Finite and infinite product transformations

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Several infinite products are studied along with their finite counterparts. For certain values of the parameters these infinite products reduce to modular forms. The finite product formulas give an elementary proof of a particular modular transformation.

1. B. Cais in an unpublished manuscript [1] shows how to get the product transformation formula

$$\prod_{n=1}^{\infty}\left(\frac{1-e^{-\pi\alpha n}}{1+e^{-\pi\alpha n}}\right)^{(-1)^n}=\frac{1}{\sqrt{\alpha}}\prod_{n=1}^{\infty}\left(\frac{1-e^{-\pi n/\alpha}}{1+e^{-\pi n/\alpha}}\right)^{(-1)^n} \tag{1}$$

from the well known series transformation [2]

$$\sum_{n=1}^{\infty}\frac{n(-1)^n}{\sinh\pi\alpha n}+\alpha^{-2}\sum_{n=1}^{\infty}\frac{\alpha n(-1)^n}{\sinh\pi n/\alpha}=-\frac{1}{2\pi\alpha} \tag{2}$$

by integration with respect to α using the indefinite integration

$$\int\frac{ds}{\sinh s}=\ln\left(\frac{\sinh s}{1+\cosh s}\right).$$

He also generalizes this approach to more complicated products like

$$\prod_{n=1}^{\infty}\left(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh\frac{2\pi\alpha n}{5}}\right)^{\left(\frac{n}{5}\right)}=\prod_{n=1}^{\infty}\left(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh\frac{2\pi n}{5\alpha}}\right)^{\left(\frac{n}{5}\right)}, \tag{3}$$

where $\left(\frac{n}{5}\right)$ denotes the Legendre symbol. Recall also the functional equation for the Dedekind eta product

$$e^{-\frac{\pi\alpha}{12}}\prod_{n=1}^{\infty}\left(1-e^{-2\pi\alