mu\nu}\bar{\psi}(\gamma^\mu i\partial^\nu - \frac{1}{4}g^{\mu\nu}m_\psi)\psi \xrightarrow[E.O.M]{on-shell} 0 \tag{196}$$

is just the full Lagrangian of a matter field, so we need set $m \neq \frac{1}{4}m_\psi$ in (195), and, we can just set

$$m = 0 \tag{197}$$

according to the following result in (203).

For future convenience, we write the interaction terms for U_1^μ obviously, as

$$\mathcal{L}_I = -\alpha\frac{\Lambda}{M}U_{1\mu}\bar{\psi}i\partial^\mu\psi - \beta\frac{1}{M}\overline{F}^{(U_1)}_{\mu\nu}\bar{\psi}(\underline{-2g^{\mu\nu}m} + \gamma^\nu i\partial^\mu + \gamma^\mu i\partial^\nu)\psi + \ldots . \tag{198}$$

2. The E.O.M for tensor theory of Version II

See (388) in Appendix D, we can get the dynamical equation for the field U^μ as

$$\partial^4 U_\sigma = m_U^4 U^\sigma + J^\sigma , \tag{199}$$

with the gauge fixed condition $\partial_\sigma U^\sigma = 0$, and

$$\begin{aligned}J^\sigma &= +\alpha\frac{\Lambda}{M}\bar{\psi}\bar{\psi}i\partial^\sigma\psi \\ &\quad + -2\beta\frac{1}{M}\partial_\tau[\bar{\psi}(\underline{-2g^{\tau\sigma}m} + \gamma^\sigma i\partial^\tau + \gamma^\tau i\partial^\sigma)\psi] + \ldots .\end{aligned} \tag{200}$$

3. Quantization and the effective potential for Version II

Now we can write an effective potential mediated by U^μ. The amplitude of the process in Fig. 4-(a) could be written out(with the underlined terms are generated by the underlined

[9]The symmetric energy-momentum tensor should be

$$T^{\mu\nu} = \bar{\psi}\gamma^\nu i\partial^\mu\psi - i\partial^\mu\bar{\psi}\gamma^\nu\psi$$

rather than here

$$\bar{\psi}(\gamma^\nu i\partial^\mu + \gamma^\mu i\partial^\nu)\psi ,$$

but the results would be the same for NR approximation case.

term in (195)), as[10]

$$
\begin{aligned}
i\mathcal{M}_a &= \bar{u}^{s'} i\{\alpha_1 \frac{\Lambda}{M} i i k_\rho + \beta_1 \frac{1}{M}(i q_\sigma g_{\kappa\rho} + i q_\kappa g_{\sigma\rho})[\underline{-2g^{\kappa\sigma}m} + \gamma^\kappa i i k^\sigma + \gamma^\sigma i i k^\kappa]\} u^s \cdot \frac{+ig^{\rho\rho'}}{q^4} \\
&\quad \cdot \bar{u}^{r'} i\{\alpha_2 \frac{\Lambda}{M} i i p_{\rho'} - \beta_2 \frac{1}{M}(i q_{\sigma'} g_{\kappa'\rho'} + i q_{\kappa'} g_{\sigma'\rho'})[\underline{-2g^{\kappa'\sigma'}m} + \gamma^{\kappa'} i i p^{\sigma'} + \gamma^{\sigma'} i i p^{\kappa'}]\} u^r \\
&= -\bar{u}^{s'}\{-\alpha_1\frac{\Lambda}{M}k_\rho - i\beta_1\frac{1}{M}2[\underline{2q_\rho m} + q\cdot k\gamma_\rho + q_\sigma\gamma^\sigma k_\rho]\}u^s\cdot\frac{+ig^{\rho\rho'}}{q^4} \\
&\quad \cdot\bar{u}^{r'}\{-\alpha_2\frac{\Lambda}{M}p_{\rho'} + i\beta_2\frac{1}{M}2[\underline{2q_{\rho'}m} + q\cdot p\gamma_{\rho'} + q_{\sigma'}\gamma^{\sigma'}p_{\rho'}]\}u^r \\
&= \frac{-i}{q^4}\cdot\{\frac{\Lambda^2\alpha_1\alpha_2 k\cdot p}{M^2} \\
&\quad +i\frac{2\Lambda\alpha_1\beta_2}{M^2}\{-[\underline{2q\cdot km} + q\cdot p\gamma_{\rho'}k^\rho + q_{\sigma'}\gamma^{\sigma'}p\cdot k] + [\underline{2q\cdot pm} + q\cdot k\gamma_\rho p^{\rho'} + q_\sigma\gamma^\sigma p\cdot k]\} \\
&\quad +\frac{4\beta_1\beta_2}{M^2}[\underline{4q^2m^2 + 4q\cdot pm\not{q} + 4q\cdot km\not{q}} + q\cdot kq\cdot p + q\cdot k\not{p}\not{q} + q\cdot p\not{q}\not{k} + q^2 k\cdot p]\} \\
&\quad \cdot 2m\delta^{ss'}2m\delta^{rr'} \\
&= \frac{-i}{q^4}\cdot\{\frac{\Lambda^2\alpha_1\alpha_2 k\cdot p}{M^2} + i\frac{2\Lambda\alpha_1\beta_2 p\cdot k}{M^2}(-q_{\sigma'}\gamma^{\sigma'} + q_\sigma\gamma^\sigma) \\
&\quad +\frac{4\beta_1\beta_2}{M^2}(\underline{4m^2} + k\cdot p)q^2\}\cdot 2m\delta^{ss'}2m\delta^{rr'} \\
&= -i\cdot\{\frac{\Lambda^2\alpha_1\alpha_2 k\cdot p}{M^2}\frac{1}{q^4} + i\frac{2\Lambda\alpha_1\beta_2 k\cdot p}{M^2}(-q_{\sigma'}\gamma^{\sigma'} + q_\sigma\gamma^\sigma)\frac{1}{q^4} \\
&\quad +\frac{4\beta_1\beta_2}{M^2}(\underline{4m^2} + k\cdot p)\frac{1}{q^2}\}\cdot 2m\delta^{ss'}2m\delta^{rr'} \\
&\simeq -i\cdot\{\frac{\Lambda^2\alpha_1\alpha_2 k^0p^0}{M^2}\frac{1}{|\boldsymbol{q}|^4} + i\frac{\lambda 2\Lambda\alpha_1\beta_2 k^0p^0}{M^2}\frac{1}{|\boldsymbol{q}|^3} \\
&\quad -\frac{4\beta_1\beta_2}{M^2}(\underline{4m^2} + k^0p^0)\frac{1}{|\boldsymbol{q}|^2}\}\cdot 2m\delta^{ss'}2m\delta^{rr'}
\end{aligned}
\tag{201}
$$

with $\alpha_1\beta_2 = \alpha_2\beta_1$, and the N.R. approximation $q\cdot k \simeq q\cdot p \simeq 0$, and the definition for λ as in (67)

$$
-q_{\sigma'}\gamma^{\sigma'} + q_\sigma\gamma^\sigma \simeq (v_1 - v_2)\cdot q \equiv \lambda|\boldsymbol{q}|\ , -\infty < \lambda < +\infty\,, \tag{202}
$$

particularly, for NR case, $\lambda \simeq 0$.

Then, by comparing with the Born approximation to the scattering amplitude in non-relativistic quantum mechanics, see (69), by omitting the λ term according to the optical theorem, we can get the effective potential as

$$
V(r) = -\frac{\Lambda^2\alpha_1\alpha_2 k^0p^0}{8\pi M^2}r - \frac{4\beta_1\beta_2(\underline{4m^2} + k^0p^0)}{4\pi M^2}\frac{1}{r}\,, \tag{203}
$$

where we have set $m = 0$, see (197).

As our expectation, for two particles with the same kind of charges, this potential is apparently constituted with a linear impulsive part, and a Coulomb-type attractive part, each of which might correspond to the dark energy effects [7] (since this effect should only

[10] As in (170), for simplicity, here we can only consider the contributions from U_1, and, for the contributions from U_2, the result just need a double.

display at the cosmological level, by treating the upper limit of the distance $L \sim \frac{1}{\Lambda}$ for the theory corresponding to the I.R. energy scale Λ as just the size of universe), and the ordinary Newton's gravity, respectively. The special relativity effects are automatically served by the momentum p and the spinor basis $u^s(p)$.

Now we can say, the introducing for phantom in Ref. [6] [7] and for U or U^μ field in this paper could give equivalent results for impulsive force rather than attractive force, where the former one contributed a imaginary unit i in the current(the "charge"), or a whole minus sign in the kinetic energy, with the propagator of graviton maintaining, while the latter one contributed a minus sign in the propagator of U or U^μ, as $\left(\frac{1}{p^2} \simeq \frac{1}{-|\boldsymbol{p}|^2}\right) \to \left(\frac{1}{p^4} \simeq \frac{1}{|\boldsymbol{p}|^4}\right)$ for a space-like momentum p. Indeed, phantom and $\{U, U^\mu\}$ are all non-traditional field, the former one has a negative kinetic energy term, while the latter one has a negative mass term(as a higgs field rather than the conventional quantum fluctuation fields).

If $m \neq 0$, it might partly serve as the $\sqrt{-g}$ term in Einstein's General Relativity(GR). We will introduce the $\sqrt{-g}$ term in next section.

3.3.1 Matching for the d.o.f: generation of a linear gravity theory

Can $\overline{F}(U) \sim \partial U$ serve for the graviton?

As discussed for $F(U)$ in Section (3.2.2), according to the square form of the kinetic energy term $(\partial\overline{F})^2$ and the Coulomb-type interaction arising from the term $\bar{\psi}\overline{F}(U)\psi$ in (195), we can say, if U^μ is a field with propagator $\sim 1/p^4$, then its field strength

$$\overline{F}(U) \sim \partial U \tag{204}$$

could be treated as a particle d.o.f with propagator $\sim 1/p^2$ serving for the gravity, that means, $\overline{F}(U)$ could be the graviton.

If our E.O.M with ∂^4 is true, maybe we can partly understand why Einstein gravity is secretly the square of Yang-Mills theory. [12] That is, gravity is generated by a field A^μ with a E.O.M $\partial^4 A^\mu = 0$, which is the square of ordinary gauge field with a E.O.M $\partial^2 A^\mu = 0$. There is another viewpoint for this "square", see Section 4.1.

3.3.2 Unification III: which is for gravity, $F_{\mu\nu}$ or $\overline{F}_{\mu\nu}$?

By comparing (185) and (203), we can see that both $F_{\mu\nu}$ and $\overline{F}_{\mu\nu}$ could generate a gravitation, with a crucial reason that, a vector field U^μ, either with a field strength $F_{\mu\nu}$ or $\overline{F}_{\mu\nu}$, is corresponding to the same propagator at the tree level with the same gauge fixed condition. Now we would ask: "which one would be the real origin of gravitation, $F_{\mu\nu}$ or $\overline{F}_{\mu\nu}$?"

If we treat $F_{\mu\nu}$ being the only origin of both E.M. and gravitation, then the matter current would be an anti-symmetric one, which is not consistent with the traditional energy-momentum tensor, unless it's allowed for a new type of current to generate gravity. If it's truly allowed, then, within this framework, we can discuss the unification of E.M. and gravitation in the viewpoint of currents. As we know, these versions of gravity theory would be non-renormalizable because of the vertex including a momentum in, however, if we adopt the procedure in the Section 2.9.2, they would become a renormalizable one. Moreover, another interesting result would come with this renormalization, that is, a unification, which could be illustrated with the "excitation of d.o.f" again: with treating the two currents,

$$\bar{\psi}[\beta(\varepsilon^{\mu\nu}+\sigma^{\mu\nu})]\psi \quad \text{and} \quad \bar{\psi}[\frac{\beta}{M}(\gamma^{\mu}i\partial^{\nu}-\gamma^{\nu}i\partial^{\mu})]\psi$$

in (166), as two field d.o.f of a same complex field (see Section 2.9.2), we might say, the differentiation between the two coefficients, the E.M. constant β and the gravitation constant $\frac{\beta}{M}$, happened in the case of the latter one "field" frozen, which would serve for the gravity, while the unification between them happened in the case of the latter one "field" excited, with the former one "field" was always excited serving for a E.M. force.

On the other hand, if we take $F_{\mu\nu}$ and $\overline{F}_{\mu\nu}$ to serve for E.M. and gravitation respectively, then the field U^{μ} would not have a field strength defined in a definite irreducible representation, that is, the field U^{μ} would not be a pure Maxwell field. However, this property of U^{μ} would also take us a chance to unify E.M. and gravitation, that is, to combine them into a single field. Within this framework, U^{μ} has a mixed-type field strength, $aF_{\mu\nu}+b\overline{F}_{\mu\nu}$, which would give a different magnitude for the coupling coefficients when $F_{\mu\nu}$ and $\overline{F}_{\mu\nu}$ coupled to two currents with the same magnitude, say,

$$(aF_{\mu\nu})\cdot\beta(\bar{\psi}\sigma^{\mu\nu}\psi) \quad \text{and} \quad (b\overline{F}_{\mu\nu})\cdot\beta[\bar{\psi}(\gamma^{\mu}i\partial^{\nu}+\gamma^{\nu}i\partial^{\mu})\psi]\,.$$

Anyway, in both the two viewpoints above, we might say, with respect to the E.M. constant e, the smallness of the gravitation constant (related to the energy scale M_{Planck}), was from the smallness of coefficient b with respect to a in (149), that is, we can consider a relation for $M_{Planck}=\frac{a}{b}M_{EW}$.

And, under our consideration, for the **black holes**, there is nothing extraordinary for it by contrast with the ordinary matter, since the black hole is just a particular concept in the sense of "the first cosmic speed" of this object at the classic mechanics level, which would not be a well-defined concept at the quantum level since the "force" on a particle/field would be ill-defined. Of course, there might be some correspondence for the black holes concept at the quantum level:

a. which might be the non-perturbative/confined point of the coupling constant, that is, $\frac{4\beta_1\beta_2k^0p^0}{4\pi M^2}\simeq 1$, according to (203). Let's take an estimation for it: if there is a logarithm enhancement for the running/walking (asymptotic behavior) of the coupling constant, then the non-perturbative point need not be at $k^0\simeq M$ at all!

b. which might be the formation of a stable or meta-stable bound state, including massless particles and driven by only the gravity, in that case, one can always treat a "black hole" as a short-lived ("fast-evaporated") bound state!

3.4 U^{μ} out a nutshell: generation of a non-linear QED

As for the scalar U field shown in (82), with the boundary condition $[\partial^2U]_B\neq 0$ for the nontrivial P-4 type differential E.O.M $\partial^4U^{\mu}=...$ of U^{μ}, certainly we can't generate a P-2 type **linear** E.O.M $\partial^2U^{\mu}=...$ for U^{μ}.

For short, see (165), if $i\partial\ll\Lambda_U$ and $\langle F^{\lambda\rho}\rangle\gg F^{\lambda\rho}-\langle F^{\lambda\rho}\rangle$, then we can get an approximately linear E.O.M for a free "P-2" type field U^{μ}. For detail, that is, if the characteristic energy scale $\mu \sim p_U$ for physical processes is far less than the energy scale Λ_U, then (165) would be reduced to

$$\mathcal{L}_U \to -\Lambda_U \left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\beta\mu}\partial^\alpha F^{\beta\mu}_{(U+U^\dagger)} + (\text{cyclic for indices})\right] - m_U^4 U^\dagger_\mu U^\mu\,, \quad (205)$$

and now, for an intuitively view, if $F^{\lambda\rho}$ has a large nonzero vacuum expectation value $\langle F^{\lambda\rho}\rangle = F\epsilon^{\lambda\rho} \gg F^{\lambda\rho} - \langle F^{\lambda\rho}\rangle$, then the main part of this kinetic energy term would include only two ∂ symbol, with the form $\sim U\partial\partial U$, so we can get an approximately linear E.O.M for a free "P-2" type $U(1)$ gauge field U^μ, see Appendix D.

Firstly we get the nonlinear E.O.M for $m_U = 0$,

$$-\Lambda_U X_{(\mu\sigma)(\alpha\beta)} \cdot \partial^\alpha F^{\beta\mu}_{(U+U^\dagger)} = 0\,, \quad (206)$$

through the Euler-Lagrangian equation, where the tensor

$$\begin{aligned} X_{(\mu\sigma)(\alpha\beta)} &\equiv \left[-\epsilon_{\mu\beta}F^{(U+U^\dagger)}_{\sigma\alpha} - \epsilon_{\sigma\alpha}F^{(U+U^\dagger)}_{\mu\beta}\right] + \epsilon_{\rho\lambda}F^{\lambda\rho}_{(U+U^\dagger)}g_{\alpha\beta}g_{\mu\sigma} \\ &\neq \epsilon_{\rho\lambda}F^{\lambda\rho}\left[-\delta_{\rho\mu}\delta_{\lambda\beta}\delta_{\lambda\sigma}\delta_{\rho\alpha} - \delta_{\rho\sigma}\delta_{\lambda\alpha}\delta_{\lambda\mu}\delta_{\rho\beta} + g_{\alpha\beta}g_{\mu\sigma}\right] \end{aligned} \quad (207)$$

is symmetric for the indices $(\mu\sigma)$ and $(\alpha\beta)$ (denoted with a round bracket), respectively.

Now, we can return to the point expressed at the beginning of this section, that is, if $F^{\lambda\rho}$ has a large nonzero vacuum expectation value, $\langle F^{\lambda\rho}\rangle = F\epsilon^{\lambda\rho} + ... \gg F^{\lambda\rho} - \langle F^{\lambda\rho}\rangle$, then we can get the approximately linear E.O.M for a free "P-2" type $U(1)$ gauge field U^μ,

$$\partial_\alpha F^{\alpha\sigma} = 0\ ,\ \text{with}\ \langle F^{\lambda\rho}\rangle \gg F^{\lambda\rho} - \langle F^{\lambda\rho}\rangle\,, \quad (208)$$

otherwise, we can only get a nonlinear E.O.M (206) with self-interaction for even a $U(1)$ gauge field U^μ.

Note, if we roughly set

$$\partial^\alpha F^{\beta\mu}(U+U^\dagger) = 0\,,\ F^{\beta\mu}_{(U+U^\dagger)} \neq 0 \quad (209)$$

that might not give an nontrivial E.O.M, since the constraint of (209) is too strict, by contrast with the Bianchi identity, $\partial^\alpha F^{\beta\mu} + (\text{cyclic for indices}) = 0$.

4 Field $U^{\mu\nu}$

4.1 Lagrangian for $U^{\mu\nu}$: a form like GR

By taking the Minkovski space as background space, if we take the generalization of $U \to U^{\mu\nu} = U^{\nu\mu}$, with the transform property of $U^{\mu\nu}$ under the **global** group element V as

$$U^{\mu\nu} \to VU^{\mu\nu}V^\dagger\,,\ U^{\mu\nu} = U_1^{\mu\nu} - iU_2^{\mu\nu}\,, \quad (210)$$

where the indices $\mu\nu$ means the transform is for each component of $U^{\mu\nu}$; besides, the gauge fixed condition and E.O.M are

$$\hat{p}_\mu U^{\mu\nu} = \hat{p}_\nu U^{\mu\nu} = 0 \Rightarrow -\hat{p}^4 U^{\mu\nu} = -m_U^4 U^{\mu\nu}, \quad (211)$$

with the propagator in momentum space as [2]

$$D_F^{\mu\nu\rho\sigma}(U) = \frac{-i}{p^4+i\epsilon}\frac{\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}}{2}\,, \tag{212}$$

then the corresponding Lagrangian for the higgs-type tensor field $U^{\mu\nu}$ should be

$$\begin{aligned}\mathcal{L}_U &= -\frac{1}{3}\partial_\alpha F^\dagger_{\beta\mu\nu}\partial^\alpha F^{\beta\mu\nu} + m_U^4 U^\dagger_{\mu\nu}U^{\mu\nu} \\ &\quad -\Lambda_U^2\left[(U+U^\dagger)_{\alpha\nu}(U+U^\dagger)_{\beta\mu}\partial^\alpha F^{\beta\mu\nu}_{(U+U^\dagger)} + (...)_{i(U-U^\dagger)}\right]\,,\ U^\dagger_{\mu\nu}U^{\mu\nu}<1\,, \end{aligned} \tag{213}$$

where

$$F_{\beta\mu\nu} = +\partial_\beta U_{\mu\nu} + \partial_\nu U_{\beta\mu} + \partial_\mu U_{\beta\nu}, \tag{214}$$
$$F^\dagger_{\alpha\mu\nu} = +\partial_\alpha U^\dagger_{\mu\nu} + \partial_\nu U^\dagger_{\alpha\mu} + \partial_\mu U^\dagger_{\alpha\nu}, \tag{215}$$
$$F^\beta_{\mu\nu} = \eta^{\beta\rho}F_{\rho\mu\nu} = \eta^{\beta\rho}\left(+\partial_\beta U_{\mu\nu} + \partial_\nu U_{\beta\mu} + \partial_\mu U_{\beta\nu}\right)\,. \tag{216}$$

The term $U^\dagger_{\alpha\nu}U^\dagger_{\beta\mu}\partial^\alpha F^{\beta\mu\nu}$ in (213) is a special 3-particle interaction term, and we will return to consider this term in sections below, while for here we set $\Lambda_U = 0$.

Formally, by coincidence, if

$$U^{\mu\nu} = \widetilde{g}^{\mu\nu} = -g^{\mu\nu} \tag{217}$$

is the metric tensor of the space-time, we can find

$$F^\beta_{\mu\nu} \sim 2\Gamma^\beta_{\mu\nu} \text{ or } F_{\beta\mu\nu} \sim 2\Gamma_{\beta\mu\nu} \tag{218}$$

where Γ is just the Christoffel connection

$$\Gamma_{\beta\mu\nu} = \frac{1}{2}\left(\underline{-\partial_\beta\widetilde{g}_{\mu\nu}} + \partial_\nu\widetilde{g}_{\beta\mu} + \partial_\mu\widetilde{g}_{\beta\nu}\right)\,, \tag{219}$$
$$\Gamma^\beta_{\mu\nu} = \widetilde{g}^{\beta\rho}\Gamma_{\rho\mu\nu} = \frac{1}{2}\widetilde{g}^{\beta\rho}\left(\underline{-\partial_\beta\widetilde{g}_{\mu\nu}} + \partial_\nu\widetilde{g}_{\beta\mu} + \partial_\mu\widetilde{g}_{\beta\nu}\right)\,, \tag{220}$$

where the minus sign in the underlined term is from the different definition for the metric tensor, see (217). For the detail, we just need see the definition of the affine connection Γ: firstly, see the definition of a "translation"

$$A^\mu(P\to Q) = A^\mu(P) - \Gamma^\mu_{\nu\lambda}A^\nu(P)dx^\lambda\,, \tag{221}$$

and, secondly, with the unitary property $A^2(P\to Q) = A^2(P)$, we can get a D.E. (to the order of $(dx)^1$),

$$\partial_\mu\widetilde{g}_{\nu\lambda} - \widetilde{g}_{\alpha\lambda}\Gamma^\alpha_{\nu\mu} - \widetilde{g}_{\nu\alpha}\Gamma^\alpha_{\lambda\mu} = 0 \tag{222}$$

so, the different definition for the metric tensor would influence the contraction of indices and then give an extra minus sign.

However, there is a new result, that is, if we still take the form $\Gamma \sim U\partial U$ as the gauge field to construct a gravity theory, then, from the Lagriagian (213), the EOM of gravity wave would be

$$\partial^2\Gamma^\beta_{\mu\nu} = 0 \Rightarrow \partial^4\widetilde{g}_{\mu\nu} = 0, \tag{223}$$

rather than the Einstein gravity wave equation in weak-field approximation as

$$\partial^2\widetilde{g}_{\mu\nu} = 0, \tag{224}$$

which is deduced from the Einstein equation or the corresponding action:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\widetilde{g}_{\mu\nu} = 0 \Leftrightarrow S = \int d^4x\sqrt{-\widetilde{g}}R\,, \tag{225}$$

where

$$\begin{aligned} R &= \widetilde{g}^{\mu\nu}R_{\mu\nu}, && (226)\\ R_{\mu\nu} &= R^{\lambda}_{\mu\nu\lambda}, && (227)\\ R^{\rho}_{\lambda\mu\nu} &= -\partial_\nu\Gamma^{\rho}_{\lambda\mu} + \partial_\mu\Gamma^{\rho}_{\lambda\nu} - \Gamma^{\sigma}_{\lambda\mu}\Gamma^{\rho}_{\sigma\nu} + \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\rho}_{\sigma\mu} && (228)\end{aligned}$$

are the scalar curvature, Ricci curvature tensor and Riemann curvature tensor, respectively. If our E.O.M with ∂^4 is true, maybe we can partly understand why Einstein gravity is secretly the square of Yang-Mills theory. [12] That is, if we treat the E.O.M $\partial^4U^{\mu\nu} = 0$ for the classic gravity field, it is just the square of a E.O.M $\partial^2A^\mu = 0$ for ordinary gauge fields, see (23,24), differing from the case in Section 3.3.1.

For the self-interaction of $U^{\mu\nu}$, like the definition of the field strength for a non-Abelian gauge field A^i_α as

$$F^i_{\alpha\beta}t^i = \partial_\alpha A^i_\beta t^i - \partial_\beta A^i_\alpha t^i - ig[A^i_\alpha t^i, A^j_\beta t^j]\,, \tag{229}$$

and the "field strength $F_{\alpha\beta\mu}$ of field strength $F_{\beta\mu}$ of U^μ" in (159), we would give a definition of the field strength for $F_{\alpha\beta\mu\nu}$ of field strength $F_{\beta\mu\nu}$ of $U^{\mu\nu}$ as (anti-symmetric for $\alpha\nu$)

$$F_{\alpha\beta\mu\nu} = (-\partial_\alpha F_{\beta\mu\nu} + \partial_\nu F_{\beta\mu\alpha}) - \eta^{\sigma\sigma'}\left(F_{\sigma\mu\nu}F_{\beta\sigma'\alpha} - F_{\sigma\mu\alpha}F_{\beta\sigma'\nu}\right)\,. \tag{230}$$

with the correspondence to (226,227,228) for the indices as

$$\begin{aligned} F_{\alpha\beta\mu\nu} \equiv R_{U\,\alpha\beta\mu\nu} &\sim R_{\beta\mu\nu\alpha}\,, && (231)\\ U^{\alpha\beta}F_{\alpha\beta\mu\nu} = R_{U\,\mu\nu} &\sim R_{\mu\nu}\,, && (232)\\ U^{\mu\nu}R_{U\,\mu\nu} = R_U &\sim R\,. && (233)\end{aligned}$$

We could also designate

$$F_{\sigma\mu[\underline{\nu}}F_{\beta\sigma'\underline{\alpha}]} \equiv F_{\sigma\mu\nu}F_{\beta\sigma'\alpha} - F_{\sigma\mu\alpha}F_{\beta\sigma'\nu} \tag{234}$$

for some simplicity. Then the Lagrangian could be written as

$$\begin{aligned}\mathcal{L}_U &= -\frac{1}{3}F^\dagger_{\alpha\beta\mu\nu}F^{\alpha\beta\mu\nu} + m_U^4U^\dagger_{\mu\nu}U^{\mu\nu}\\ &\quad -\Lambda_U^2\left[(U+U^\dagger)_{\alpha\nu}(U+U^\dagger)_{\beta\mu}i\partial^\alpha iF^{\beta\mu\nu}_{(U+U^\dagger)} + (\ldots)_{i(U-U^\dagger)}\right]\,. \end{aligned} \tag{235}$$

However, here we would ignore this kind of construction for (230) and the Lagrangian form in (235), and that would not influence the results we concerned in this paper (that is, the kinetic energy term of of $U^{\mu\nu}$ and the interaction term of $U^{\mu\nu}$ and matters in the Lagrangian). For the detail, although the first term for the r.h.s of (230)

$$\begin{aligned} &-\partial_\alpha F_{\beta\mu\nu} + \partial_\nu F_{\beta\mu\alpha}\\ =\ &-\partial_\alpha\left(-\partial_\beta U_{\mu\nu} + \partial_\nu U_{\beta\mu} + \partial_\mu U_{\beta\nu}\right) + \partial_\nu\left(-\partial_\beta U_{\mu\alpha} + \partial_\alpha U_{\beta\mu} + \partial_\mu U_{\beta\alpha}\right)\\ =\ &\partial_\alpha\partial_\beta U_{\mu\nu} + \partial_\nu\partial_\mu U_{\beta\alpha} - \partial_\alpha\partial_\mu U_{\beta\nu} - \partial_\nu\partial_\beta U_{\mu\alpha} \end{aligned} \tag{236}$$

would not be a trivial construction, however, if we choose the gauge fixed condition $\partial_\mu U^{\mu\nu} = \partial_\nu U^{\mu\nu} = 0$ for deducing the E.O.M, then the construction for (236) is really trivial.

4.2 Version I: $U^{\mu\nu}$ coupled to intrinsic charges

1. The interaction Lagrangian

Like the case of U in (72), we can write the dynamical EOM of $U^{\mu\nu}$ in the form as

$$\partial^4 U_{\mu\nu} = J_{\mu\nu} \tag{237}$$

where $J_{\mu\nu}$ is not the energy-momentum tensor $T_{\mu\nu}$ in Einstein equation, and the corresponding interaction term

$$\begin{aligned}\mathcal{L}_I &= -\alpha Q\Lambda\,[(U+U^\dagger)+i(U-U^\dagger)]_{\mu\nu}\bar{\psi}\eta^{\mu\nu}\psi \\ &\quad -\beta Q\bar{\psi}[F^{(U+U^\dagger)}_{\alpha\mu\nu}+iF^{(U-U^\dagger)}_{\alpha\mu\nu}](\gamma^\alpha\eta^{\mu\nu}+\gamma^\mu\eta^{\alpha\nu}+\gamma^\nu\eta^{\mu\alpha})\psi \\ &\quad +(\text{higher-order operators})\,.\end{aligned} \tag{238}$$

The second term in (238) means that $U^{\mu\nu}$ doesn't affect the kinetic energy of matter field ψ through its VEV $\langle U^{\mu\nu}\rangle \sim \eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)$. For the definition of field strength $F^{(U+U^\dagger)}_{\alpha\mu\nu} = F_{\alpha\mu\nu}(U+U^\dagger)$, see (216).

We don't consider terms as

$$\mathcal{L}_I = -\alpha\Lambda\, U_1^{\mu\nu}\bar{\psi}[\sigma_{\mu\alpha}\sigma_{\nu\beta}(\gamma^\alpha i\partial^\beta+\eta^{\alpha\beta})]\psi \tag{239}$$

in this work.

2. The E.O.M for $U^{\mu\nu}$ of Version I

With the Lagrangian (213,238), we can get the dynamical equation of $U^{\mu\nu}$ from the Euler-Lagrangian equation (37), see (416) in Appendix D, as

$$-\partial^4 U_{\sigma\rho} = -m_U^4 U^{\sigma\rho} + J^{\sigma\rho}\,, \tag{240}$$

with the gauge fixed condition $\partial_\mu U^{\mu\nu} = \partial_\nu U^{\mu\nu} = 0$, and

$$\begin{aligned}J^{\sigma\rho} &= +\alpha Q\Lambda\,\bar{\psi}\eta^{\sigma\rho}\psi \\ &\quad -3\beta Q\partial^\tau\bar{\psi}(\gamma_\tau\eta_{\sigma\rho}+\gamma_\sigma\eta_{\tau\rho}+\gamma_\rho\eta_{\sigma\tau})\psi\,.\end{aligned} \tag{241}$$

3. Quantization and the effective potential for Version I

Now we can also write an effective potential mediated by $U^{\mu\nu}$. The amplitude of the process in Fig. 4-(a) could be written as[11]

$$\begin{aligned}i\mathcal{M}_a &= \bar{u}^{s'} i\Bigg\{\alpha_1\Lambda\frac{\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}}{2}\eta^{\mu\nu} \\ &\quad +(\gamma^\beta\eta^{\mu\nu}+\gamma^\mu\eta^{\beta\nu}+\gamma^\nu\eta^{\mu\beta}) \\ &\quad \cdot[iq_\beta\frac{\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}}{2}+(\beta\mu\nu\to\nu\beta\mu)+(\beta\mu\nu\to\mu\beta\nu)]\Bigg\}u^s \\ &\quad \cdot\frac{-i}{q^4+i\epsilon}\frac{\eta^{\rho\rho'}\eta^{\sigma\sigma'}+\eta^{\sigma\rho'}\eta^{\rho\sigma'}}{2}\end{aligned}$$

[11] As in (65,170,201), for simplicity, here we can only consider the contributions from U_1, and, for the contributions from U_2, the result just need a double.

$$
\begin{aligned}
& \cdot\bar{u}^{r'} i \Big\{ \alpha_2 \, \Lambda \frac{\eta^{\mu'\rho'}\eta^{\nu'\sigma'} + \eta^{\mu'\sigma'}\eta^{\nu'\rho'}}{2} \eta^{\mu'\nu'} \\
& -(\gamma^{\beta'}\eta^{\mu'\nu'} + \gamma^{\mu'}\eta^{\beta'\nu'} + \gamma^{\nu'}\eta^{\mu'\beta'}) \\
& \cdot[iq_{\beta'} \frac{\eta^{\mu'\rho'}\eta^{\nu'\sigma'} + \eta^{\mu'\sigma'}\eta^{\nu'\rho'}}{2} + (\beta'\mu'\nu' \to \nu'\beta'\mu') + (\beta'\mu'\nu' \to \mu'\beta'\nu')] \Big\} u^r \\
= & -i \cdot \Big\{ -\frac{4\alpha_1\alpha_2\Lambda^2}{q^4} + i\frac{18\alpha_1\beta_2\Lambda(v_2 - v_1)\cdot q}{q^4} - \frac{108\beta_1\beta_2}{q^2} \Big\} \cdot 2m\delta^{ss'} 2m\delta^{rr'} \\
= & -i \cdot \Big\{ -\frac{4\alpha_1\alpha_2\Lambda^2}{|\boldsymbol{q}|^4} + i\frac{18\lambda\alpha_1\beta_2\Lambda|\boldsymbol{q}|}{|\boldsymbol{q}|^4} + \frac{108\beta_1\beta_2}{|\boldsymbol{q}|^2} \Big\} \cdot 2m\delta^{ss'} 2m\delta^{rr'} \qquad (242)
\end{aligned}
$$

with $(v_2 - v_1)\cdot q \equiv \lambda|\boldsymbol{q}|$, $\alpha_1\beta_2 = \alpha_2\beta_1$, and, $-\infty < \lambda < +\infty$ was defined in (67), particularly, for NR case, $\lambda \simeq 0$.

Then, by comparing with the Born approximation to the scattering amplitude in non-relativistic quantum mechanics, see (69), according to the optical theorem, we can get the effective potential, as

$$
V(r) = +\frac{\Lambda^2 4\alpha_1\alpha_2}{4\pi} r + \frac{108\beta_1\beta_2}{4\pi}\frac{1}{r}\,, \qquad (243)
$$

Apparently, it's impulsive for two particles with the same charge. And, we can find the effect potential form of $U_{\mu\nu}$ in (243) is same as the one of U in (72).

4.3 Version II: $U^{\mu\nu}$ coupled to momentum

1. The interaction Lagrangian

In the case of weak field approximation, still by taking the Minkovski space as background space, the action is written as

$$
S = \int d^4x \, \mathcal{L}. \qquad (244)
$$

and, another version of the interaction term of $U^{\mu\nu}$ could be written as

$$
\begin{aligned}
\mathcal{L}_I = & -\alpha Q \, \Lambda[(U + U^\dagger) + i(U - U^\dagger)]_{\mu\nu}) \cdot \bar{\psi}\frac{1}{M}(\eta^{\mu\nu} m + \gamma^\mu i\partial^\nu + \gamma^\nu i\partial^\mu)\psi \\
& -\beta Q[F^{(U+U^\dagger)} + iF^{(U-U^\dagger)}]_{\alpha\mu\nu} \cdot \bar{\psi}[\frac{1}{M}(\eta^{\mu\nu} i\partial^\alpha + \eta^{\alpha\nu} i\partial^\mu + \eta^{\alpha\mu} i\partial^\nu) \\
& +\frac{1}{M^2}(\gamma^\alpha i\partial^\mu i\partial^\nu + \gamma^\mu i\partial^\alpha i\partial^\nu + \gamma^\nu i\partial^\alpha i\partial^\mu) + \frac{1}{M^3}(i\partial^\alpha i\partial^\mu i\partial^\nu)]\psi \\
& -\alpha Q \frac{\Lambda}{M}\,[(U + U^\dagger) + i(U - U^\dagger)]_{\mu\nu}) V^{\mu\nu}(\psi) \\
& -\beta Q \frac{1}{M}\,[F^{(U+U^\dagger)} + iF^{(U-U^\dagger)}]_{\alpha\mu\nu} V^{\alpha\mu\nu}(\psi) \\
& +(\text{higher-order operators}), \qquad (245)
\end{aligned}
$$

with the definition of field strength $F_{\alpha\mu\nu}^{(U+U^\dagger)} = F_{\alpha\mu\nu}(U + U^\dagger)$ in (216). As in (197), we set $m = 0$ in (245). And, for a simple calculation, we only consider the terms below,

$$
\begin{aligned}
\mathcal{L}_I = & -\alpha Q \, \Lambda[(U + U^\dagger) + i(U - U^\dagger)]_{\mu\nu}) \cdot \bar{\psi}\frac{1}{M}(\gamma^\mu i\partial^\nu + \gamma^\nu i\partial^\mu)\psi \\
& -\beta Q[F^{(U+U^\dagger)} + iF^{(U-U^\dagger)}]_{\alpha\mu\nu} \cdot \bar{\psi}[\frac{1}{M}(\eta^{\mu\nu} i\partial^\alpha + \eta^{\alpha\nu} i\partial^\mu + \eta^{\alpha\mu} i\partial^\nu)]\psi\,. \qquad (246)
\end{aligned}
$$

The $V^{(\dots)}(\psi)$ tensors are constructed from the interactive term $V(\psi)$, such as:

a. the scalar-scalar type current-current-coupled term $V(\psi) = g\bar{\psi}\psi\phi$ (including the mass term $V(\psi) = m_\psi\bar{\psi}\psi$ as particular case),

b. the vector-vector type current-source-coupled term $V(\psi) = e\bar{\psi}\gamma_\mu\psi A^\mu$,

c. the vector-vector type current-current-coupled term $V(\psi) = G\bar{\psi}\gamma_\mu\psi \cdot \bar{\psi}\gamma^\mu\psi$,

with the corresponding construction for

a. $V^{\mu\nu}(\psi) = g\bar{\psi}v^\mu v^\nu\psi\phi$,

b. $V^{\mu\nu}(\psi) = e\bar{\psi}\gamma^\mu\psi A^\nu$,

c. $V^{\mu\nu}(\psi) = G\bar{\psi}\gamma^\mu\psi \cdot \bar{\psi}\gamma^\nu\psi$,

respectively. Similarly, we construct

a. $V^{\alpha\mu\nu}(\psi) = g\bar{\psi}v^\alpha v^\mu v^\nu\psi\phi$,

b. $V^{\alpha\mu\nu}(\psi) = e\bar{\psi}\gamma^\alpha\gamma^\mu\psi A^\nu$,

c. $V^{\alpha\mu\nu}(\psi) = G\bar{\psi}\gamma^\alpha\gamma^\mu\psi \cdot \bar{\psi}\gamma^\nu\psi$,

respectively, where $\{g, e, G\}$ are coupling parameters and v is the velocity of matter field ψ. Apparently, if $\{g, e, G\}$ are small, then contributions from these terms could be omitted. We might designate the operation of $V(\psi) \to V^{\mu\nu}(\psi)$ as "current-ization" (converting an energy density scalar to an energy current tensor).

We don't construct the $U_{\mu\nu}V^{\mu\nu}(\psi)$ terms as

$V(\psi) \cdot \mathrm{Tr}\left[U_{\mu\nu} + U^\dagger_{\mu\nu}\right]$

or

$V(\psi) \cdot \sqrt{-det(U_{\mu\nu} + U^\dagger_{\mu\nu})}$,

since the two forms might lose the information of $U_{\mu\nu}$ for the strong field cases.

2. The E.O.M for $U^{\mu\nu}$ of Version II

With the Lagrangian (213,246), we can get the dynamical equation from the Euler-Lagrangian equation (37), see (422) in Appendix D, as

$$-\partial^4 U_{\sigma\rho} = -m_U^4 U^{\sigma\rho} + J^{\sigma\rho}\,, \tag{247}$$

with the gauge fixed condition $\partial_\mu U^{\mu\nu} = \partial_\nu U^{\mu\nu} = 0$, and

$$\begin{aligned} J^{\sigma\rho} &= +\alpha Q\,\Lambda\bar{\psi}\frac{1}{M}(\gamma_\sigma i\partial_\rho + \gamma_\rho i\partial_\sigma)\psi \\ &\quad -3\beta Q\partial_\tau(\bar{\psi}[\frac{1}{M}(\eta_{\sigma\rho}i\partial_\tau + \eta_{\tau\rho}i\partial_\sigma + \eta_{\tau\sigma}i\partial_\rho)]\psi) + \dots\,. \end{aligned} \tag{248}$$

Actually, we can treat the Λ term as the cosmological constant term, see the corresponding linear impulsive potential in following (250).

3. Quantization and the effective potential for Version II

Now, omitting the contributions from $U_{\mu\nu}V^{\mu\nu}(\psi)$ terms in (245), with the Lagrangian (213,246), we can write the effective potential mediated by $U^{\mu\nu}$ corresponding to the amplitude of the process in Fig. 4-(a), as[12]

$$i\mathcal{M}_a = \bar{u}^{s'} i\left\{\alpha_1\,\Lambda\frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}}{2}\frac{1}{M}(\gamma^\mu iik^\nu + \gamma^\nu iik^\mu)\right.$$

[12] As in (242), for simplicity, here we can only consider the contributions from U_1, and, for the contributions from U_2, the result just need a double.

$$
\begin{aligned}
&+\frac{1}{M}(\eta^{\mu\nu}iik^{\beta}+\eta^{\beta\nu}iik^{\mu}+\eta^{\mu\beta}iik^{\nu})\\
&\cdot[iq_{\beta}\frac{\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}}{2}+(\beta\mu\nu\to\nu\beta\mu)+(\beta\mu\nu\to\mu\beta\nu)]\Big\}\, u^{s}\\
&\cdot\frac{-i}{q^4+i\epsilon}\frac{\eta^{\rho\rho'}\eta^{\sigma\sigma'}+\eta^{\sigma\rho'}\eta^{\rho\sigma'}}{2}\\
&\cdot\bar{u}^{r'}i\Big\{\alpha_2\,\Lambda\frac{\eta^{\mu'\rho'}\eta^{\nu'\sigma'}+\eta^{\mu'\sigma'}\eta^{\nu'\rho'}}{2}\frac{1}{M}(\gamma^{\mu'}iip^{\nu'}+\gamma^{\nu'}iip^{\mu'})\\
&-\frac{1}{M}(\eta^{\mu'\nu'}iip^{\beta'}+\eta^{\beta'\nu'}iip^{\mu'}+\eta^{\mu'\beta'}iip^{\nu'})\\
&\cdot[iq_{\beta'}\frac{\eta^{\mu'\rho'}\eta^{\nu'\sigma'}+\eta^{\mu'\sigma'}\eta^{\nu'\rho'}}{2}+(\beta'\mu'\nu'\to\nu'\beta'\mu')+(\beta'\mu'\nu'\to\mu'\beta'\nu')]\Big\}\, u^{r}\\
\simeq\;& -i\cdot\Big\{-\frac{\Lambda^2 4\alpha_1\alpha_2 k^0p^0}{M^2}\frac{1}{q^4}-i\frac{6\Lambda\alpha_1\beta_2k^0p^0(v_2-v_1)\cdot q}{M}\frac{1}{q^4}-\frac{18k^0p^0\beta_1\beta_2}{M^2}\frac{1}{q^2}\Big\}\\
&\cdot 2m\delta^{ss'}2m\delta^{rr'}\\
\simeq\;& -i\cdot\Big\{-\frac{\Lambda^2 4\alpha_1\alpha_2 k^0p^0}{M^2}\frac{1}{|\boldsymbol{q}|^4}-i\frac{6\Lambda\alpha_1\beta_2k^0p^0\lambda|\boldsymbol{q}|}{M}\frac{1}{|\boldsymbol{q}|^4}+\frac{18k^0p^0\beta_1\beta_2}{M^2}\frac{1}{|\boldsymbol{q}|^2}\Big\}\\
&\cdot 2m\delta^{ss'}2m\delta^{rr'} \qquad (249)
\end{aligned}
$$

with $(v_2-v_1)\cdot q\equiv\lambda|\boldsymbol{q}|$ and $\alpha_1\beta_2=\alpha_2\beta_1$.

Then, by comparing with the Born approximation to the scattering amplitude in non-relativistic quantum mechanics, see (69), we can get a similar form for the effective potential as the one in (242),

$$
V(r) \;=\; +\frac{\Lambda^2 4\alpha_1\alpha_2 k^0p^0}{M^2 4\pi}r+\frac{18\beta_1\beta_2k^0p^0}{M^2 4\pi}\frac{1}{r}\,, \qquad (250)
$$

where $\alpha_{1,2},\beta_{1,2}>0$, and, $-\infty<\lambda<+\infty$, particularly, for NR case, $\lambda\simeq 0$, as in (242).

Apparently, it's impulsive for two particles with the same charge. And, we can find the effect potential form of $U_{\mu\nu}$ is same as the one of U in (72).

4.4 Matching of d.o.f: generation of a linear 3rd-order tensor version QED

1. Detection and matching of d.o.f

If the metric tensor $g^{\mu\nu}$ corresponds the $U^{\mu\nu}$ field in this section, then the gauge fields would not be $g^{\mu\nu}$ but $F^{\alpha}_{\mu\nu}$ defined in (216) or $\Gamma^{\lambda}_{\mu\nu}$ defined in (220), and, $g^{\mu\nu}$ is just the BKG effect (rather than field strength) of $F^{\lambda}_{\mu\nu}$ or $\Gamma^{\lambda}_{\mu\nu}$.

For the detection of gravity wave, by a analogy of the relation between the gauge field A_μ and BKG field U, we can say that the Christoffel connection field $\Gamma^{\lambda}_{\mu\nu}$ is a good d.o.f (the real graviton) rather than $g_{\mu\nu}$ in the weak coupling case, however, since $g_{\mu\nu}$ with two tensor indices corresponding to a quadrupole moment effect has been quite difficult to detect, the $\Gamma^{\lambda}_{\mu\nu}$ with three tensor indices would be more difficult to detect!

2. The generation of a linear 3rd-order tensor version QED

In the weak field approximation, we can parameterize $U^{\mu\nu}$ as

$$U^{\mu\nu}(x, x+\epsilon) = 1 - ig\epsilon n_{\alpha\mu\nu} A^{\alpha\mu\nu} + \ldots\,, \tag{251}$$

where $A^{\alpha\mu\nu}$ is a spin-1 particle for the index α.

By comparing (243) and (250) with (72), we can find that the effect potential form of $U_{\mu\nu}$ is same as the one of U. So, by treating each component of $U_{\mu\nu}$ as a scalar U field, and taking the field strength $F^{\alpha}_{\mu\nu}$ defined in (216) to correspond to the gauge field, like (106), maybe we can use the new field

$$A^{\alpha}_{\mu\nu} \equiv \frac{(\epsilon^{[\rho\sigma\alpha]} - \epsilon^{[\sigma\rho\alpha]})}{2} A_{[\rho\sigma](\mu\nu)} = \phi(x) F^{\alpha}_{\mu\nu}\,,\ \phi(x)^{\dagger}\phi(x) \simeq 1\,, \tag{252}$$

to generate a linear QED with respect to the index α for each $\mu\nu$ component of $A^{\alpha}_{\mu\nu}$, where the square and round brackets are used for the anti-symmetric and symmetric indices, respectively. And, with the kinetic energy term form $(\partial F)^2$, the E.O.M of a free $A^{\alpha}_{\mu\nu}$ would be

$$\hat{p}^2 A^{\alpha}_{\mu\nu} = m^2 A^{\alpha}_{\mu\nu}\,. \tag{253}$$

4.5 $U^{\mu\nu}$ out a nutshell: the generation of Einstein's GR

4.5.1 A classic field $U^{\mu\nu}$

Assuming $U^{\mu\nu}$ is a real-valued field, we can define the decompositions

$$\begin{aligned} u_1^{\mu\nu} &\equiv g^{\mu\nu} \equiv \eta^{\mu\nu} + h^{\mu\nu}\,, && (254)\\ u_2^{\mu\nu} &\equiv \sqrt{-g}\eta^{\mu\nu} = \eta^{\mu\nu} + (\sqrt{-g}-1)\eta^{\mu\nu}\,, && (255)\\ U^{\mu\nu} &\equiv u_1^{\mu\nu} + u_2^{\mu\nu} = 2\eta^{\mu\nu} + h^{\mu\nu} + (\sqrt{-g}-1)\eta^{\mu\nu}\,, && (256) \end{aligned}$$

for a general case of (256), it's allowed to set the mixing angle as tuning parameter, as

$$U^{\mu\nu} \equiv \cos\theta \cdot u_1^{\mu\nu} + \sin\theta \cdot u_2^{\mu\nu}\,. \tag{257}$$

Here, for dealing with the complex term $\sqrt{-g} = \sqrt{-det(g^{\mu\nu})} = \sqrt{-det(u_1^{\mu\nu})}$, we introduced a new tensor field $u_2^{\mu\nu}$(actually a scalar one, nevertheless with the consistent propagator form and attractive forces with $u_1^{\mu\nu}$) as a "gauge singlet" (while the spin-2 u_2 in a adjoint representation) of the underlying gauge group corresponding to the interaction mediated by $U = u_1 + u_2$.

We don't introduce a spin-1 field for $U^{\mu\nu}$ in (256), since that would correspond to an anti-symmetric tensor, which could be corresponding to the vector U^{μ} case, as shown in (106) or (175).

Then, for the weak field approximation, we could concentrate only on the fluctuation part of $U^{\mu\nu}$, as

$$\begin{aligned} U^{\mu\nu} &= 2\eta^{\mu\nu} + V^{\mu\nu}\,, && (258)\\ V^{\mu\nu} &= h^{\mu\nu} + (\sqrt{-g}-1)\eta^{\mu\nu}\,. && (259) \end{aligned}$$

Based on a flat space background, with the coupling constant such as gravitational constant G absorbed in the kinetic energy term of $U^{\mu\nu}$ or $V^{\mu\nu}$, the Lagrangian term $\mathcal{L}_0$ without $V^{\mu\nu}$ and the interaction term $\mathcal{L}_V$ with coupling to $V^{\mu\nu}$ could be written as

$$\begin{aligned}
\mathcal{L}_0 &= \eta^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi - \bar{\psi}m\psi\,, &(260)\\
\mathcal{L}_V &= h^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi \\
&\quad +(\sqrt{-g}-1)\eta^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi - (\sqrt{-g}-1)\bar{\psi}m\psi\,, &(261)\\
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_V &= g^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi - \bar{\psi}m\psi \\
&\quad +(\sqrt{-g}-1)(g-h)^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi - (\sqrt{-g}-1)\bar{\psi}m\psi \\
&= \sqrt{-g}g^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi - \sqrt{-g}\bar{\psi}m\psi \\
&\quad -(\sqrt{-g}-1)h^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi \\
\xrightarrow{\sqrt{-g}\simeq 1} &\simeq \sqrt{-g}g^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi - \sqrt{-g}\bar{\psi}m\psi\,, &(262)
\end{aligned}$$

then the action would be

$$S = \int d^4x\sqrt{-g}\left(g^{\mu\nu}\bar{\psi}\gamma_\mu i\partial_\nu\psi - \bar{\psi}m\psi\right)\,, \tag{263}$$

like the term given in Ref. [13].

Besides, we list two more points about u_1 and u_2:

1. Are u_1 and u_2 independent? Yes! Since that u_1 could determine u_2, but not vice versa.

2. What is u_2, and how to detect u_2? Surely u_1 and u_2 are both massless particles, however, from the interaction Lagrangian term, their effects are mixed together and not able to isolated from each other.

Generally to say, there could be many different ways to construct the action (or kinetic energy term) of the field $U^{\mu\nu}$ or $V^{\mu\nu}$, which surely depends on one's interpretation on the interaction and the d.o.f, for instance, we might treat (262) as a nonlinear σ model type theory, then the transition to a theory based on a curved space background would be s-traightforward by taking $u_1^{\mu\nu} = g^{\mu\nu}$ as the metric tensor and hiding the effect of $u_2^{\mu\nu}$ in the space measure, just as the Einstein's version.

4.5.2 $U^{\mu\nu}$ out a nutshell: generation of a non-linear tensor field theory

As in (205,206) in Section 3.4, Einstein's GR would be a non-linear E.O.M for tensor field $U^{\mu\nu}$ out of a nutshell.

In our framework, the Einstein's version indeed holds just as a particular case, with some evidences listed below:

1. From (225), Einstein's action

$$S = \int d^4x\sqrt{-g}R(g^{\mu\nu}) = \int d^4x\,\frac{1}{4}\eta_{\mu\nu}u_2^{\mu\nu}R(u_1) \tag{264}$$

is a mixed term of u_1 and u_2, without the pure terms for u_1 and u_2.

2. From (225), even in the flat space, Einstein's action and equation

$$S = \int \left(d^4x\sqrt{-g}\right) R \quad \rightarrow \quad \int d^4xR = \int d^4x g_{\alpha\nu} g_{\beta\mu} R^{\alpha\beta\mu\nu}$$
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad \rightarrow \quad R_{\mu\nu} = 0 \sim (g\partial\partial g)_{\mu\nu} = 0 \ \ \text{(nonlinear D.E.)} \tag{265}$$

is a non-linear differential equation(D.E.) constituent with only multi-field coupled terms(for kinetics information mixed with interaction information) rather than an linear differential E.O.M of field $g_{\mu\nu}$ with the crucial term $\hat{p}^N g_{\mu\nu}$ with N a positive integer (for pure kinetics information)

$$\hat{p}^N g_{\mu\nu} = 0 \ \ \text{(linear D.E.)}\,, \tag{266}$$

which seems like the former one (265), Einstein's GR, is an uncompleted version for the E.O.M, but only a part picked from a complete dynamical E.O.M. For a more obvious view, just like (205, 208), we can take the E.O.M of a non-Abelian gauge field for example, as

$$\begin{aligned} \partial_\mu F^{i\mu\nu} \quad &+ \quad g\epsilon^{ijk} A^j_\mu F^{k\mu\nu} = J^\nu \\ = \partial_\mu(\partial^\mu A^{i\nu} - \partial^\nu A^{i\mu}) \quad &+ \quad g\epsilon^{ijk} A^j_\mu (\partial^\mu A^{k\nu} - \partial^\nu A^{k\mu})\,, \\ (\text{if } i\partial \sim k \ll 1) \rightarrow \qquad & \quad g\epsilon^{ijk} A^j_\mu (\partial^\mu A^{k\nu} - \partial^\nu A^{k\mu}) = J^\nu\,. \end{aligned} \tag{267}$$

Actually, the action for free graviton in flat space in (265), is just corresponding to the second term in the complete Lagrangian (213)! Well, the term $U_{\mu\nu} U_{\alpha\beta} R_U^{\alpha\beta\mu\nu}$ is truly a 3-particle interaction term, but, it's really a term with the least number of derivative for generating the E.O.M, $R_{U\mu\nu} = 0$, although which is a non-linear one as in (265).

3. Besides, if we define $U = u_1 + u_2$ as in (256) for the complete information of gravity, then the term

$$\mathcal{L} = u_2 \cdot R_{(u_1)} = \frac{1}{4} \eta_{\rho\lambda} u_2^{\rho\lambda} \cdot u_{1\mu\nu} u_{1\alpha\beta} R_{u_1}^{\alpha\beta\mu\nu} \tag{268}$$

is really the term with the least number of derivative for including complete information from both u_1 and u_2, otherwise, for instance, either $\eta_{\alpha\beta} R_{u_1}^{\alpha\beta\mu\nu}$ or $\eta_{\mu\nu} R_{u_1}^{\alpha\beta\mu\nu}$ would lose the complete tensor-information of u_1.

4.5.3 Is that true for $g_{\mu\nu} = U_{\mu\nu}$?

Note that, in (258),

$$U^{\mu\nu} = 2\eta^{\mu\nu} + V^{\mu\nu}\,, \tag{269}$$

strictly speaking, the $\eta^{\mu\nu}$ term separated from $U^{\mu\nu}$ would contribute to the complete interaction rather than be roughly omitted for the strong field cases especially, or contribute to mass corrections for particles for the weak-coupled cases.

I. Indistinguishability between metric $g_{\mu\nu}$ and field $U_{\mu\nu}$.

If we define the Minkovski metric tensor as

$$\eta_{\mu\nu} = \bar{\eta}_{\mu\nu} + \langle U_{\mu\nu} \rangle, \tag{270}$$

where $\bar{\eta}_{\mu\nu}$ could be treated as the absolute flat background space metric tensor independent of $\langle U_{\mu\nu} \rangle$, then the complete metric tensor for the space could be

$$g_{\mu\nu} = \bar{\eta}_{\mu\nu} + U_{\mu\nu}. \tag{271}$$

That means, the VEV of $U_{\mu\nu}$ could contribute to the mass of particles. Besides, if there is multi-vacuum structure for $U_{\mu\nu}$, then all the different VEV of $U_{\mu\nu}$ might contribute to the mass of particles.

However, if we don't configure a flat background space metric tensor $\bar{\eta}_{\mu\nu}$ independent of $\langle U_{\mu\nu}\rangle$, that is, $\bar{\eta}_{\mu\nu} = 0$, then, in the case of very strong gravity, since $\langle U_{\mu\nu}\rangle = 0$, all matters even the graviton, do not have the kinetic energy term $\bar{\eta}_{\mu\nu}\partial^\mu\phi^\dagger\partial^\nu\phi$ hence the E.O.M. At that time, we would ask what is the d.o.f, and how to define the particle? Maybe we should turn to Section 2.9, that is, now the current is the real field!

It's difficult to check whether $\bar{\eta}_{\mu\nu} = 0$ or not, since it is usually mixed with $U_{\mu\nu} \simeq \langle U_{\mu\nu}\rangle$ for most cases. However, the point of Einstein's **equivalence principle** is just to **define** $\bar{\eta}_{\mu\nu} = 0$ and $\eta_{\mu\nu} = \langle U_{\mu\nu}\rangle$, that means, the flat background space was defined as gravity field takeing the VEV. So, in the case of very strong gravity, when the decomposion of

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

is unavailable, with $h_{\mu\nu}$ the fluctuation effects corresponding to gravity, that might generate the picture that all matters do not have a kinetics term $\eta_{\mu\nu}\partial^\mu\phi^\dagger\partial^\nu\phi$, with a doubt on what matters should be. And, just in the case of very strong gravity, we might detect the breaking of Einstein's equivalence principle.

II. what is $g^{\mu\nu}$, metric tensor or gravity field? Either-or!

The energy-momentum tensor are part of generators ("charge") of the Lorentz group [14], whose corresponding global group parameters could be a constant 2nd-order tensor $\varepsilon_{\mu\nu}$, which could be parameterized to be proportional to the metric tensor of Minkovski space, $\varepsilon_{\mu\nu} = C\eta_{\mu\nu}$, rather than the coordinates. In this sense, the physical correspondence of $\varepsilon_{\mu\nu}$ could be treated as the "light speed", which corresponding different kind of space, and might not be unique. However, after localization, $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu}(x)$, it is not metric any more but just a field. So, there is no necessary reason for us to interpret the field $g_{\mu\nu}(x) = \eta_{\mu\nu} + \varepsilon_{\mu\nu}(x)$ as the metric tensor. In analogy with the case of electromagnetic field, that is, a global group parameter, labeling the electric quantum number of particles, becomes the electromagnetic field after localization.

So, what is $g^{\mu\nu}$, metric tensor or gravity field? Now, it's Either-or! If $g^{\mu\nu}$ is metric tensor, only being used to raise and lower the indices, then the theory is purely a geometric dynamics (dynamics in curved space), without the need for a gravity field, but just a need for constraint conditions to represent the curved character of the world, for instance, the Einstein equation is just a most fundamental constraint. If $g^{\mu\nu}$ is a field, then, it isn't innate to be all the metric tensor but just a field defined on the space background. Ultimately, the two kind of description should be equivalent. To be a field, $g_{\mu\nu}$ doesn't directly impact the curvature of space, but indirectly impact the space by coupling to the kinetic energy term of matter, according to the equivalence principle(kinetic energy for inertia, interaction for acceleration). Or, we can say, to take $g^{\mu\nu}$ for a covariant derivative and for a curved space background metric tensor, should be equivalent.

The mass terms with no tensor indices interact with gravity field by coupling to a scalar field, $\sqrt{-g}$, constructed with $g_{\mu\nu}$. Why is $\sqrt{-g}$ rather than other scalar variables? Because $\sqrt{-g}$ represent the core properties of $g_{\mu\nu}$, that is, on one hand, the core character of a constant tensor $\varepsilon_{\mu\nu}$ as the group parameter is the light speed, which could be corresponding to g, the determinant of $\varepsilon_{\mu\nu}$ for the most simpleness, on the other hand, the substantial deviation of $g_{\mu\nu}$ from $\varepsilon_{\mu\nu}$, after a local Lorentz transformation, is the diagonal elements, more especially, the g_{00} and g_{33} component. Like the generalization of $U \to U^\mu$, we can treat $\sqrt{-g} \to g_{\mu\nu}$ as a similar generalization.

Now that we have treated $g_{\mu\nu}$ just as a ordinary field, we should forget the concept of curved space, and take the integration measure just as d^4x rather than $d^4x\sqrt{-g}$, for avoiding double counting of the effects of field $g_{\mu\nu}$. And, all the indices should be raised and lowered by $\eta_{\mu\nu}$, as they are defined in the flat Minkovski background space.

III. An $\eta^{\mu\nu}$ for convenience.

Nevertheless, taking a flat space background independent from the gravity field is a good choice for convenience to construct theories, on the other hand, the evolution of the (local and topological) structure of space could be represented in a flat background space framework by modifying the parameters and constraints of a theory, including the dimension of space which might inevitably need to evolute as a dynamical operator in some cases, rather than treating the metric tensor $g^{\mu\nu}$ with a definite rank as all the mixed information for space and gravity, which might be troublesome and probably insufficient for some cases. In a word, the metric tensor $g^{\mu\nu}$ serves for all the information of the space, but the gravity field $U^{\mu\nu}$ isn't innate to be all the $g^{\mu\nu}$ but just a field defined on the space background.

5 Massive $\{U, U^\mu, U^{\mu\nu}\}$ for superconductor

5.1 Effective potential for Cooper pair

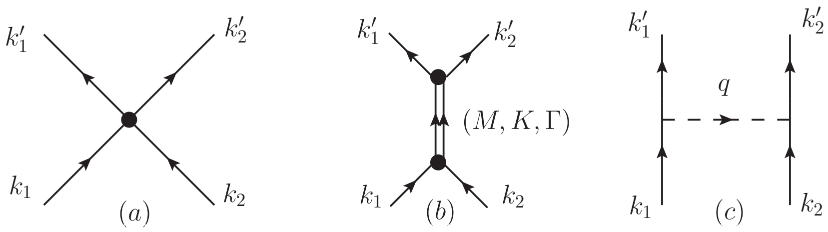

Figure 5: The Feynman diagrams for the evolution of a Cooper pair.

In Ref. [15], a constant valued evolution amplitude(or, a square-well type potential in momentum space) was introduced as

$$\begin{aligned} i\mathcal{M} \;&\sim\; -i\frac{e^2}{M^2}[\theta(|\boldsymbol{k}| - |\boldsymbol{k}_0|) - \theta(|\boldsymbol{k}| - |\boldsymbol{k}_m|)] \\ \text{with} \qquad & k = \frac{1}{2}(k_2 - k_1)\,,\; |\boldsymbol{k}_0| \le |\boldsymbol{k}| \le |\boldsymbol{k}_m| \end{aligned} \tag{272}$$

for the evolution of electron pair, that means, the pair maintains through an effective 4-fermion-contacted interaction or an mutual interaction mediated by the vacuum, see Fig. 5-(a). Now we write (272) in the q space, where $q = k_1 - k_1' = -(k_2 - k_2')$ is the momentum transfer between two electrons. With the relations

$$k = \frac{1}{2}(k_2 - k_1) = \frac{1}{2}[(k_2' - q) - (k_1' + q)] = k' - q\,,\; |\boldsymbol{k}_0| \le \{|\boldsymbol{k}|, |\boldsymbol{k}'|\} \le |\boldsymbol{k}_m| \tag{273}$$

$$\Rightarrow 0 \le |\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}| \le (|\boldsymbol{k}_m| - |\boldsymbol{k}_0|)\,, \tag{274}$$

we have

$$i\mathcal{M} \sim -i\frac{e^2}{M^2}[\theta(|\boldsymbol{q}|-|\boldsymbol{q}_1|)-\theta(|\boldsymbol{q}|-|\boldsymbol{q}_2|)],\ \text{with}\ |\boldsymbol{q}_1|<|\boldsymbol{q}_2|, \tag{275}$$

with $|\boldsymbol{q}_1|=0$ and $|\boldsymbol{q}_2|=|\boldsymbol{k}_m|-|\boldsymbol{k}_0|\equiv m$, M an unknown mass for balancing the dimension, and the corresponding effective potential in coordinate space as

$$\begin{aligned} V(r) &\sim \frac{e^2}{M^2}\frac{1}{r^3}\left[(|\boldsymbol{q}_2|r)\cdot\sin(|\boldsymbol{q}_2|r)-(|\boldsymbol{q}_1|r)\cdot\sin(|\boldsymbol{q}_1|r)+\cos(|\boldsymbol{q}_2|r)-\cos(|\boldsymbol{q}_1|r)\right] \\ &= \frac{e^2}{M^2}\frac{1}{r^3}\left[(mr)\cdot\sin(mr)+\cos(mr)-1\right]. \end{aligned} \tag{276}$$

It's allowed to model the evolution amplitude (275) as a "$2\to 2$" scattering amplitude, with the time-like s-channel and space-like t-channel as

$$\begin{aligned} i\mathcal{M}_{s-channel} &\sim (ie)^2\frac{i}{K^2-(M-i\frac{\Gamma}{2})^2} \sim (ie)^2(2\pi)\delta(K^2-(M-i\frac{\Gamma}{2})^2) \\ &\sim -ie^2\frac{2\pi}{\Gamma}\{-\theta[K^2+(M-i\frac{\Gamma}{2})^2]+\theta[K^2-(M-i\frac{\Gamma}{2})^2]\} \\ &\quad \text{with}\ K^2>0, \end{aligned} \tag{277}$$

$$i\mathcal{M}_{t-channel} \sim (ie)^2\frac{-ig^{00}}{q^2-m^2} \sim \frac{ie^2}{q^2-m^2} \sim -ie^2\frac{1}{|\mathrm{q}|^2+m^2},\ \text{with}\ q^2<0 \tag{278}$$

respectively. Then we get two points:

1. In the time-like channel (277), see Fig. 5-(b), M could be seemed as the mass and Γ the width for an effective quasi-state $|M\rangle$ which mediated the evolution of electron pair, or the M in (275), and K the time-like momentum approximately constrained in the vicinity of the shell $K^2=M^2$ to concentrate only the resonance enhancement part of the whole amplitude (or, only the effects from the pole of the propagator $\sim i/(K^2-M^2)$) rather than the $K^2 \ll M^2$ part as in Ref. [15].

2.On the other hand, if we treat the quasi-state $|M\rangle$ as a bound state constituted by the electron pair, then we can model (275) to be (278), which is corresponding to a scattering process mediated by a scalar particle, see Fig. 5-(c), with the propagator $\sim i/(q^2-m^2)$ rather than a symmetry-broken version QED with a massive photon propagator $\sim -ig^{\mu\nu}/(q^2-m^2)$, for the reason that the Coulomb type potential is impulsive for two electrons while the Yukawa type is attractive.

Now we will move on to our model in (7) and (166). As in (72,172,243), we can get the amplitudes for massive $\{U, U^\mu, U^{\mu\nu}\}$,

$$\begin{aligned} i\mathcal{M}_0 &\simeq -i\left(-\Lambda^2\alpha_1\alpha_2\frac{1}{|\boldsymbol{q}|^4-m^4+i\epsilon}-i\Lambda\lambda\alpha_1\beta_2\frac{|\boldsymbol{q}|}{|\boldsymbol{q}|^4-m^4+i\epsilon}+\beta_1\beta_2 g^{00}\frac{|\boldsymbol{q}|^2}{|\boldsymbol{q}|^4-m^4+i\epsilon}\right) \\ &\quad \cdot 2m\delta^{ss'}2m\delta^{rr'}, \end{aligned} \tag{279}$$

$$\begin{aligned} i\mathcal{M}_1 &\simeq -i\cdot\left(4\Lambda^2\alpha_1\alpha_2\frac{1}{|\boldsymbol{q}|^4-m^4+i\epsilon}-2\Lambda\lambda\alpha_1\beta_2\frac{|\boldsymbol{q}|}{|\boldsymbol{q}|^4-m^4+i\epsilon}+4\beta_1\beta_2\frac{|\boldsymbol{q}|^2}{|\boldsymbol{q}|^4-m^4+i\epsilon}\right) \\ &\quad \cdot 2m\delta^{ss'}2m\delta^{rr'}, \end{aligned} \tag{280}$$

$$\begin{aligned} i\mathcal{M}_2 &\simeq -i\cdot\left(-4\alpha_1\alpha_2\Lambda^2\frac{1}{|\boldsymbol{q}|^4-m^4+i\epsilon}-i\cdot 18\Lambda\lambda\alpha_1\beta_2\frac{|\boldsymbol{q}|}{|\boldsymbol{q}|^4-m^4+i\epsilon}+108\beta_1\beta_2\frac{|\boldsymbol{q}|^2}{|\boldsymbol{q}|^4-m^4+i\epsilon}\right) \\ &\quad \cdot 2m\delta^{ss'}2m\delta^{rr'}, \end{aligned} \tag{281}$$

and, **in the finite temperature case**,[13] with picking up the real parts of the integration results in Appendix C we can get the non-relativistic effective potentials (according to the optical theorem),

$$\begin{aligned} V_0(r) &= -\frac{\Lambda^2\alpha_1\alpha_2}{8\pi m^2 r}(\cos mr - e^{-mr}) - \frac{\lambda\Lambda\alpha_1\beta_2}{8\pi mr}[-(e^{-mr}+\sin mr)] \\ &\quad +\frac{\beta_1\beta_2}{8\pi r}(\cos mr + e^{-mr}), \qquad (282) \\ V_1(r) &= \frac{4\Lambda^2\alpha_1'\alpha_2'}{8\pi m^2 r}(\cos mr - e^{-mr}) - \frac{2\Lambda\lambda'\alpha_1'\beta_2'}{8\pi mr}\cos mr \\ &\quad +\frac{4\beta_1'\beta_2'}{8\pi r}(\cos mr + e^{-mr}), \qquad (283) \\ V_2(r) &= -\frac{4\Lambda^2\alpha_1''\alpha_2''}{8\pi m^2 r}(\cos mr - e^{-mr}) + \frac{18\Lambda\lambda''\alpha_1''\beta_2''}{8\pi mr}[-(e^{-mr}+\sin mr)] \\ &\quad +\frac{108\beta_1''\beta_2''}{8\pi r}(\cos mr + e^{-mr}), \qquad (284) \end{aligned}$$

where we can still treat $\Lambda \simeq 0$ so that the potentials would include only the $\beta_1\beta_2$ terms. Now, we write two new potentials: one for a sign opposite,

$$V(r) = -V_0(r)\,, \qquad (285)$$

and one for a combination,

$$V(r) = V_0(r) + V_1(r)\,, \qquad (286)$$

which would be used for a comparison in the figure below. The shape of potentials in (276), (282- 283) and (285-286) are roughly plotted in Fig.(6).

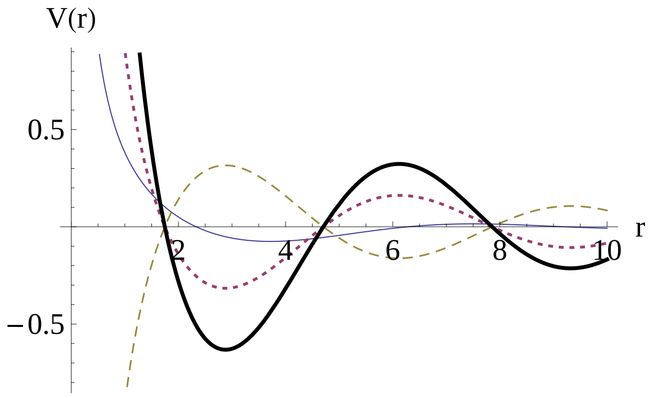

Figure 6: The shape of different potentials. With setting $\frac{e^2}{M^2} = \frac{\beta_1\beta_2}{8\pi} = \frac{4\beta_1'\beta_2'}{8\pi} = \frac{108\beta_1''\beta_2''}{8\pi} = 1$ and $m = 1$, the thin-solid, dotted, dashed and thick-solid lines are for (276), (282), (285) and (286), respectively.

Mathematically, the reason for why the massive photon A^μ in a symmetry broken version QED couldn't give attractive potential as U^μ, is the different forms so that the singularity structure of the propagator for A^μ and U^μ, for the former one a single pole while the latter one two poles.

[13]See Section 2.2.2, the discussions about poles in propagator.

5.2 What are $\{U, U^\mu, U^{\mu\nu}\}$?

What are the quasi-particles $\{U, U^\mu, U^{\mu\nu}\}$ with mass m?

1.According to Fig. 6, the shapes of (276) and (282) are truly similar, so, we can employ U to serve for the attractive forces in the s-wave pairing superconductors(low-temperature superconductors).

2. Where is the effects from photon/phonon? Both U and U^μ aren't photon, and, in our framework, "photon/phonon" is a false concept, while only the $\{U, U^\mu, U^{\mu\nu}\}$ fields are real. However, as in (101) we treat ∂U for massless U as the photon, here, from (282), based on the fact that it's attractive for even the same kind of charges, we can consider the fluctuation ∂U for massive U as the phonon, and U as a kind of field(or quasi-particle) with its corresponding "charge" for interaction is electric charge.

3. From (172,283), we can consider U^μ as a kind of field(or quasi-particle) with mass m, and and its corresponding "charge" for interaction is a kind of moment, see (166).

4. Surely it's allowed for the combination of the two effects from U and U^μ in the same system. From Fig.(6), we can see that both the field U and U_μ could mediate attractive interaction for two particles, the former one only for the charges while the latter one only for the magnetic moments. We define the combined potential (286) to serve for the attractive forces in the d-wave pairing superconductors(high-temperature superconductor).

5.The field U filled the coordinate space as a media for transferring interactions between electrons in the condensate matters, with impulsive forces generated by massless U when $m \sim k_m - k_0$ was small enough to be ignored, while attractive forces generated by massive U when $m \sim k_m - k_0$ couldn't to be ignored.

6. The core differences between U and U^μ are their masses and the couplings β. If the couplings are corresponding to electric charges, then what is it that determined the mass? As in Ref. [15], there is $m \sim k_m - k_0$, where k_0 corresponds to the Fermi surface, so that, we might get the relations below:

$$\begin{aligned} & \text{the stronger fluctuation of Fermi surface} \\ \Leftrightarrow \quad & \text{the larger } m \ (m \sim k_m - k_0) \\ \Leftrightarrow \quad & \text{the larger region for the allowed momentum } q (\text{since } q < m) \ . \end{aligned} \tag{287}$$

However, there should be

$$\begin{aligned} (287) \not\Leftrightarrow \quad & \text{the closer distance allowed for the two electrons} \\ \Leftrightarrow \quad & \text{the stronger interaction between the two electrons} \, , \end{aligned} \tag{288}$$

because of the non-perturbative properties for the Cooper-pair system, which would be discussed below.

5.3 Supplement to the matching for d.o.f

1. $A \sim \partial U \Rightarrow U \sim \int dx A$.

Now that $A \sim \partial U$ could be corresponding to the photon, see (101), we can again confirm that $U \sim \int dx A$ could be corresponding to a string, see (122).

2. $U \sim AA$?
That is to ask:
"Could $\{U, U^{\mu}, U^{\mu\nu}\}$ be treated as spin-0, spin-1 and spin-2 bound states of di-photon, respectively? Or, is there exist the effects for massive/massless photon condensation in a symmetry maintained/broken version for QED?"

We should note that, as discussed in Section 2.8.3,only in the sense of that U is a group element or a classic field, the parameterization, of taking $U\partial_\mu U^\dagger$ as the very one allowed gauge particle degree of freedom Γ_μ, $U\partial_\mu U^\dagger = \Gamma_\mu$, is valid. However, in the sense of U is a particle, the term $\bar{\psi}(U\partial\!\!\!/ U^\dagger)\psi$ in fact gives an effects mediated by two particles, U and $U^\dagger$. Or, in another viewpoint , the parameterization of $U\partial_\mu U^\dagger = \Gamma_\mu$ indicate the existence of a composite gauge particle Γ_μ with the flavor quantum number $\{U, U^\dagger\}$, although a massless bound states(with zero bound energy) is very bizarre.

3. $U \sim A\widetilde{A}$?
In the viewpoint of anti-particle, according to (26) and (27), U might be a composite of $A\widetilde{A}$, with $\tilde{A}$ a tachyon(possibly corresponding to the quantum entanglement effects) or a phantom.

5.4 An origin of non-perturbative property.

With the Schrodinger equation, Cooper had given the binding energy of the ground state for an clectron pair [15], as

$$E_0 \sim \frac{1}{e^{b/g^2} - 1}. \tag{289}$$

The exponential form rather than a polynomial form indicates the sensitivity of the dependence on the coupling g for the energy E_0. For the reason, we might concentrate on the multi-vacuum structure of the potential, see Fig. 6, that is, when g is small, the "position" and existence of the ground state would even be sensitive on the depths of the potential well(corresponding to the magnitude of g), needlessly to say the energy of the state or the perturbative calculation in the vicinity of $g = 0$. Contrarily, when g is large, the "position" of the ground state would be "frozen" in a definite well, so now the energy E_0 seems able to be computed perturbatively in the power expansion of $1/g$ in the vicinity of $g = \infty$.

5.5 Cooper pair → Cooper cluster?

Now that we have the effective potential, it's allowed to take further consideration on the multi-body problems. We won't give a complete calculation, but give some discussions.

As discussed in Section 2.6.2, the multi-body processes in Fig. 2-(d) could be renormalized by the restricted kinetics phase space in our framework, and, since there exist an attractive force between each two electrons, it should be allowed for the existence of Cooper's electron chain or cluster, just like the baryons with attractive three quarks for the non-Abelian gauge theory case.

Is it in a stronger or weaker binding for (tetra/multi)-electron clusters than for Cooper's electron pairs? Well, that need a calculation to give a certain answer.

6 Conclusion

An introduction for a new class of higgs type fields $\{U, U^{\mu}, U^{\mu\nu}\}$, with a fourth order (P-4 type) differential equation as its equation of motion, motivated by the linear potential in the lattice gauge theory, could provide a wealth of interaction forms, with some postulations on convergence being taken.

In the case of U coupled to the intrinsic charges of matter fields, electromagnetic (E.M.) Coulomb potential with an extra linear potential and Newton's gravitation could be generated with the operators of different orders from the dynamics of U, respectively; that the two kinds of forces appear in a single model with a relation on the coupling coefficients, might be seemed as a kind of unification; besides, the linear potential generated in the E.M. case, would correspond to the confinement/dark energy effects. Meanwhile, a nonlinear Klein-Gordon equation could be generated as a low energy approximation of the dynamics of U. Moreover, in the weak field case, the gauge symmetry could superficially arise, and a linear QED could be generated by relating the field strength of U to the corresponding gauge field.

For the matter fields, with the multi-vacuum structure of a sine-Gordon type vector field A^{μ} induced from U, a seesaw mechanism for gauge symmetry and flavor symmetry of fermions could be generated, in which the heavy fermions could be produced. Besides, after a P-4 generalization, by treating the fermion current as a P-2 type field, a possible way for a renormalizable gravitation could be proposed.

The Coulomb potential in electromagnetism and gravitation could be generated by the anti-symmetric part $F^{\mu\nu}$ of the field strength of a P-4 type vector field U^{μ}, when $F^{\mu\nu}$ is coupled to the intrinsic charge current and momentum current of the matter fields, respectively; and, it might be seemed as the second kind of unification for E.M. and gravitation by generalizing the two currents as the two d.o.f of a same field; except for the Coulomb part in each case, there is a linear and a logarithmic part in the E.M. case which might correspond to the confinement in strong QED, while there is a linear and a logarithmic part in the gravitation case which might correspond to the dark energy effects in the impulsive case and dark matter effects in the attractive case, respectively. Besides, the symmetric part $\overline{F}^{\mu\nu}$ of the field strength of U^{μ} could also generate the same gravitation form as the $F^{\mu\nu}$ case; so, it might be the third version for unification if we consider $F^{\mu\nu}$ only for E.M. and $\overline{F}^{\mu\nu}$ only for gravitation as two parts of a same field strength with different proportions, respectively. Moreover, a nonlinear version QED could be generated as a low energy approximation of the dynamics of U^{μ}, and a linear gravitation could be generated by relating the field strength of U^{μ} to the corresponding gauge field.

There could be a linear 3rd-order tensor version QED generated, with the field strength $F^{\alpha\mu\nu}$ of a P-4 type tensor field $U^{\mu\nu}$ corresponding to the gauge fields; for something out our expectation, one thing should be noted, that is, at the quantum level, there wasn't an attractive potential generated by $U^{\mu\nu}$ as in the case of GR at classic level, but an impulsive Coulomb-type one combined with a linear part. Besides, Einstein's general relativity could be generated as a low energy approximation of the dynamics of $U^{\mu\nu}$. For an antisymmetric tensor $U^{\mu\nu}$, that could be corresponding to the vector U^{μ} case.

For the massive $\{U, U^{\mu}\}$ in finite temperature case, attractive potentials for particles with the same kind of charges could be generated, which might serve as candidate for interactions maintaining Cooper pairs in superconductors, with the U case for the s-wave pairing ones by taking the electric charge as interaction charge, and the U^μ case for the d-wave pairing ones by taking the magnetic moment as interaction charge; etc.

About the framework of model-building itself, if the results in our calculations are effective for the real physical processes, then it would be said that the $1/p^4$ framework is a more effective and more general one, by contrast to the the $1/p^2$ one.

From the redefinition for the d.o.f, $x \to \phi = e^{ip\cdot x}$ for the first quantization in quantum mechanics, and $\phi \to U = e^{i\phi}$ in this paper, could we ask, whether there is a principle about this redefinition of d.o.f (maybe we can call it "exponential-ization" or "wave-lization")?

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A The covariant derivative

See Ref. [2] (the Chapter 15 in it).

The comparator $U(y,x)$ and the definition of D_μ

Define the transformation property of the matter field $\psi(x)$ as

$$\psi(x) \to V(x)\psi(x) = e^{i\alpha(x)}\psi(x). \tag{290}$$

The derivative of $\psi(x)$ in the direction of the vector n^μ is defined by the limiting procedure

$$n^\mu \partial_\mu \psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\psi(x+\epsilon n) - \psi(x)\right], \tag{291}$$

while the covariant derivative as

$$n^\mu D_\mu \psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\right], \tag{292}$$

where

$$U(y,x) = e^{i\phi(y,x)}, \tag{293}$$

is defined as an abstract comparator $U(y,x)$, with the restriction

$$U(x,x) = 1,\ [U(x,y)]^\dagger = U(y,x), \tag{294}$$

and

$$U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}. \tag{295}$$

Note that, generally the value of $U(x,z)$ can't be directly derived from the product $U(x,y)U(y,z)$, and we denote that by $U(x,y)U(y,z) \nRightarrow U(x,z)$ for simplicity.

$U(y,x)$ defined as Wilson line and Wilson loop

The $U(y,x)$ can be realized in different forms, for example, the so-called Wilson line, defined as

$$U_P(y,x) \equiv \exp\left[-ig\int_P dz^\mu A_\mu(x)\right], \tag{296}$$

where the subscript P means the integral is taken along any path P that runs from x to $y$or, the expansion form

$$U(x+\epsilon n, x) = 1 - ig\epsilon n^\mu A_\mu(x) + \mathcal{O}((g\epsilon)^2) \tag{297}$$

when an arbitrarily extracted constant g is small.

Sometimes, to obtain locally gauge invariant bricks, we take the path P in (296) to be a closed one, and then we get the Wilson loop, defined as

$$U_P(x,x) \equiv \exp\left[-ig\oint_P dz^\mu A_\mu(x)\right], \tag{298}$$

where P is a closed path that returns to x. Similarly, one can work out the the expansion form for $U_P(x,x)$ according to the Stokes's theorem, for instance, by setting the path is the small square in the $(1,2)$ plane

$$U_P(x,x) = \exp\left[-i\frac{g}{2}\int_\Sigma d\sigma^{\mu\nu}F_{\mu\nu}\right] \tag{299}$$

$$= 1 - i\epsilon^2 gF_{12} + \mathcal{O}(\epsilon^3). \tag{300}$$

where Σ is a surface that spans the closed loop P, $d\sigma^{\mu\nu}$ is an area element on this surface, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{301}$$

is the field tensor.

And now, consequently, we can get the transformation property

$$U_P(x, x) \to e^{i\alpha(x)} U_P(x, x) e^{-i\alpha(x)}, \tag{302}$$

and, particularly for the Abelian group case here, we have

$$U_P(x, x) \to U_P(x, x), \tag{303}$$

showing the gauge invariance of $U_P(x, x)$, and, for the non-Abelian group case, we just have

$$Tr[U_P(x, x)] \to Tr[U_P(x, x)]. \tag{304}$$

Apparently, $U(y, x)$ is introduced as the simplified version of $U_P(y, x)$ (since $U_P(y, x)$ is not only the function of x and y but also of the path P, while $U(y, x)$ is only the function of x and y corresponding to ϵ and n), and, Wilson loop $U_P(x, x)$ is a kind of special Wilson line $U_P(y, x)$(since the value is invariant when the endpoint, where the integrand function $A_\mu(x)$ takes a limited value, was taken out).

$U(y, x)$ in the expression of D_μ

Apparently, there are

$$\begin{aligned} n^\mu D_\mu \psi &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\psi(x + \epsilon n) - U(y, x)\psi(x)\right], \; (y = x + \epsilon n) \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\psi(x + \epsilon n) - U(x, x)\psi(x) + U(x, x)\psi(x) - U(y, x)\psi(x)\right] \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\psi(x + \epsilon n) - \psi(x) + U(y, x)[U(x, y) - U(x, x)]\psi(x)\right] \qquad (305) \\ &= n^\mu \left[\partial_\mu \psi(x) + \underline{U(y, x)\partial_\mu U^\dagger(y, x)}\psi(x)\right], . \qquad (306) \end{aligned}$$

It should be paid some attention to (305), in which the factor

$$\begin{aligned} &U(x, y) - U(x, x) = U(x, y) - U(y, y) = [U(y, x)]^{-1} - [U(x, x)]^{-1} \\ &(\text{with } U(y, y) = U(x, x) = 1), \end{aligned} \tag{307}$$

was picked out since we have treat the second variable x to be the only argument of $U_P(y, x)$, that is, we should keep the first variable the same.

Actually, $U_P(x, y)$ is more general than $U_P(x, x)$, since for some $U_P(x, x)$ there would be $U_P(x, x) = 0$ in the case of lattice size $a \to 0$.

Reminding the definition of the connection field as a MaurerCCartan 1-form of the gauge group element V,

$$B_\mu(x) = V(x)\partial_\mu V^\dagger(x), \tag{308}$$

we can get the traditional definition of covariant derivative from (306)

$$D_\mu = \partial_\mu + B_\mu\,, \tag{309}$$

and get a map from (306), see the underlined term, as

$$U(x, y) = e^{i\phi(y,x)} \to V(x) = e^{i\alpha(x)}\,. \tag{310}$$

B Integration formulas for Fourier transformation

B.1 $m_U = 0$

See Ref. [2].

We have

$$\begin{aligned}
&\int \frac{d^3q}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{|\boldsymbol{q}|^2} \\
=& \frac{1}{4\pi^2}\int_0^\infty dq\, q^2 \frac{e^{iqr}-e^{-iqr}}{iqr}\frac{1}{q^2} \\
=& \frac{1}{4\pi^2 ir}\int_0^\infty dq\,(e^{iqr}-e^{-iqr})\frac{1}{q} \\
=& \frac{1}{4\pi^2 ir}\int_{-\infty}^{+\infty} dq\, e^{iqr}\frac{1}{q} \\
=& \frac{1}{4\pi^2 ir}(i\pi) \qquad (311)\\
=& \frac{1}{4\pi r}\,, \qquad (312)
\end{aligned}$$

and

$$\begin{aligned}
&\int \frac{d^3q}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{|\boldsymbol{q}|^4} \\
=& \frac{1}{4\pi^2}\int_0^\infty dq\, q^2 \frac{e^{iqr}-e^{-iqr}}{iqr}\frac{1}{q^4} \\
=& \frac{1}{4\pi^2 ir}\int_0^\infty dq\,(e^{iqr}-e^{-iqr})\frac{1}{q^3} \\
=& \frac{1}{4\pi^2 ir}\int_{-\infty}^{+\infty} dq\, e^{iqr}\frac{1}{q^3} \\
=& \frac{1}{4\pi^2 ir}(\frac{-i\pi r^2}{2}) \qquad (\sim i\pi\cdot\frac{(ir)^2}{2} \neq i2\pi\cdot\frac{(ir)^2}{2})\\
=& -\frac{1}{8\pi}r\,, \qquad (313)
\end{aligned}$$

here we use the differential property of the Fourier transformation to get (313) from (312), as[14]

---

[14]Note, for the integration

$$\begin{aligned}
&\frac{1}{4\pi^2 ir}\int_0^\infty dq\,(e^{iqr}-e^{-iqr})\frac{1}{q^3} \\
=& \frac{1}{4\pi^2 ir}\int_0^\infty dq\,\frac{i2\sin qr}{q^3} \\
=& \frac{1}{2\pi^2 r}\int_0^\infty dq\,\frac{\sin qr}{q^3} \\
=& \frac{1}{2\pi^2 r}r^2\int_0^\infty d(qr)\,\frac{\sin qr}{(qr)^3} \\
=& \frac{1}{2\pi^2 r}r^2\int_0^\infty dz\,\frac{\sin z}{z^3} = ? > 0? \qquad (314)
\end{aligned}$$

if we choose the contour contain $z = 0$, then Res=0; but, the integration shouldn't be 0 ! So, that reminds us the contour may not be the right one!

$$\begin{aligned}
\frac{1}{q} &\leftrightarrow i\pi\cdot\mathrm{sign}(x) \quad ,\text{(see Eq. (311))} \\
i^n \frac{d^n}{dq^n} F(q) &\leftrightarrow x^n f(x) \qquad (315)\\
i^2\frac{d^2}{dq^2}\frac{1}{q} = -2\cdot\frac{1}{q^3} &\leftrightarrow x^2\cdot i\pi\cdot\mathrm{sign}(x) \\
\frac{1}{q^3} &\leftrightarrow \frac{-i\pi x^2}{2}\cdot\mathrm{sign}(x) \qquad (316)
\end{aligned}$$

Besides, we have

$$\begin{aligned}
&\int\frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{1}{|\boldsymbol{q}|^3} \\
=&\ \frac{1}{4\pi^2}\int_0^\infty dq\, q^2\frac{e^{iqr}-e^{-iqr}}{iqr}\frac{1}{q^3} \\
=&\ \frac{1}{4\pi^2 ir}\int_0^\infty dq\,(e^{iqr}-e^{-iqr})\frac{1}{q^2} \\
=&\ \frac{1}{4\pi^2 ir}\int_{-\infty}^{+\infty} dq\, e^{iqr}\left[\frac{\theta(q)-\theta(-q)}{q^2}\right] \quad (\theta \text{ is the Heaviside step funtion}) \\
=&\ \frac{1}{4\pi^2 ir}[-i2r(\log r+\gamma_E-1)] \quad (\text{Euler constant } \gamma_E\simeq 0.577) \\
=&\ -\frac{1}{2\pi^2}(\log r+\gamma_E-1)\,. \qquad (317)
\end{aligned}$$

In a intuitive view, since there is $\frac{1}{q^2}\left[\theta(q)-\theta(-q)\right]_{q=0}\sim\delta(q)$, the constant in (317) would arise from the $\delta(q)$ in the integrand function; and, in the sense of derivative for the Fourier transformation, as shown in (315), a logarithm is allowed to arise in this integration.

B.2 $m_U\neq 0$

Firstly, we do the expansion

$$\frac{1}{q^4-m^4+i\epsilon}=\frac{1}{(q-q_{(1)})(q-q_{(2)})(q-q_{(3)})(q-q_{(4)})} \qquad (318)$$

$$q_{(1)}=-im-\epsilon,\ q_{(2)}=im+\epsilon,\ q_{(3)}=-m+i\epsilon,\ q_{(4)}=m-i\epsilon\,. \qquad (319)$$

Then, with

$$(q_{(2)}-q_{(1)})(q_{(2)}-q_{(3)})(q_{(2)}-q_{(4)})=-i4m^3, \qquad (320)$$

$$(q_{(3)}-q_{(1)})(q_{(3)}-q_{(2)})(q_{(3)}-q_{(4)})=-4m^3, \qquad (321)$$

we can get

$$\begin{aligned}
&\int\frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{1}{|\boldsymbol{q}|^4-m^4+i\epsilon} \\
=&\ \frac{1}{4\pi^2}\int_0^\infty dq\, q^2\frac{e^{iqr}-e^{-iqr}}{iqr}\frac{1}{q^4-m^4+i\epsilon} \\
=&\ \frac{1}{4\pi^2 ir}\int_0^\infty dq\,(e^{iqr}-e^{-iqr})\frac{q}{q^4-m^4+i\epsilon}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2 ir}\int_{-\infty}^{+\infty} dq\, e^{iqr}\frac{q}{q^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2 ir}\oint_C dq\, e^{iqr}\frac{q}{q^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2 ir}2\pi i(Res_{(2)}+Res_{(3)}) \\
&= \frac{1}{4\pi^2 ir}2\pi i(\frac{q_{(2)}e^{iq_{(2)}r}}{-i4m^3}+\frac{q_{(3)}e^{iq_{(3)}r}}{-4m^3}) \\
&= \frac{1}{8\pi m^2 r}(-e^{-mr}+e^{-imr}) \\
&= \frac{1}{8\pi m^2 r}[(\cos mr - e^{-mr}) - i\sin mr], \qquad (322)
\end{aligned}
$$

where the contour of the integrals $\oint_C$ was closed above in the complex plane and the residue of the simple pole at $q_{(2)}$ and $q_{(3)}$ were picked up.

$$
\begin{aligned}
&\int \frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{|\boldsymbol{q}|}{|\boldsymbol{q}|^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2}\int_0^{\infty} dq\, q^2\frac{e^{iqr}-e^{-iqr}}{iqr}\frac{q}{q^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2 ir}\int_0^{\infty} dq\,(e^{iqr}-e^{-iqr})\frac{q^2}{q^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2 ir}\int_{-\infty}^{+\infty} dq\, e^{iqr}\left[\frac{q^2[\theta(q)-\theta(-q)]}{q^4-m^4+i\epsilon}\right] \quad (\theta \text{ is the Heaviside step funtion}) \\
&= \frac{1}{4\pi^2 ir}2\pi i(Res_{(2)}+Res_{(3)}) \\
&= \frac{1}{4\pi^2 ir}2\pi i(\frac{q_{(2)}^2 e^{iq_{(2)}r}}{-i4m^3}+\frac{-q_{(3)}^2 e^{iq_{(3)}r}}{-4m^3}) \\
&= \frac{1}{8\pi mr}(-ie^{-mr}+e^{-imr}) \\
&= \frac{1}{8\pi mr}[\cos mr - i(e^{-mr}+\sin mr)], \qquad (323)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int \frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{|\boldsymbol{q}|^2}{|\boldsymbol{q}|^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2}\int_0^{\infty} dq\, q^2\frac{e^{iqr}-e^{-iqr}}{iqr}\frac{q^2}{q^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2 ir}\int_0^{\infty} dq\,(e^{iqr}-e^{-iqr})\frac{q^3}{q^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2 ir}\int_{-\infty}^{+\infty} dq\, e^{iqr}\frac{q^3}{q^4-m^4+i\epsilon} \\
&= \frac{1}{4\pi^2 ir}2\pi i(Res_{(2)}+Res_{(3)}) \\
&= \frac{1}{4\pi^2 ir}2\pi i(\frac{q_{(2)}^3 e^{iq_{(2)}r}}{-i4m^3}+\frac{q_{(3)}^3 e^{iq_{(3)}r}}{-4m^3}) \\
&= \frac{1}{4\pi^2 ir}2\pi i(\frac{-im^3e^{-mr}}{-i4m^3}+\frac{-m^3e^{-imr}}{-4m^3})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{8\pi r}(e^{-mr} + e^{-imr}) \\
&= \frac{1}{8\pi r}[(\cos mr + e^{-mr}) - i\sin mr]. \qquad (324)
\end{aligned}
$$

C Loop integrations

C.1 Loop integrations A

Define

$$
I_l = \int d^4l \, \frac{1}{l^4(l-q)^4}. \qquad (325)
$$

With the Feynman parameters(F.P.) and Wick rotation(W.R.), we can compute I_l as

$$
\begin{aligned}
I_l &= \int d^4l \, \frac{1}{(l^2)^2[(l-q)^2]^2} \\
&\overset{\text{(F.P.)}}{=\!=} \int d^4l \int_0^1 dx \frac{x(1-x)}{[xl^2 + (1-x)(l-q)^2]^4} \frac{\Gamma(2+2)}{\Gamma(2)\Gamma(2)} \\
&= 6\int_0^1 dx\, x(1-x) \int d^4l \, \frac{1}{[xl^2 + (1-x)(l-q)^2]^4} \\
&= 6\int_0^1 dx\, x(1-x) \int d^4l \, \frac{1}{\{[l-(1-x)q]^2 + [x(1-x)]q^2\}^4} \\
&\overset{l\to l'}{=\!=} 6\int_0^1 dx\, x(1-x) \int d^4l' \, \frac{1}{\{l'^2 + [x(1-x)]q^2\}^4} \\
&\overset{\text{(W.R.)}}{=\!=} i6\int_0^1 dx x(1-x) \int d^4l_E \, \frac{1}{\{-l_E^2 + [x(1-x)]q^2\}^4} \\
&= i6\int_0^1 dx\, x(1-x) \int d\Omega \int d|l_E| \frac{|l_E|^3}{(l_E^2 + \Delta^2)^4} \\
&= i12\pi^2 \int_0^1 dx\, x(1-x) \int_0^{+\infty} d|l_E| \frac{|l_E|^3}{(l_E^2 + \Delta^2)^4} \\
&\xrightarrow[(l_E \geq \mu)]{\text{(I.R.)}} i12\pi^2 \int_0^1 dx\, x(1-x) \, \frac{\Delta^2 + 3\mu^2}{12\,(\Delta^2 + \mu^2)^3} \\
&= i\pi^2 \int_0^1 dx\, x(1-x) \, \frac{\Delta^2 + 3\mu^2}{(\Delta^2 + \mu^2)^3}\,, \qquad (326)
\end{aligned}
$$

with the variables

$$
q^2 < 0 \;\Rightarrow \Delta^2 = -[x(1-x)]q^2 > 0\,, \qquad (327)
$$

$$
l \to l' = l - (1-x)q \Leftrightarrow l = l' + (1-x)q, \qquad (328)
$$

and the relations

$$
q\cdot l = -q_E \cdot l_E \;\;,\;\; q_E \cdot l_E = \sum_{i=0}^{3} q_E^i l_E^i, \qquad (329)
$$

where the 4-dimensional Euclidean l_E-integration was taken in the spherical coordinates, denoted as [2]

$$
\begin{aligned}
x &= r(\sin\omega\sin\theta\cos\phi, \sin\omega\sin\theta\sin\phi, \sin\omega\cos\theta, \cos\omega) \\
&\quad \text{with } \ 0 \le \omega, \theta < \pi, \ 0 \le \phi < 2\pi, \qquad (330) \\
d^4x &= drr^3 \cdot d\Omega = drr^3 \cdot (d\omega \sin^2\omega d\theta \sin\theta d\phi), \qquad (331) \\
&\int d\Omega = 2\pi^2 \,. \qquad (332)
\end{aligned}
$$

Since there is only infrared divergences in this integration like (79), here we impose a very large μ (say, $\mu > \alpha\Lambda$) as the infrared cutoff to renormalize the infrared divergences, with the result of the integration

$$
\begin{aligned}
&\int_{\mu}^{+\infty} \frac{{l_E}^3}{({l_E}^2+\Delta^2)^4}\, dl_E \\
= \ & \int_{\mu}^{+\infty} \frac{\frac{{l_E}^2}{2}\cdot(2l_E)}{({l_E}^2+\Delta^2)^4}\, dl_E = \frac{1}{2}\int_{\mu^2}^{+\infty} \frac{{l_E}^2}{({l_E}^2+\Delta^2)^4}\, d{l_E}^2 \\
= \ & \frac{1}{2}\int_{\mu^2}^{+\infty} \frac{t}{(t+\Delta^2)^4}\, dt = \frac{1}{2}\int_{\mu^2+\Delta^2}^{+\infty} \frac{z-\Delta^2}{z^4} dz \quad (z = t+\Delta^2 \ , \, t = z-\Delta^2) \\
= \ & \frac{1}{2}\int_{\mu^2+\Delta^2}^{+\infty} \frac{1}{z^3} dz - \frac{\Delta^2}{2}\int_{\mu^2+\Delta^2}^{+\infty} \frac{1}{z^4} dz \\
= \ & \frac{1}{2}\cdot\left[-\frac{1}{2z^2}\right]_{\mu^2+\Delta^2}^{+\infty} - \frac{\Delta^2}{2}\cdot\left[-\frac{1}{3z^3}\right]_{\mu^2+\Delta^2}^{+\infty} \\
= \ & \frac{1}{2}\cdot\left[\frac{1}{2\left(\mu^2+\Delta^2\right)^2}\right] - \frac{\Delta^2}{2}\cdot\left[\frac{1}{3\left(\mu^2+\Delta^2\right)^3}\right] \\
= \ & \frac{1}{4\left(\mu^2+\Delta^2\right)^2} - \frac{\Delta^2}{6\left(\mu^2+\Delta^2\right)^3} \\
= \ & \frac{\Delta^2+3\mu^2}{12\left(\Delta^2+\mu^2\right)^3}\,. \qquad (333)
\end{aligned}
$$

Let's finish the integration of x, as

$$
\begin{aligned}
I_l &= i\pi^2 \int_0^1 dx\, x(1-x)\, \frac{\Delta^2+3\mu^2}{\left(\Delta^2+\mu^2\right)^3} \\
&= i\pi^2 \int_0^1 dx\, x(1-x)\, \frac{q^2(x-1)x+3\mu^2}{\left(q^2(x-1)x+\mu^2\right)^3} \\
&= i\pi^2 \int_0^1 dx\, f(x)\,, \\
&= \frac{i\pi^2}{(q^2)^3}\cdot\frac{1}{2}[(2a_1\log|1-x_a| - 2a_1\log|-x_a|) \\
&\quad + \left(\frac{2a_2}{x_a-1} - \frac{2a_2}{x_a}\right) + \left(\frac{a_3}{x_a^2} - \frac{a_3}{(x_a-1)^2}\right) \\
&\quad + (2b_1\log|1-x_b| - 2b_1\log|-x_b|) \\
&\quad + \left(\frac{2b_2}{x_b-1} - \frac{2b_2}{x_b}\right) + \left(\frac{b_3}{x_b^2} - \frac{b_3}{(x_b-1)^2}\right)] \\
&\xrightarrow[(l_E \ge \mu)]{\text{(I.R.)}} \frac{i\pi^2}{(q^2)^3}\cdot\left(\frac{q^4}{4\mu^2}\right) = -i\frac{\pi^2}{4}\frac{1}{q^4}\cdot\left(\frac{-q^2}{\mu^2}\right)\,, \qquad (334)
\end{aligned}
$$

where

$$f(x) = x(1-x)\,\frac{q^2(x-1)x+3\mu^2}{(q^2(x-1)x+\mu^2)^3} = \frac{1}{(q^2)^3}\left[\frac{a_1}{(x-x_a)}+\frac{a_2}{(x-x_a)^2}+\frac{a_3}{(x-x_a)^3} + \frac{b_1}{(x-x_b)}+\frac{b_2}{(x-x_b)^2}+\frac{b_3}{(x-x_b)^3}\right], \quad (335)$$

with

$$x_a = \frac{1}{2} - \sqrt{\frac{1}{4}-\frac{\mu^2}{q^2}} < 0\,,\; x_b = \frac{1}{2} + \sqrt{\frac{1}{4}-\frac{\mu^2}{q^2}} > 1\,, \quad (336)$$

and

$$\begin{aligned}
a_3 &= -\frac{(x_a-1)x_a[3\mu^2+q^2(x_a-1)x_a]}{(x_a-x_b)^3}\,,\\
a_2 &= \frac{3\mu^2\left[x_a^2+2x_a(x_b-1)-x_b\right]-q^2(x_a-1)x_a\left(x_a^2-4x_ax_b+x_a+2x_b\right)}{(x_a-x_b)^4}\,,\\
a_1 &= -\frac{3\mu^2\left[x_a^2+x_a(4x_b-3)+(x_b-3)x_b\right]+q^2\left[x_a^2\left(6x_b^2-6x_b+1\right)+2x_a(2-3x_b)x_b+x_b^2\right]}{(x_a-x_b)^5}\,,\\
b_3 &= \frac{(x_b-1)x_b[3\mu^2+q^2(x_b-1)x_b]}{(x_a-x_b)^3}\,,\\
b_2 &= \frac{3\mu^2[x_a(2x_b-1)+(x_b-2)x_b]-q^2(x_b-1)x_b\left(-4x_ax_b+2x_a+x_b^2+x_b\right)}{(x_a-x_b)^4}\,,\\
b_1 &= -a_1\,.
\end{aligned} \quad (337)$$

C.2 Loop integrations B.

Define

$$I_l = \int d^4l\,\frac{1}{l^2(l-q)^4}. \quad (338)$$

With the Feynman parameters(F.P.) and Wick rotation(W.R.), we can compute I_l as

$$\begin{aligned}
I_l &= \int d^4l\,\frac{1}{l^2[(l-q)^2]^2}\\
&\overset{\text{(F.P.)}}{=\!=} \int d^4l\int_0^1 dx\frac{2(1-x)}{[xl^2+(1-x)(l-q)^2]^3}\\
&= \int_0^1 dx\,[2(1-x)]\int d^4l\,\frac{1}{\{[l-(1-x)q]^2+[(1-x)-(1-x)^2]q^2\}^3}\\
&\overset{l\to l'}{=\!=} \int_0^1 dx\,[2(1-x)]\int d^4l'\,\frac{1}{\{l'^2-\Delta^2\}^3}\\
&\overset{\text{(W.R.)}}{=\!=} i\int_0^1 dx\,[2(1-x)]\int d^4l_E\,\frac{1}{\{-l_E^2-\Delta^2\}^3}\\
&= -i\int_0^1 dx\,[2(1-x)]\int d^4l_E\,\frac{1}{\{l_E^2+\Delta^2\}^3}
\end{aligned}$$

$$
\begin{aligned}
&= -i\int_0^1 dx\,[2(1-x)]\int d\Omega\int d|l_E|\,\frac{|l_E|^3}{\{|l_E|^2+\Delta^2\}^3}\\
&= -i(2\pi^2)\int_0^1 dx\,[2(1-x)]\int_0^{+\infty} d|l_E|\,\frac{|l_E|^3}{\{|l_E|^2+\Delta^2\}^3}\\
&\xrightarrow[(l_E\geq\mu)]{\text{(I.R.)}} -i(2\pi^2)\int_0^1 dx\,[2(1-x)]\left[\frac{\Delta^2+2\mu^2}{4\left(\Delta^2+\mu^2\right)^2}\right]\\
&= -i\pi^2\int_0^1 dx\,(1-x)\left[\frac{\Delta^2+2\mu^2}{\left(\Delta^2+\mu^2\right)^2}\right] \qquad (339)
\end{aligned}
$$

where

$$
l' = l-(1-x)q \;\;\Leftrightarrow\;\; l = l'+(1-x)q\,, \qquad (340)
$$

$$
q^2<0 \;\;\Rightarrow\;\; \Delta^2 = -[(1-x)-(1-x)^2]q^2>0\,. \qquad (341)
$$

Since there is only infrared divergences in this integration like (79), here we impose a very large μ (say, $\mu>\alpha\Lambda$) as the infrared cutoff to renormalize the infrared divergences, with the result of the integration

$$
\begin{aligned}
&\int_\mu^{+\infty}\frac{{l_E}^3}{({l_E}^2+\Delta^2)^3}\,dl_E\\
&= \int_\mu^{+\infty}\frac{\frac{{l_E}^2}{2}*(2l_E)}{({l_E}^2+\Delta^2)^3}\,dl_E = \frac{1}{2}\int_{\mu^2}^{+\infty}\frac{{l_E}^2}{({l_E}^2+\Delta^2)^3}\,d{l_E}^2\\
&= \frac{1}{2}\int_{\mu^2}^{+\infty}\frac{t}{(t+\Delta^2)^3}\,dt = \frac{1}{2}\int_{\mu^2+\Delta^2}^{+\infty}\frac{z-\Delta^2}{z^3}dz \quad (z=t+\Delta^2,\, t=z-\Delta^2)\\
&= \frac{1}{2}\int_{\mu^2+\Delta^2}^{+\infty}\frac{1}{z^2}dz-\frac{\Delta^2}{2}\int_{\mu^2+\Delta^2}^{+\infty}\frac{1}{z^3}dz\\
&= \frac{1}{2}\left(-\frac{1}{z}\right)_{\mu^2+\Delta^2}^{+\infty}-\frac{\Delta^2}{2}\left(-\frac{1}{2z^2}\right)_{\mu^2+\Delta^2}^{+\infty}\\
&= \frac{1}{2}\left(\frac{1}{(\mu^2+\Delta^2)}\right)-\frac{\Delta^2}{2}\left(\frac{1}{2\left(\mu^2+\Delta^2\right)^2}\right)\\
&= \left(\frac{1}{2\left(\mu^2+\Delta^2\right)}-\frac{\Delta^2}{4\left(\mu^2+\Delta^2\right)^2}\right)\\
&= \frac{\Delta^2+2\mu^2}{4\left(\Delta^2+\mu^2\right)^2}\,. \qquad (342)
\end{aligned}
$$

Let's finish the integration of x, as

$$
\begin{aligned}
I_l &= -i\pi^2\int_0^1 dx\,(1-x)\left[\frac{\Delta^2+2\mu^2}{\left(\Delta^2+\mu^2\right)^2}\right]\\
&= -i\pi^2\int_0^1 dx\,(1-x)\left[\frac{q^2(x-1)x+2\mu^2}{\left(q^2(x-1)x+\mu^2\right)^2}\right]\\
&= -i\pi^2\int_0^1 dx\,f(x)\\
&= -i\pi^2\cdot\frac{1}{(q^2)^2}\left\{[a_1\log(1-x_a)-a_1\log(-x_a)]+\left(\frac{a_2}{x_a-1}-\frac{a_2}{x_a}\right)\right.
\end{aligned}
$$

$$+ [b_1 \log(1-x_b) - b_1 \log(-x_b)] + \left(\frac{b_2}{x_b - 1} - \frac{b_2}{x_b}\right)\Bigg\}$$

$$\xrightarrow[(l_E \geq \mu)]{\text{(I.R.)}} \quad -i\pi^2 \cdot \frac{1}{(q^2)^2}\left(\frac{q^4}{\mu^2}\right) = +i\pi^2 \cdot \frac{1}{q^2}\left(\frac{-q^2}{\mu^2}\right), \tag{343}$$

where

$$\begin{aligned} f(x) &= (1-x)\left[\frac{q^2(x-1)x + 2\mu^2}{(q^2(x-1)x+\mu^2)^2}\right] \\ &= \frac{1}{(q^2)^2}\left[\frac{a_1}{(x-x_a)} + \frac{a_2}{(x-x_a)^2} \right. \\ &\quad \left. + \frac{b_1}{(x-x_b)} + \frac{b_2}{(x-x_b)^2}\right], \end{aligned} \tag{344}$$

with

$$x_a = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\mu^2}{q^2}} < 0\,,\; x_b = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\mu^2}{q^2}} > 1\,, \tag{345}$$

and

$$\begin{aligned} a2 &= -\frac{(x_a-1)[2\mu^2 + q^2(x_a-1)x_a]}{(x_a-x_b)^2}, \\ a1 &= \frac{2\mu^2(x_a+x_b-2) - q^2(x_a-1)\left(x_a^2 - 3x_a x_b + x_a + x_b\right)}{(x_a-x_b)^3}, \\ b2 &= -\frac{(x_b-1)[2\mu^2 + q^2(x_b-1)x_b]}{(x_a-x_b)^2}, \\ b1 &= -\frac{2\mu^2(x_a+x_b-2) + q^2\left(3x_a x_b^2 - 4x_a x_b + x_a - x_b^3 + x_b\right)}{(x_a-x_b)^3}. \end{aligned} \tag{346}$$

C.3 Loop integrations C

Define

$$I_l = \int d^4l\, \frac{v\cdot l}{l^4(l-q)^4}. \tag{347}$$

With the Feynman parameters(F.P.) and Wick rotation(W.R.), we can compute I_l as

$$\begin{aligned} I_l &= \int d^4l\,(v\cdot l)\frac{1}{(l^2)^2[(l-q)^2]^2} \\ &\overset{\text{(F.P.)}}{=\!=} \int d^4l\,(v\cdot l)\int_0^1 dx \frac{x(1-x)}{[xl^2 + (1-x)(l-q)^2]^4}\frac{\Gamma(2+2)}{\Gamma(2)\Gamma(2)} \\ &= 6\int_0^1 dx\, x(1-x)\int d^4l\,(v\cdot l)\frac{1}{[xl^2+(1-x)(l-q)^2]^4} \\ &= 6\int_0^1 dx\, x(1-x)\int d^4l\,(v\cdot l)\frac{1}{\{[l-(1-x)q]^2 + [x(1-x)]q^2\}^4} \\ &\overset{l\to l'}{=\!=} 6\int_0^1 dx\, x(1-x)\int d^4l'\,[v\cdot(l'+(1-x)q)]\frac{1}{\{l'^2 + [x(1-x)]q^2\}^4} \\ &= 6\int_0^1 dx\, x(1-x)\int d^4l\,\frac{v\cdot l}{\{l^2+[x(1-x)]q^2\}^4} \end{aligned}$$

$$
\begin{aligned}
& +6\int_0^1 dx\, x(1-x)\cdot[(1-x)v\cdot q]\int d^4l\,\frac{1}{\{l^2+[x(1-x)]q^2\}^4} \\
= \quad & 0+6\int_0^1 dx\, x(1-x)\cdot[(1-x)v\cdot q]\int d^4l\,\frac{1}{\{l^2+[x(1-x)]q^2\}^4} \\
\underset{\text{(W.R.)}}{=\!=} \quad & i6\int_0^1 dx x(1-x)\cdot[(1-x)v\cdot q]\int d^4l_E\,\frac{1}{\{-l_E^2+[x(1-x)]q^2\}^4} \\
= \quad & i6\int_0^1 dx\, x(1-x)\cdot[(1-x)v\cdot q]\int d\Omega\int d|l_E|\frac{|l_E|^3}{(l_E^2+\Delta^2)^4} \\
= \quad & i12\pi^2\int_0^1 dx\, x(1-x)\cdot[(1-x)v\cdot q]\int d|l_E|\frac{|l_E|^3}{(l_E^2+\Delta^2)^4} \\
\xrightarrow[(l_E\geq\mu)]{\text{(I.R.)}} \quad & i12\pi^2(v\cdot q)\cdot\int_0^1 dx\,[x(1-x)^2]\,\frac{\Delta^2+3\mu^2}{12\,(\Delta^2+\mu^2)^3} \\
= \quad & i\pi^2(v\cdot q)\cdot\int_0^1 dx\,[x(1-x)^2]\,\frac{\Delta^2+3\mu^2}{(\Delta^2+\mu^2)^3}\,,
\end{aligned}
\tag{348}
$$

with

$$
l'=l-(1-x)q \Leftrightarrow l=l'+(1-x)q. \tag{349}
$$

Let's finish the integration of x, as

$$
\begin{aligned}
I_l \quad = \quad & i\pi^2(v\cdot q)\cdot\int_0^1 dx\,[x(1-x)^2]\,\frac{\Delta^2+3\mu^2}{(\Delta^2+\mu^2)^3} \\
= \quad & i\pi^2(v\cdot q)\cdot\int_0^1 dx\,[x(1-x)^2]\,\frac{q^2(x-1)x+3\mu^2}{(q^2(x-1)x+\mu^2)^3} \\
= \quad & i\pi^2(v\cdot q)\cdot\int_0^1 dx\, f(x)\,, \\
= \quad & i\pi^2(v\cdot q)\cdot\frac{1}{(q^2)^3}\frac{1}{2}\{[2a_1\log(1-x_a)-2a_1\log(-x_a)] \\
& +\left(\frac{2a_2}{x_a-1}-\frac{2a_2}{x_a}\right)+\left(\frac{a_3}{x_a^2}-\frac{a_3}{(x_a-1)^2}\right) \\
& +[2b_1\log(1-x_b)-2b_1\log(-x_b)] \\
& +\left(\frac{2b_2}{x_b-1}-\frac{2b_2}{x_b}\right)+\left(\frac{b_3}{x_b^2}-\frac{b_3}{(x_b-1)^2}\right)\Bigg\} \\
\xrightarrow[(l_E\geq\mu)]{\text{(I.R.)}} \quad & i\pi^2(v\cdot q)\cdot\frac{1}{(q^2)^3}\left(-\frac{q^6-3q^4\mu^2}{8\mu^4}\right) \\
\xrightarrow{(q\ll\mu)} \quad & i\pi^2(v\cdot q)\cdot\frac{1}{(q^2)^3}\left(\frac{3q^4}{8\mu^2}\right)=-i\frac{3\pi^2}{8}\cdot\frac{v\cdot q}{(q^2)^2}\left(\frac{-q^2}{\mu^2}\right)\,,
\end{aligned}
\tag{350}
$$

where

$$
\begin{aligned}
f(x) \quad = \quad & [x(1-x)^2]\,\frac{q^2(x-1)x+3\mu^2}{(q^2(x-1)x+\mu^2)^3} \\
= \quad & \frac{1}{(q^2)^3}\left[\frac{a_1}{(x-x_a)}+\frac{a_2}{(x-x_a)^2}+\frac{a_3}{(x-x_a)^3}\right. \\
& \left.+\frac{b_1}{(x-x_b)}+\frac{b_2}{(x-x_b)^2}+\frac{b_3}{(x-x_b)^3}\right]\,,
\end{aligned}
\tag{351}
$$

with

$$x_a = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\mu^2}{q^2}} < 0\,,\; x_b = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\mu^2}{q^2}} > 1\,, \tag{352}$$

and

$$\begin{aligned}
a3 &= \frac{(x_a-1)^2 x_a\,[3\mu^2 + q^2(x_a-1)x_a]}{(x_a-x_b)^3}\,,\\
a2 &= \frac{(x_a-1)\,[3\mu^2[x_a(2-3x_b)+x_b] + q^2(x_a-1)x_a\,(2x_a^2 - 5x_a x_b + x_a + 2x_b)]}{(x_a-x_b)^4}\,,\\
a1 &= \frac{1}{(x_a-x_b)^5}\left\{3\mu^2\left[x_a^2(3x_b-2) + x_a\left(3x_b^2 - 8x_b + 3\right) + (3-2x_b)x_b\right]\right.\\
&\quad \left.+q^2(x_a-1)\left[x_a^4 + x_a^3(1-5x_b) + x_a^2\left(10x_b^2 - 5x_b + 1\right) + 4x_a(1-2x_b)x_b + x_b^2\right]\right\}\,,\\
b3 &= -\frac{(x_b-1)^2 x_b\,[3\mu^2 + q^2(x_b-1)x_b]}{(x_a-x_b)^3}\,,\\
b2 &= -\frac{(x_b-1)\,[\mu^2[x_a(9x_b-3) - 6x_b] - q^2(x_b-1)x_b\,(-5x_a x_b + 2x_a + 2x_b^2 + x_b)]}{(x_a-x_b)^4}\,,\\
b1 &= -\frac{1}{(x_a-x_b)^5}\left\{3\mu^2\left[x_a^2(3x_b-2) + x_a\left(3x_b^2 - 8x_b + 3\right) + (3-2x_b)x_b\right]\right.\\
&\quad \left.+q^2(x_b-1)\left[x_a^2\left(10x_b^2 - 8x_b + 1\right) + x_a x_b\left(-5x_b^2 - 5x_b + 4\right) + x_b^2\left(x_b^2 + x_b + 1\right)\right]\right\}\,.
\end{aligned} \tag{353}$$

D The derivation for E.O.M

We write the Euler-Lagrange equation in (37) here, as

$$\frac{\partial \mathcal{L}_U}{\partial U} - \partial_\mu \frac{\partial \mathcal{L}_U}{\partial(\partial_\mu U)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}_U}{\partial(\partial_\mu \partial_\nu U)} = 0\,. \tag{354}$$

D.1 Scalar U

From (5) and (7), we can have the expansion

$$\begin{aligned}
\mathcal{L}_U &= -\partial^\mu \partial^\nu U^\dagger \partial_\mu \partial_\nu U - \Lambda_U^4(U + U^\dagger) + m_U^4 U^\dagger U\\
&= -\partial_\mu \partial_\nu U^\dagger g^{\mu\mu'} g^{\nu\nu'} \partial_{\mu'} \partial_{\nu'} U - \Lambda_U^4(U + U^\dagger) + m_U^4 U^\dagger U\,,
\end{aligned} \tag{355}$$

and the variational derivative

$$\frac{\partial \mathcal{L}_U}{\partial(\partial^\lambda \partial^\tau U^\dagger)} = -\delta^{\mu\lambda}\delta^{\nu\tau} g^{\mu\mu'} g^{\nu\nu'} \partial_{\mu'} \partial_{\nu'} U = -\partial_\lambda \partial_\tau U\,, \tag{356}$$

then the term

$$\partial^\lambda \partial^\tau \frac{\partial \mathcal{L}_U}{\partial(\partial^\lambda \partial^\tau U^\dagger)} = -\partial^4 U\,. \tag{357}$$

For the interaction part, we have the expansion

$$\begin{aligned}
\mathcal{L}_I &= -\alpha Q^{-1}\Lambda\,\bar{\psi}(U + U^\dagger)\psi - \beta Q\,\bar{\psi}(\partial\!\!\!/ U + \partial\!\!\!/ U^\dagger)\psi\\
&\quad +(\text{higher-order operators})\\
&= -\alpha Q^{-1}\Lambda\,\bar{\psi}(U^\dagger)\psi - \beta Q\,\bar{\psi}(+\partial_\mu U^\dagger) g^{\mu\mu'} \gamma_{\mu'} \psi + \ldots\,,
\end{aligned} \tag{358}$$

where the "..." corresponding to the higher order terms and irrelevant terms in all this section, and the variational derivative

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial(\partial^\tau U^\dagger)}\\ = &-\beta Q\,\bar{\psi}(+\delta_{\mu\tau})g^{\mu\mu'}\gamma_{\mu'}\psi+...\\ = &-\beta Q\,\bar{\psi}\gamma^\tau\psi+...\,,\end{aligned} \tag{359}$$

then the term

$$\partial^\tau\frac{\partial\mathcal{L}}{\partial(\partial^\tau U^\dagger)}=-\beta Q\,\partial^\tau\left(\bar{\psi}\gamma^\tau\psi\right)+...=0\,, \tag{360}$$

with the current conservation law $\partial_\tau\left(\bar{\psi}\gamma^\tau\psi\right)=0$. Indeed, if we do the variational derivative

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = -\alpha Q^{-1}\Lambda\,(U+U^\dagger)\psi-\beta Q\,\partial_\mu(U+U^\dagger)\gamma^\mu\psi+...\,, \tag{361}$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -\alpha Q^{-1}\Lambda\,\bar{\psi}(U+U^\dagger)-\beta Q\,\bar{\psi}\partial_\mu(U+U^\dagger)\gamma^\mu+...\,, \tag{362}$$

we can get the E.O.M for the matter field ψ as

$$(i\partial\!\!\!/-m_\psi)\psi = \alpha Q^{-1}\Lambda\,(U+U^\dagger)\psi+\beta Q\,\partial_\mu(U+U^\dagger)\gamma^\mu\psi \tag{363}$$

$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/}-m_\psi) = \alpha Q^{-1}\Lambda\,\bar{\psi}(U+U^\dagger)+\beta Q\,\bar{\psi}\partial_\mu(U+U^\dagger)\gamma^\mu \tag{364}$$

then, with the E.O.M, we can truly get the current conservation law [2]

$$\begin{aligned}\partial_\tau\left(\bar{\psi}\gamma^\tau\psi\right) &= \,:\partial^\tau\bar{\psi}\gamma^\tau\psi+\bar{\psi}\gamma^\tau\partial^\tau\psi:=:-\bar{\psi}\overleftarrow{\partial}\gamma^\tau\psi+\bar{\psi}\gamma^\tau\partial^\tau\psi:\\ &= \,:i\left[m_\psi\bar{\psi}+\alpha Q^{-1}\Lambda\,\bar{\psi}(U+U^\dagger)+\beta Q\,\bar{\psi}\partial_\mu(U+U^\dagger)\gamma^\mu\right]\psi\\ &\quad -i\bar{\psi}\left[m_\psi\psi+\alpha Q^{-1}\Lambda\,(U+U^\dagger)\psi+\beta Q\,\partial_\mu(U+U^\dagger)\gamma^\mu\psi\right]:\\ &= \,:i\left[\bar{\psi}m_\psi+\alpha Q^{-1}\Lambda\,\bar{\psi}(U+U^\dagger)+\beta Q\,\bar{\psi}\partial_\mu(U+U^\dagger)\gamma^\mu\right]\psi\\ &\quad -i\bar{\psi}\left[m_\psi\psi+\alpha Q^{-1}\Lambda\,(U+U^\dagger)\psi+\beta Q\,\partial_\mu(U+U^\dagger)\gamma^\mu\psi\right]:\\ &= 0\,,\end{aligned} \tag{365}$$

with the definition of [2]

$$\bar{\psi}\overleftarrow{\partial}\equiv-\partial\bar{\psi}\,. \tag{366}$$

Besides, we have the term

$$\frac{\partial(\mathcal{L}_U+\mathcal{L}_I)}{\partial U^\dagger}=-\Lambda_U^4+m_U^4U-\alpha Q^{-1}\Lambda\,\bar{\psi}\psi\,. \tag{367}$$

Then, we can get the dynamical equation for the field U as

$$-\,\partial^4U=\Lambda_U^4-m_U^4U+\alpha Q^{-1}\Lambda\,\bar{\psi}\psi\,. \tag{368}$$

D.2 Vector U^μ

D.2.1 $\Lambda_U = 0$ case

From (152,165)and (166), by setting $\Lambda_U = 0$, we have the expansion

$$\begin{aligned}
\mathcal{L}_U &= \frac{1}{2}\partial_\alpha F^\dagger_{\beta\mu}\partial^\alpha F^{\beta\mu} - m_U^4 U^\dagger_\mu U^\mu \\
&= \frac{1}{2}\left[\partial_\alpha(\partial_\beta U^\dagger_\mu - \partial_\mu U^\dagger_\beta)\right]\left[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)\right] - m_U^4 U^\dagger_\mu U^\mu \\
&= \frac{1}{2}\left[\partial_\alpha(\partial_\beta U^\dagger_\mu - \partial_\mu U^\dagger_\beta)\right] g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\left[\partial_{\alpha'}(\partial_{\beta'}U_{\mu'} - \partial_{\mu'}U_{\beta'})\right] \\
&\quad - m_U^4 U^\dagger_\mu g^{\mu\mu'}U_{\mu'} \\
&= \frac{1}{2}\Big[\partial_\alpha\partial_\beta U^\dagger_\mu g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\partial_{\alpha'}\partial_{\beta'}U_{\mu'} - \partial_\alpha\partial_\mu U^\dagger_\beta g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\partial_{\alpha'}\partial_{\beta'}U_{\mu'} \\
&\quad - \partial_\alpha\partial_\beta U^\dagger_\mu g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'} + \partial_\alpha\partial_\mu U^\dagger_\beta g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'}\Big] \\
&\quad - m_U^4 U^\dagger_\mu g^{\mu\mu'}U_{\mu'}\,,
\end{aligned} \tag{369}$$

and the variational derivative

$$\begin{aligned}
&\frac{\partial\mathcal{L}_U}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma})} \\
&= \frac{1}{2}\Big[\delta_{\alpha\lambda}\delta_{\beta\tau}\delta_{\mu\sigma}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\partial_{\alpha'}\partial_{\beta'}U_{\mu'} - \delta_{\alpha\lambda}\delta_{\mu\tau}\delta_{\beta\sigma}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\partial_{\alpha'}\partial_{\beta'}U_{\mu'} \\
&\quad - \delta_{\alpha\lambda}\delta_{\beta\tau}\delta_{\mu\sigma}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'} + \delta_{\alpha\lambda}\delta_{\mu\tau}\delta_{\beta\sigma}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'}\Big] \\
&= \frac{1}{2}\left[\partial_\lambda\partial_\tau U_\sigma - \partial_\lambda\partial_\sigma U_\tau - \partial_\lambda\partial_\sigma U_\tau + \partial_\lambda\partial_\tau U_\sigma\right] \\
&= \left[\partial_\lambda\partial_\tau U_\sigma - \partial_\lambda\partial_\sigma U_\tau\right]
\end{aligned} \tag{370}$$

then the term

$$\begin{aligned}
&\partial^\lambda\partial^\tau\frac{\partial\mathcal{L}_U}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma})} \\
&= \left[\partial^4 U_\sigma - \partial^2\partial_\sigma\partial^\tau U_\tau\right] = \partial^2\left[\partial^2 g^\tau_\sigma - \partial_\sigma\partial^\tau\right]U_\tau \\
&\overset{(\partial\cdot U=0)}{=\!=} \partial^4 U_\sigma = (i\partial)^4 U_\sigma
\end{aligned} \tag{371}$$

with the gauge fixed condition $\partial_\sigma U^\sigma = 0$.

1. $F_{\mu\nu}$ part: a generation of QED

For the interaction Lagrangian in (166), we have the expansion

$$\begin{aligned}
\mathcal{L}_I &= -\alpha\,\Lambda\bar\psi(\not{U} + \not{U}^\dagger)\psi - \beta\,\bar\psi(\varepsilon^{\mu\nu} + \sigma^{\mu\nu})[F_{\mu\nu}(U) + F_{\mu\nu}(U^\dagger)]\psi \\
&\quad + (\text{higher-order operators}) \\
&= -\alpha\,\Lambda\bar\psi(+U^\dagger_\mu g^{\mu\mu'}\gamma_{\mu'})\psi - \beta\,\bar\psi(\varepsilon_{\mu\nu} + \sigma_{\mu\nu})g^{\mu\mu'}g^{\nu\nu'}(\partial_{\mu'}U^\dagger_{\nu'} - \partial_{\nu'}U^\dagger_{\mu'})\psi \\
&\quad + \ldots\,,
\end{aligned} \tag{372}$$

and the variational derivative

$$\begin{aligned}
\frac{\partial\mathcal{L}_I}{\partial(\partial^\tau U^{\dagger\sigma})} &= -\beta\,\bar\psi(\varepsilon_{\mu\nu} + \sigma_{\mu\nu})g^{\mu\mu'}g^{\nu\nu'}(\delta_{\mu'\tau}\delta_{\nu'\sigma} - \delta_{\nu'\tau}\delta_{\mu'\sigma})\psi + \ldots\,,\,, \\
&= -2\beta\,\bar\psi(\varepsilon^{\tau\sigma} + \sigma^{\tau\sigma})\psi + \ldots\,,
\end{aligned} \tag{373}$$

then the term

$$\partial_\tau \frac{\partial \mathcal{L}_I}{\partial(\partial^\tau U^{\dagger\sigma})} = -2\beta\, \partial_\tau \left[\bar{\psi}(\varepsilon^{\tau\sigma} + \sigma^{\tau\sigma})\psi\right] . \tag{374}$$

Besides, we have the term

$$\begin{aligned}\frac{\partial(\mathcal{L}_U + \mathcal{L}_I)}{\partial U^{\dagger\sigma}} &= -m_U^4 \delta_{\mu\sigma} g^{\mu\mu'} U_{\mu'} - \alpha\, \Lambda \bar{\psi}(\delta_{\mu\sigma} g^{\mu\mu'} \gamma_{\mu'})\psi \\ &= -m_U^4 U^\sigma - \alpha\, \Lambda \bar{\psi}\gamma^\sigma \psi\, .\end{aligned} \tag{375}$$

Then, we can get the dynamical equation for the field U^μ as

$$\partial^4 U_\sigma = m_U^4 U^\sigma + J^\sigma \tag{376}$$

with

$$J^\sigma = \alpha\, \Lambda \bar{\psi}\gamma^\sigma\psi - 2\beta\, \partial_\tau \left[\bar{\psi}(\varepsilon^{\tau\sigma} + \sigma^{\tau\sigma})\psi\right] \tag{377}$$

and the gauge fixed condition $\partial_\sigma U^\sigma = 0$.

For the E.O.M, another convenient method is rewrite the action as(ignoring the mass term here) [2]

$$\begin{aligned}S_0 &= \int d^4x \mathcal{L}_U \\ &= \int d^4x \left[\frac{1}{2}\partial_\alpha F^\dagger_{\beta\mu}\partial^\alpha F^{\beta\mu}\right] \\ &= \frac{1}{2}\int d^4x \left[\partial_\alpha(\partial_\beta U^\dagger_\mu - \partial_\mu U^\dagger_\beta)\right]\left[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)\right] \\ &= \frac{1}{2}\int d^4x \Big\{\underline{\partial_\alpha}[(\partial_\beta U^\dagger_\mu - \partial_\mu U^\dagger_\beta)\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)] \\ &\quad -(\partial_\beta U^\dagger_\mu - \partial_\mu U^\dagger_\beta)\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)]\Big\} \\ &= -\frac{1}{2}\int d^4x \,\{(\partial_\beta U^\dagger_\mu)\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)] \\ &\quad -(\partial_\mu U^\dagger_\beta)\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)]\Big\} \\ &= -\frac{1}{2}\int d^4x \Big\{\underline{\partial_\beta}\{U^\dagger_\mu\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)]\} \\ &\quad -U^\dagger_\mu\partial_\beta\{\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)]\} \\ &\quad -\underline{\partial_\mu}\{U^\dagger_\beta\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)]\} \\ &\quad +U^\dagger_\beta\partial_\mu\{\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)]\}\Big\} \\ &= \frac{1}{2}\int d^4x \,\{+U^\dagger_\mu\partial_\beta\{\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)]\} \\ &\quad -U^\dagger_\beta\partial_\mu\{\partial_\alpha[\partial^\alpha(\partial^\beta U^\mu - \partial^\mu U^\beta)]\}\Big\}\end{aligned} \tag{378}$$

where the underlined derivative term were omitted, and then we can get the E.O.M by the Euler-Lagrangian equation.

For the free propagator, with the E.O.M in (376), we can get the equation for Feynman propagator $D_F^{\nu\rho}$

$$
\begin{aligned}
(\partial^4 g_{\mu\nu} - m_U^4 + ...)D_F^{\nu\rho}(x-y) &= i\delta_\mu^\rho \delta^{(4)}(x-y) && (379)\\
\text{or } (p^4 g_{\mu\nu} - m_U^4 + ...)\widetilde{D}_F^{\nu\rho}(p) &= i\delta_\mu^\rho && (380)
\end{aligned}
$$

which has the solution

$$
\widetilde{D}_F^{\mu\nu}(p) = \frac{+i(g^{\mu\nu} + ...)}{p^4 - m_U^4 + i\epsilon}, \qquad (381)
$$

where the "..." denoted the gauge free term which would be omitted in the Lorentz gauge condition. As the case for (53,59), please pay attention to the extra minus sign in $\widetilde{D}_F^{\mu\nu}(p)$ by contrast to the propagator of a P-2 type vector field.

2. $\overline{F}_{\mu\nu}$ part: a generation of SR gravity
The E.O.M for a free U^μ in $\overline{F}_{\mu\nu}$ case would be the same as in $F_{\mu\nu}$ case.

The Lagrangian for free particle U^μ in an full-symmetric irreducible representation could be written as

$$
\begin{aligned}
\mathcal{L}_U &= \frac{1}{12}\left[(\partial_\alpha \overline{F}_{\beta\mu} + \partial_\beta \overline{F}_{\alpha\mu} + \partial_\mu \overline{F}_{\alpha\beta})^\dagger (\partial^\alpha \overline{F}^{\beta\mu} + \partial^\beta \overline{F}^{\alpha\mu} + \partial^\mu \overline{F}^{\alpha\beta})\right] - m_U^4 U_\mu^\dagger U^\mu \\
&= \frac{1}{12}\Big[(\partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\alpha \overline{F}^{\beta\mu} + \partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\beta \overline{F}^{\alpha\mu} + \partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\mu \overline{F}^{\alpha\beta}) \\
&\quad +(\partial_\beta \overline{F}^\dagger_{\alpha\mu}\partial^\alpha \overline{F}^{\beta\mu} + \partial_\beta \overline{F}^\dagger_{\alpha\mu}\partial^\beta \overline{F}^{\alpha\mu} + \partial_\beta \overline{F}^\dagger_{\alpha\mu}\partial^\mu \overline{F}^{\alpha\beta}) \\
&\quad +(\partial_\mu \overline{F}^\dagger_{\alpha\beta}\partial^\alpha \overline{F}^{\beta\mu} + \partial_\mu \overline{F}^\dagger_{\alpha\beta}\partial^\beta \overline{F}^{\alpha\mu} + \partial_\mu \overline{F}^\dagger_{\alpha\beta}\partial^\mu \overline{F}^{\alpha\beta})\Big] - m_U^4 U_\mu^\dagger U^\mu \\
&= +\frac{1}{12}\Big[(\partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\alpha \overline{F}^{\beta\mu} + \partial_\beta \overline{F}^\dagger_{\alpha\mu}\partial^\beta \overline{F}^{\alpha\mu} + \partial_\mu \overline{F}^\dagger_{\alpha\beta}\partial^\mu \overline{F}^{\alpha\beta}) \\
&\quad +(\partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\beta \overline{F}^{\alpha\mu} + \partial_\beta \overline{F}^\dagger_{\alpha\mu}\partial^\alpha \overline{F}^{\beta\mu} + \partial_\mu \overline{F}^\dagger_{\alpha\beta}\partial^\beta \overline{F}^{\alpha\mu}) \\
&\quad +(\partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\mu \overline{F}^{\alpha\beta} + \partial_\beta \overline{F}^\dagger_{\alpha\mu}\partial^\mu \overline{F}^{\alpha\beta} + \partial_\mu \overline{F}^\dagger_{\alpha\beta}\partial^\alpha \overline{F}^{\beta\mu})\Big] - m_U^4 U_\mu^\dagger U^\mu \\
&= \frac{3}{12}(\partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\alpha \overline{F}^{\beta\mu} + \partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\beta \overline{F}^{\alpha\mu} + \partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\mu \overline{F}^{\beta\alpha}) - m_U^4 U_\mu^\dagger U^\mu \\
&= \frac{3}{12}\partial_\alpha \overline{F}^\dagger_{\beta\mu}(\partial^\alpha \overline{F}^{\beta\mu} + \partial^\beta \overline{F}^{\alpha\mu} + \partial^\mu \overline{F}^{\beta\alpha}) - m_U^4 U_\mu^\dagger U^\mu \\
&= \frac{3}{12}\partial_\alpha \overline{F}^\dagger_{\beta\mu}[\partial^\alpha(\partial^\beta U^\mu + \partial^\mu U^\beta) + \partial^\beta(\partial^\alpha U^\mu + \partial^\mu U^\alpha) + \partial^\mu(\partial^\beta U^\alpha + \partial^\alpha U^\beta)] \\
&\quad - m_U^4 U_\mu^\dagger U^\mu \\
&= \frac{6}{12}\partial_\alpha \overline{F}^\dagger_{\beta\mu}(\partial^\alpha\partial^\beta U^\mu + \partial^\alpha\partial^\mu U^\beta + \partial^\beta\partial^\mu U^\alpha) - m_U^4 U_\mu^\dagger U^\mu \\
&= \frac{1}{2}\partial_\alpha \overline{F}^\dagger_{\beta\mu}(\partial^\alpha \overline{F}^{\beta\mu} + \underline{\partial^\beta\partial^\mu U^\alpha}) - m_U^4 U_\mu^\dagger U^\mu, \quad U_\mu^\dagger U^\mu < 1, \qquad (382)
\end{aligned}
$$

where the underlined term could be omitted with the Lorentz gauge condition $\partial_\alpha U^\alpha = 0$. After an expansion,

$$
\begin{aligned}
\mathcal{L}_U &= +\frac{1}{2}\partial_\alpha \overline{F}^\dagger_{\beta\mu}\partial^\alpha \overline{F}^{\beta\mu} - m_U^4 U_\mu^\dagger U^\mu \\
&= +\frac{1}{2}\left[\partial_\alpha(\partial_\beta U_\mu + \partial_\mu U_\beta)^\dagger \partial^\alpha(\partial_\beta U_\mu + \partial_\mu U_\beta)\right] - m_U^4 U_\mu^\dagger U^\mu \\
&= +\frac{1}{2}\left[\partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\beta U^\mu + \partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\mu U^\beta + \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\alpha\partial^\beta U^\mu + \partial_\alpha\partial_\mu U_\beta^\dagger \partial^\alpha\partial^\mu U^\beta\right]
\end{aligned}
$$

$$
\begin{aligned}
& -m_U^4 U_\mu^\dagger U^\mu \\
= \; & +\frac{1}{2}\left[2\partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\beta U^\mu + 2\partial_\alpha\partial_\beta U_\mu^\dagger \partial^\alpha\partial^\mu U^\beta\right] - m_U^4 U_\mu^\dagger U^\mu \\
= \; & +\partial_\alpha\partial_\beta U_\mu^\dagger(\partial^\alpha\partial^\beta U^\mu + \partial^\alpha\partial^\mu U^\beta) - m_U^4 U_\mu^\dagger U^\mu \,, \qquad (383)
\end{aligned}
$$

we can know that the propagator would now be the same as that in the $F_{\mu\nu}$ case, with the Lorentz gauge condition being taken.

For the interaction Lagrangian in (195), we have the expansion

$$
\begin{aligned}
\mathcal{L}_I \; = \; & -\alpha\frac{\Lambda}{M}U_\mu\bar{\psi}\bar{\psi}i\partial^\mu\psi \\
& -\beta\frac{1}{M}\overline{F}^{(U)}_{\mu\nu}\bar{\psi}(\underline{-2g^{\mu\nu}m} + \gamma^\nu i\partial^\mu + \gamma^\mu i\partial^\nu)\psi + \ldots \\
= \; & -\alpha\frac{\Lambda}{M}U_\mu\bar{\psi}g^{\mu\mu'}\bar{\psi}i\partial_{\mu'}\psi \\
& -\beta\frac{1}{M}\overline{F}^{(U)}_{\mu\nu}g^{\mu\mu'}g^{\nu\nu'}\bar{\psi}(\underline{-2g_{\mu'\nu'}m} + \gamma_{\nu'}i\partial_{\mu'} + \gamma_{\mu'}i\partial_{\nu'})\psi + \ldots \\
= \; & -\alpha\frac{\Lambda}{M}U_\mu\bar{\psi}g^{\mu\mu'}\bar{\psi}i\partial_{\mu'}\psi \\
& -\beta\frac{1}{M}(\partial_\mu U_\nu + \partial_\nu U_\mu)g^{\mu\mu'}g^{\nu\nu'}\bar{\psi}(\underline{-2g_{\mu'\nu'}m} + \gamma_{\nu'}i\partial_{\mu'} + \gamma_{\mu'}i\partial_{\nu'})\psi + \ldots \,, \qquad (384)
\end{aligned}
$$

and the variational derivative

$$
\begin{aligned}
\frac{\partial\mathcal{L}_I}{\partial(\partial_\tau U_\sigma)} \; = \; & -\beta\frac{1}{M}(\delta_{\mu\tau}\delta_{\nu\sigma} + \delta_{\nu\tau}\delta_{\mu\sigma})g^{\mu\mu'}g^{\nu\nu'}\bar{\psi}(\underline{-2g_{\mu'\nu'}m} + \gamma_{\nu'}i\partial_{\mu'} + \gamma_{\mu'}i\partial_{\nu'})\psi + \ldots \\
= \; & -2\beta\frac{1}{M}\bar{\psi}(\underline{-2g^{\tau\sigma}m} + \gamma^\sigma i\partial^\tau + \gamma^\tau i\partial^\sigma)\psi + \ldots \,, \qquad (385)
\end{aligned}
$$

then the term

$$
\partial_\tau\frac{\partial\mathcal{L}_I}{\partial(\partial_\tau U_\sigma)} \; = \; -2\beta\frac{1}{M}\partial_\tau\left[\bar{\psi}(\underline{-2g^{\tau\sigma}m} + \gamma^\sigma i\partial^\tau + \gamma^\tau i\partial^\sigma)\psi\right] + \ldots \,, \qquad (386)
$$

Besides, we have the term

$$
\frac{\partial\mathcal{L}_I}{\partial U^\sigma} = -\alpha\frac{\Lambda}{M}\delta_{\mu\sigma}\bar{\psi}g^{\mu\mu'}\bar{\psi}i\partial_{\mu'}\psi = -\alpha\frac{\Lambda}{M}\bar{\psi}\bar{\psi}i\partial^\sigma\psi \,, \qquad (387)
$$

Then, we can get the dynamical equation for the field U^μ as

$$
\partial^4 U_\sigma = m_U^4 U^\sigma + J^\sigma \qquad (388)
$$

with

$$
\begin{aligned}
J^\sigma \; = \; & +\alpha\frac{\Lambda}{M}\bar{\psi}\bar{\psi}i\partial^\sigma\psi \\
& + -2\beta\frac{1}{M}\partial_\tau[\bar{\psi}(\underline{-2g^{\tau\sigma}m} + \gamma^\sigma i\partial^\tau + \gamma^\tau i\partial^\sigma)\psi] + \ldots \,, \qquad (389)
\end{aligned}
$$

and the gauge fixed condition $\partial_\sigma U^\sigma = 0$.

D.2.2 $\Lambda_U \neq 0$ case: generation of a nonlinear QED

For the Λ_U term, we can have the expansion

$$\mathcal{L}_U \to -\Lambda_U \left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\beta\mu}\partial^\alpha F^{\beta\mu}_{(U+U^\dagger)} + (\text{cyclic for indices})\right] - m_U^4 U^\dagger_\mu U^\mu \,, \quad (390)$$

so we have the variational derivative

$$\begin{aligned}
&\frac{\partial \mathcal{L}_U}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma})} \\
&= \frac{\partial}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma})}\left\{-\Lambda_U\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\beta\mu}\partial^\alpha F^{\beta\mu}_{(U^\dagger)}\right]+...\right\} \\
&= -\Lambda_U g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\frac{\partial}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma})}\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\beta\mu}(\partial_{\alpha'}\partial_{\beta'}U^\dagger_{\mu'} - \partial_{\alpha'}\partial_{\mu'}U^\dagger_{\beta'})\right]+... \\
&= -\Lambda_U g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\beta\mu}(\delta_{\alpha'\lambda}\delta_{\beta'\tau}\delta_{\mu'\sigma} - \delta_{\alpha'\lambda}\delta_{\mu'\tau}\delta_{\beta'\sigma})\right]+... \\
&= -\Lambda_U\epsilon_{\rho\lambda}\left[(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\tau\sigma} - (U+U^\dagger)^\rho i F^{(U+U^\dagger)}_{\sigma\tau}\right]+... \\
&= -2\Lambda_U\epsilon_{\rho\lambda}\left[(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\tau\sigma}\right]+...\,, \quad (391)
\end{aligned}$$

with the term

$$\begin{aligned}
\partial^\lambda\partial^\tau\frac{\partial \mathcal{L}_U}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma})} &= -2\Lambda_U\epsilon_{\rho\lambda}\partial^\lambda\partial^\tau\left[(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\tau\sigma}\right]+... \\
&= -2\Lambda_U\epsilon_{\rho\lambda}\partial^\lambda\partial^\tau\left[(U+U^\dagger)^\rho F^{(U+U^\dagger)}_{\tau\sigma}\right]+...\,, \quad (392)
\end{aligned}$$

and the variational derivative

$$\begin{aligned}
&\frac{\partial \mathcal{L}_U}{\partial(\partial^\tau U^{\dagger\sigma})} \\
&= \frac{\partial}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma})}\left\{-\Lambda_U\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho F^{(U^\dagger)}_{\beta\mu}\partial^\alpha F^{\beta\mu}_{(U+U^\dagger)}\right]+...\right\} \\
&= \frac{\partial}{\partial(\partial^\tau U^{\dagger\sigma})}\left\{-\Lambda_U\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho F^{(U^\dagger)}_{\beta\mu}\partial^\alpha F^{\beta\mu}_{(U+U^\dagger)}\right]+...\right\} \\
&= -\Lambda_U g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\frac{\partial}{\partial(\partial^\tau U^{\dagger\sigma})}\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho(\partial_{\beta'}U^\dagger_{\mu'} - \partial_{\mu'}U^\dagger_{\beta'})\partial_\alpha F^{(U+U^\dagger)}_{\beta\mu}\right]+... \\
&= -\Lambda_U g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho(\delta_{\beta'\tau}\delta_{\mu'\sigma} - \delta_{\mu'\tau}\delta_{\beta'\sigma})\partial_\alpha F^{(U+U^\dagger)}_{\beta\mu}\right]+... \\
&= -\Lambda_U\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho(\partial^\alpha F^{(U+U^\dagger)}_{\tau\sigma} - \partial^\alpha F^{(U+U^\dagger)}_{\sigma\tau})\right]+... \\
&= -2\Lambda_U\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho\partial^\alpha F^{(U+U^\dagger)}_{\tau\sigma}\right]+...\,, \quad (393)
\end{aligned}$$

with the term

$$\begin{aligned}
\partial^\tau\frac{\partial \mathcal{L}_U}{\partial(\partial^\tau U^{\dagger\sigma})} &= -2\Lambda_U\partial^\tau\left[\epsilon_{\rho\alpha}(U+U^\dagger)^\rho i\partial^\alpha F^{(U+U^\dagger)}_{\tau\sigma}\right]+... \\
&= -2\Lambda_U\epsilon_{\rho\alpha}\partial^\tau\left[(U+U^\dagger)^\rho\partial^\alpha F^{(U+U^\dagger)}_{\tau\sigma}\right]+...\,. \quad (394)
\end{aligned}$$

Besides, we have the term

$$\begin{aligned}
\frac{\partial \mathcal{L}_U}{\partial U^{\dagger\sigma}} &= -m_U^4\delta^{\mu\sigma}g_{\mu\mu'}U^{\mu'} - \Lambda_U\left[\epsilon_{\rho\alpha}\delta^\rho_\sigma F^{(U+U^\dagger)}_{\beta\mu}\partial^\alpha F^{\beta\mu}_{(U+U^\dagger)}\right]+... \\
&= -m_U^4 U_\sigma - \Lambda_U\left[\epsilon_{\sigma\alpha}F^{(U+U^\dagger)}_{\beta\mu}\partial^\alpha F^{\beta\mu}_{(U+U^\dagger)}\right]+... \\
&= -m_U^4 U_\sigma - \Lambda_U\epsilon_{\sigma\alpha}\left[F^{(U+U^\dagger)}_{\beta\mu}\partial^\alpha F^{\beta\mu}_{(U+U^\dagger)}\right]+...\,. \quad (395)
\end{aligned}$$

Then, we can get the dynamical equation for the field U^{μ} as

$$
\begin{aligned}
& -2\Lambda_U \epsilon_{\rho\lambda}\partial^{\lambda}\partial^{\tau}\left[(U+U^{\dagger})^{\rho}F_{\tau\sigma}^{(U+U^{\dagger})}\right] \\
& -m_U^4 U_{\sigma} - \Lambda_U \epsilon_{\sigma\alpha}\left[F_{\beta\mu}^{(U+U^{\dagger})}\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})}\right] \\
= \quad & -2\Lambda_U \epsilon_{\rho\alpha}\partial^{\tau}\left[(U+U^{\dagger})^{\rho}\partial^{\alpha}F_{\tau\sigma}^{(U+U^{\dagger})}\right] + (\text{cyclic for indices})\,. \qquad (396)
\end{aligned}
$$

Let's continue the simplification, for the first term in the l.h.s of (396), it is

$$
\begin{aligned}
& -2\Lambda_U \epsilon_{\rho\lambda}\partial^{\lambda}\partial^{\tau}\left[(U+U^{\dagger})^{\rho}F_{\tau\sigma}^{(U+U^{\dagger})}\right] + ... \\
= \quad & -2\Lambda_U \partial^{\tau}\left[\epsilon_{\rho\lambda}\partial^{\lambda}(U+U^{\dagger})^{\rho}F_{\tau\sigma}^{(U+U^{\dagger})} + \underline{\epsilon_{\rho\lambda}(U+U^{\dagger})^{\rho}\partial^{\lambda}F_{\tau\sigma}^{(U+U^{\dagger})}}\right] + ... \\
= \quad & -2\Lambda_U \partial^{\tau}\left[\frac{1}{2}(\epsilon_{\rho\lambda}-\epsilon_{\lambda\rho})\partial^{\lambda}(U+U^{\dagger})^{\rho}\right] F_{\tau\sigma}^{(U+U^{\dagger})} + ... \\
= \quad & -\Lambda_U \epsilon_{\rho\lambda}\partial^{\tau}\left[\partial^{\lambda}(U+U^{\dagger})^{\rho} - \partial^{\rho}(U+U^{\dagger})^{\lambda}\right] F_{\tau\sigma}^{(U+U^{\dagger})} + ... \\
= \quad & -\Lambda_U \epsilon_{\rho\lambda}\partial^{\tau}\left[F^{\lambda\rho}_{(U+U^{\dagger})}F_{\tau\sigma}^{(U+U^{\dagger})}\right] + ... \\
= \quad & -\Lambda_U \left[\epsilon_{\rho\lambda}\partial^{\tau}F^{\lambda\rho}_{(U+U^{\dagger})}F_{\tau\sigma}^{(U+U^{\dagger})} + \epsilon_{\rho\lambda}F^{\lambda\rho}_{(U+U^{\dagger})}\partial^{\tau}F_{\tau\sigma}^{(U+U^{\dagger})}\right] + ... \\
(\text{C.D.I}) \quad = \quad & -\Lambda_U \Big[g_{\mu\phi'}g_{\beta\varphi'}\epsilon^{\phi'\varphi'}\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})}g_{\alpha\kappa'}g_{\sigma\xi'}F^{\kappa'\xi'}_{(U+U^{\dagger})} \\
& +g_{\rho'\phi'}g_{\lambda'\varphi'}\epsilon^{\phi'\varphi'}F^{\lambda'\rho'}_{(U+U^{\dagger})}g_{\alpha\beta}g_{\sigma\mu}\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})}\Big] + ... \\
= \quad & -\Lambda_U \Big[g_{\mu\phi'}g_{\beta\varphi'}\epsilon^{\phi'\varphi'}g_{\alpha\kappa'}g_{\sigma\xi'}F^{\kappa'\xi'}_{(U+U^{\dagger})} \\
& +g_{\rho'\phi'}g_{\lambda'\varphi'}\epsilon^{\phi'\varphi'}F^{\lambda'\rho'}_{(U+U^{\dagger})}g_{\alpha\beta}g_{\sigma\mu}\Big]\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})} + ... \\
(\text{C.D.I}) \quad = \quad & -\Lambda_U \left[\epsilon_{\mu\beta}F_{\alpha\sigma}^{(U+U^{\dagger})} + \epsilon_{\rho\lambda}F^{\lambda\rho}_{(U+U^{\dagger})}g_{\alpha\beta}g_{\sigma\mu}\right]\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})} + ...\,, \qquad (397)
\end{aligned}
$$

where "C.D.I" denotes the operation for "changing the dummy indices",and the term underlined is just the r.h.s of (396). If we set $m_U = 0$, then we have the E.O.M

$$
\begin{aligned}
0 \quad = \quad & -\Lambda_U \left[\epsilon_{\mu\beta}F_{\alpha\sigma}^{(U+U^{\dagger})} + \epsilon_{\rho\lambda}F^{\lambda\rho}_{(U+U^{\dagger})}g_{\alpha\beta}g_{\sigma\mu}\right]\cdot\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})} \\
& -\Lambda_U\epsilon_{\sigma\alpha}F_{\beta\mu}^{(U+U^{\dagger})}\cdot\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})} \\
= \quad & -\Lambda_U\left[-\epsilon_{\mu\beta}F_{\sigma\alpha}^{(U+U^{\dagger})} - \epsilon_{\sigma\alpha}F_{\mu\beta}^{(U+U^{\dagger})} + \epsilon_{\rho\lambda}F^{\lambda\rho}_{(U+U^{\dagger})}g_{\alpha\beta}g_{\mu\sigma}\right]\cdot\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})} \\
\equiv \quad & -\Lambda_U X_{(\mu\sigma)(\alpha\beta)}\cdot\partial^{\alpha}F^{\beta\mu}_{(U+U^{\dagger})}\,, \qquad (398)
\end{aligned}
$$

where the terms "(cyclic for indices)" were dropped since the indices $\alpha\beta\mu$ are in fact dummy indices, and the tensor

$$
\begin{aligned}
X_{(\mu\sigma)(\alpha\beta)} \quad \equiv \quad & \left[-\epsilon_{\mu\beta}F_{\sigma\alpha}^{(U+U^{\dagger})} - \epsilon_{\sigma\alpha}F_{\mu\beta}^{(U+U^{\dagger})}\right] + \epsilon_{\rho\lambda}F^{\lambda\rho}_{(U+U^{\dagger})}g_{\alpha\beta}g_{\mu\sigma} \\
\neq \quad & \epsilon_{\rho\lambda}F^{\lambda\rho}\left[-\delta_{\rho\mu}\delta_{\lambda\beta}\delta_{\lambda\sigma}\delta_{\rho\alpha} - \delta_{\rho\sigma}\delta_{\lambda\alpha}\delta_{\lambda\mu}\delta_{\rho\beta} + g_{\alpha\beta}g_{\mu\sigma}\right] \qquad (399)
\end{aligned}
$$

is symmetric for the indices $(\mu\sigma)$ and $(\alpha\beta)$ (denoted with a round bracket), respectively.

D.3 Tensor $U^{\mu\nu}$

1. From (213-216) and (238), that is,

$$\mathcal{L}_U = -\frac{1}{3}\partial_\alpha F^\dagger_{\beta\mu\nu}\partial^\alpha F^{\beta\mu\nu} - \Lambda_U^2\left[(U+U^\dagger)_{\alpha\nu}(U+U^\dagger)_{\beta\mu} i\partial^\alpha i F^{\beta\mu\nu}\right] + m_U^4 U^\dagger_{\mu\nu}U^{\mu\nu}, \tag{400}$$

where

$$F_{\beta\mu\nu} = +\partial_\beta U_{\mu\nu} + \partial_\nu U_{\beta\mu} + \partial_\mu U_{\beta\nu}, \tag{401}$$

$$F^\dagger_{\alpha\mu\nu} = +\partial_\alpha U^\dagger_{\mu\nu} + \partial_\nu U^\dagger_{\alpha\mu} + \partial_\mu U^\dagger_{\alpha\nu}, \tag{402}$$

$$F^\beta_{\mu\nu} = \eta^{\beta\rho}F_{\rho\mu\nu} = \eta^{\beta\rho}\left(+\partial_\beta U_{\mu\nu} + \partial_\nu U_{\beta\mu} + \partial_\mu U_{\beta\nu}\right), \tag{403}$$

we can expand the first term of $\mathcal{L}_U$ as

$$\begin{aligned}
\mathcal{L}_{U1} &= -\frac{1}{3}\partial_\alpha F^\dagger_{\beta\mu\nu} g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}F_{\beta'\mu'\nu'} \\
&= -\frac{1}{3}\Big[(+\partial_\alpha\partial_\beta U^\dagger_{\mu\nu} + \partial_\alpha\partial_\nu U^\dagger_{\beta\mu} + \partial_\alpha\partial_\mu U^\dagger_{\beta\nu})g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'} \\
&\quad (+\partial_{\alpha'}\partial_{\beta'}U_{\mu'\nu'} + \partial_{\alpha'}\partial_{\nu'}U_{\beta'\mu'} + \partial_{\alpha'}\partial_{\mu'}U_{\beta'\nu'})\Big] \\
&= -\frac{1}{3}\Big[(+\partial_\alpha\partial_\beta U^\dagger_{\mu\nu})g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}(+\partial_{\alpha'}\partial_{\beta'}U_{\mu'\nu'}) \\
&\quad +(+\partial_\alpha\partial_\beta U^\dagger_{\mu\nu})g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\nu'}U_{\beta'\mu'} \\
&\quad +(+\partial_\alpha\partial_\beta U^\dagger_{\mu\nu})g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'\nu'} \\
&\quad +\partial_\alpha\partial_\nu U^\dagger_{\beta\mu}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}(+\partial_{\alpha'}\partial_{\beta'}U_{\mu'\nu'}) \\
&\quad +\partial_\alpha\partial_\nu U^\dagger_{\beta\mu}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\nu'}U_{\beta'\mu'} \\
&\quad +\partial_\alpha\partial_\nu U^\dagger_{\beta\mu}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'\nu'} \\
&\quad +\partial_\alpha\partial_\mu U^\dagger_{\beta\nu}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}(+\partial_{\alpha'}\partial_{\beta'}U_{\mu'\nu'}) \\
&\quad +\partial_\alpha\partial_\mu U^\dagger_{\beta\nu}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\nu'}U_{\beta'\mu'} \\
&\quad +\partial_\alpha\partial_\mu U^\dagger_{\beta\nu}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'\nu'}\Big],
\end{aligned} \tag{404}$$

then we can do the variational derivative

$$\begin{aligned}
&\frac{\partial\mathcal{L}_{U1}}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma\rho})} \\
&= -\frac{1}{3}\Big[(+\delta_{\alpha\lambda}\delta_{\beta\tau}\delta_{\mu\sigma}\delta_{\nu\rho})g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}(+\partial_{\alpha'}\partial_{\beta'}U_{\mu'\nu'}) \\
&\quad +(+\delta_{\alpha\lambda}\delta_{\beta\tau}\delta_{\mu\sigma}\delta_{\nu\rho})g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\nu'}U_{\beta'\mu'} \\
&\quad +(+\delta_{\alpha\lambda}\delta_{\beta\tau}\delta_{\mu\sigma}\delta_{\nu\rho})g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'\nu'} \\
&\quad +\delta_{\alpha\lambda}\delta_{\nu\tau}\delta_{\beta\sigma}\delta_{\mu\rho}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}(+\partial_{\alpha'}\partial_{\beta'}U_{\mu'\nu'}) \\
&\quad +\delta_{\alpha\lambda}\delta_{\nu\tau}\delta_{\beta\sigma}\delta_{\mu\rho}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\nu'}U_{\beta'\mu'} \\
&\quad +\delta_{\alpha\lambda}\delta_{\nu\tau}\delta_{\beta\sigma}\delta_{\mu\rho}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'\nu'} \\
&\quad +\delta_{\alpha\lambda}\delta_{\mu\tau}\delta_{\beta\sigma}\delta_{\nu\rho}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}(+\partial_{\alpha'}\partial_{\beta'}U_{\mu'\nu'}) \\
&\quad +\delta_{\alpha\lambda}\delta_{\mu\tau}\delta_{\beta\sigma}\delta_{\nu\rho}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\nu'}U_{\beta'\mu'} \\
&\quad +\delta_{\alpha\lambda}\delta_{\mu\tau}\delta_{\beta\sigma}\delta_{\nu\rho}g^{\alpha\alpha'}g^{\beta\beta'}g^{\mu\mu'}g^{\nu\nu'}\partial_{\alpha'}\partial_{\mu'}U_{\beta'\nu'}\Big] \\
&= -\frac{1}{3}\left[(+1)\cdot(+\partial_\lambda\partial_\tau U_{\sigma\rho}) + (+1)\cdot\partial_\lambda\partial_\rho U_{\tau\sigma} + (+1)\cdot\partial_\lambda\partial_\sigma U_{\tau\rho}\right.
\end{aligned}$$

$$
\begin{aligned}
&+(+\partial_\lambda\partial_\sigma U_{\rho\tau}) + \partial_\lambda\partial_\tau U_{\sigma\rho} + \partial_\lambda\partial_\rho U_{\sigma\tau} \\
&+(+\partial_\lambda\partial_\sigma U_{\tau\rho}) + \partial_\lambda\partial_\rho U_{\sigma\tau} + \partial_\lambda\partial_\tau U_{\sigma\rho}] \\
= \ & -(\partial_\lambda\partial_\tau U_{\sigma\rho}) + \partial_\lambda\partial_\sigma U_{\tau\rho} + \partial_\lambda\partial_\rho U_{\sigma\tau}) \ , \qquad (405)
\end{aligned}
$$

and get the term

$$
\begin{aligned}
\partial^\lambda\partial^\tau \frac{\partial\mathcal{L}_U}{\partial(\partial^\lambda\partial^\tau U^{\dagger\sigma\rho})} &= -\partial^\lambda\partial^\tau\left(\partial_\lambda\partial_\tau U_{\sigma\rho} + \partial_\lambda\partial_\sigma U_{\tau\rho} + \partial_\lambda\partial_\rho U_{\sigma\tau}\right) \\
&\overset{(\partial\cdot U=0)}{=\!=\!=} -\partial^\lambda\partial^\tau\partial_\lambda\partial_\tau U_{\sigma\rho} = -\partial^4 U_{\sigma\rho} = -(i\partial)^4 U_{\sigma\rho}\,, \qquad (406)
\end{aligned}
$$

with the gauge fixed condition $\partial_\mu U^{\mu\nu} = \partial_\nu U^{\mu\nu} = 0$.

2. For the mass term, we have the expansion

$$
\mathcal{L}_U = +m_U^4 U^\dagger_{\mu\nu} g^{\mu\mu'} g^{\nu\nu'} U_{\mu'\nu'}\,. \qquad (407)
$$

and the derivative

$$
\frac{\partial\mathcal{L}_U}{\partial U^{\dagger\sigma\rho}} = +m_U^4 \delta_{\mu\sigma}\delta_{\nu\rho} g^{\mu\mu'} g^{\nu\nu'} U_{\mu'\nu'} = +m_U^4 U^{\sigma\rho}\,. \qquad (408)
$$

3. For the free propagator, with the E.O.M in (422), we can get the equation for Feynman propagator $D_F^{\mu\nu\rho\sigma}$

$$
\begin{aligned}
-[\partial^4 - m_U^4 + ...]\frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})D_F^{\mu\nu\rho\sigma}(x-y) &= i\delta^\rho_\mu \delta^{(4)}(x-y) \qquad (409) \\
\text{or } -[p^4 - m_U^4 + ...]\frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})\widetilde{D}_F^{\mu\nu\rho\sigma}(p) &= i\delta^\rho_\mu \qquad (410)
\end{aligned}
$$

which has the solution

$$
\widetilde{D}_F^{\mu\nu\rho\sigma}(p) = \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} + ...)\frac{-i}{p^4 - m_U^4 + i\epsilon}\,, \qquad (411)
$$

where the ... denoted the gauge free term which would be omitted in the Lorentz gauge condition. As the case for (53,59,60), please pay attention to the extra minus sign in $\widetilde{D}_F^{\mu\nu\rho\sigma}(p)$ by contrast to the propagator of a P-2 type tensor field.

4. For the $\mathcal{L}_I$ part, we have the expansion for Version I in (238), as (for simplicity, here we treat $U^{\mu\nu}$ as a real-valued field)

$$
\begin{aligned}
\mathcal{L}_I &= -\alpha Q\Lambda\, U_{\mu\nu}\bar\psi\eta^{\mu\nu}\psi \\
&\quad -\beta Q\bar\psi F^{(U)}_{\alpha\mu\nu}(\gamma^\alpha\eta^{\mu\nu} + \gamma^\mu\eta^{\alpha\nu} + \gamma^\nu\eta^{\mu\alpha})\psi + ... \\
&= -\alpha Q\Lambda\, U_{\mu\nu} g^{\mu\mu'} g^{\nu\nu'}\bar\psi\eta_{\mu'\nu'}\psi \\
&\quad -\beta Q\bar\psi(+\partial_\alpha U_{\mu\nu} + \partial_\nu U_{\alpha\mu} + \partial_\mu U_{\alpha\nu}) g^{\alpha\alpha'} g^{\mu\mu'} g^{\nu\nu'}(\gamma_{\alpha'}\eta_{\mu'\nu'} + \gamma_{\mu'}\eta_{\alpha'\nu'} + \gamma_{\nu'}\eta_{\mu'\alpha'})\psi + ..., \qquad (412)
\end{aligned}
$$

and the variational derivative

$$
\begin{aligned}
&\frac{\partial\mathcal{L}_I}{\partial(\partial_\tau U_{\sigma\rho})} \\
= \ & -\beta Q\bar\psi(\delta_{\alpha\tau}\delta_{\mu\sigma}\delta_{\nu\rho} + \delta_{\nu\tau}\delta_{\alpha\sigma}\delta_{\mu\rho} + \delta_{\mu\tau}\delta_{\alpha\sigma}\delta_{\nu\rho}) g^{\alpha\alpha'} g^{\mu\mu'} g^{\nu\nu'}(\gamma_{\alpha'}\eta_{\mu'\nu'} + \gamma_{\mu'}\eta_{\alpha'\nu'} + \gamma_{\nu'}\eta_{\mu'\alpha'})\psi + ... \\
= \ & -\beta Q\bar\psi(g^{\tau\alpha'} g^{\sigma\mu'} g^{\rho\nu'} + g^{\sigma\alpha'} g^{\rho\mu'} g^{\tau\nu'} + g^{\sigma\alpha'} g^{\tau\mu'} g^{\rho\nu'})(\gamma_{\alpha'}\eta_{\mu'\nu'} + \gamma_{\mu'}\eta_{\alpha'\nu'} + \gamma_{\nu'}\eta_{\mu'\alpha'})\psi + ...\,, \qquad (413)
\end{aligned}
$$

with the term

$$\partial_\tau \frac{\partial \mathcal{L}_I}{\partial(\partial_\tau U_{\sigma\rho})} = -\beta Q \partial_\tau \bar{\psi}(g^{\tau\alpha'}g^{\sigma\mu'}g^{\rho\nu'} + g^{\sigma\alpha'}g^{\rho\mu'}g^{\tau\nu'} + g^{\sigma\alpha'}g^{\tau\mu'}g^{\rho\nu'}) \cdot(\gamma_{\alpha'}\eta_{\mu'\nu'} + \gamma_{\mu'}\eta_{\alpha'\nu'} + \gamma_{\nu'}\eta_{\mu'\alpha'})\psi + ... = -3\beta Q \partial^\tau \bar{\psi}(\gamma_\tau \eta_{\sigma\rho} + \gamma_\sigma \eta_{\tau\rho} + \gamma_\rho \eta_{\sigma\tau})\psi + \dots . \quad (414)$$

Besides, we have the term

$$\frac{\partial \mathcal{L}_I}{\partial U^{\sigma\rho}} = -\alpha Q \Lambda\, \delta_{\mu\sigma}\delta_{\nu\rho} g^{\mu\mu'}g^{\nu\nu'}\bar{\psi}\eta_{\mu'\nu'}\psi = -\alpha Q \Lambda\, \bar{\psi}\eta^{\sigma\rho}\psi\,. \quad (415)$$

Then, we can get the dynamical equation for the field $U_{\sigma\rho}$ with interaction of Version I as

$$-\,\partial^4 U_{\sigma\rho} = -m_U^4 U^{\sigma\rho} + J^{\sigma\rho}\,, \quad (416)$$

with

$$J^{\sigma\rho} = +\alpha Q \Lambda\, \bar{\psi}\eta^{\sigma\rho}\psi -3\beta Q \partial^\tau \bar{\psi}(\gamma_\tau \eta_{\sigma\rho} + \gamma_\sigma \eta_{\tau\rho} + \gamma_\rho \eta_{\sigma\tau})\psi\,, \quad (417)$$

and the gauge fixed condition $\partial_\mu U^{\mu\nu} = \partial_\nu U^{\mu\nu} = 0$.

5. For the $\mathcal{L}_I$ of Version II in (246), with (213-216), we have the expansion as (for simplicity, here we treat $U^{\mu\nu}$ as a real-valued field)

$$\begin{aligned}\mathcal{L}_I &= -\alpha Q\, \Lambda U_{\mu\nu}) \cdot \bar{\psi}\frac{1}{M}(\gamma^\mu i\partial^\nu + \gamma^\nu i \partial^\mu)\psi \\ &\quad -\beta Q F^{(U)}_{\alpha\mu\nu} \cdot \bar{\psi}[\frac{1}{M}(\eta^{\mu\nu}i\partial^\alpha + \eta^{\alpha\nu}i\partial^\mu + \eta^{\alpha\mu}i\partial^\nu)]\psi \\ &= -\alpha Q\, \Lambda U_{\mu\nu})g^{\mu\mu'}g^{\nu\nu'}\bar{\psi}\frac{1}{M}(\gamma_{\mu'}i\partial_{\nu'} + \gamma_{\nu'}i\partial_{\mu'})\psi \\ &\quad -\beta Q(+\partial_\alpha U_{\mu\nu} + \partial_\nu U_{\alpha\mu} + \partial_\mu U_{\alpha\nu})g^{\alpha\alpha'}g^{\mu\mu'}g^{\nu\nu'} \\ &\quad \cdot\bar{\psi}[\frac{1}{M}(\eta_{\mu'\nu'}i\partial_{\alpha'} + \eta_{\alpha'\nu'}i\partial_{\mu'} + \eta_{\alpha'\mu'}i\partial_{\nu'})]\psi + \dots,\end{aligned} \quad (418)$$

and the variational derivative

$$\begin{aligned}&\frac{\partial \mathcal{L}_I}{\partial(\partial_\tau U^{\sigma\rho})} \\ &= -\beta Q(+\delta_{\alpha\tau}\delta_{\mu\sigma}\delta_{\nu\rho} + \delta_{\nu\tau}\delta_{\alpha\sigma}\delta_{\mu\rho} + \delta_{\mu\tau}\delta_{\alpha\sigma}\delta_{\nu\rho})g^{\alpha\alpha'}g^{\mu\mu'}g^{\nu\nu'} \\ &\quad \bar{\psi}[\frac{1}{M}(\eta_{\mu'\nu'}i\partial_{\alpha'} + \eta_{\alpha'\nu'}i\partial_{\mu'} + \eta_{\alpha'\mu'}i\partial_{\nu'})]\psi + \dots \\ &= -\beta Q(g^{\tau\alpha'}g^{\sigma\mu'}g^{\rho\nu'} + g^{\sigma\alpha'}g^{\rho\mu'}g^{\tau\nu'} + g^{\sigma\alpha'}g^{\tau\mu'}g^{\rho\nu'}) \\ &\quad \bar{\psi}[\frac{1}{M}(\eta_{\mu'\nu'}i\partial_{\alpha'} + \eta_{\alpha'\nu'}i\partial_{\mu'} + \eta_{\alpha'\mu'}i\partial_{\nu'})]\psi + \dots,\end{aligned} \quad (419)$$

with the term

$$\partial_\tau \frac{\partial \mathcal{L}_I}{\partial(\partial_\tau U^{\sigma\rho})} = -3\beta Q \partial_\tau(\bar{\psi}[\frac{1}{M}(\eta_{\sigma\rho}i\partial_\tau + \eta_{\tau\rho}i\partial_\sigma + \eta_{\tau\sigma}i\partial_\rho)]\psi) + \dots . \quad (420)$$

Besides, we have the term

$$\frac{\partial \mathcal{L}_I}{\partial U^{\sigma\rho}} = -\alpha Q\, \Lambda \bar{\psi}\frac{1}{M}(\gamma_\sigma i\partial_\rho + \gamma_\rho i \partial_\sigma)\psi\,. \quad (421)$$

Then, we can get the dynamical equation for the field $U_{\sigma\rho}$ with interaction of Version II as

$$-\partial^4 U_{\sigma\rho} = -m_U^4 U^{\sigma\rho} + J^{\sigma\rho}\,, \tag{422}$$

with

$$\begin{aligned} J^{\sigma\rho} &= +\alpha Q\,\Lambda\bar{\psi}\frac{1}{M}(\gamma_\sigma i\partial_\rho + \gamma_\rho i\partial_\sigma)\psi \\ &\quad -3\beta Q\partial_\tau(\bar{\psi}[\frac{1}{M}(\eta_{\sigma\rho}i\partial_\tau + \eta_{\tau\rho}i\partial_\sigma + \eta_{\tau\sigma}i\partial_\rho)]\psi) + ...\,, \end{aligned} \tag{423}$$

and the gauge fixed condition $\partial_\mu U^{\mu\nu} = \partial_\nu U^{\mu\nu} = 0$.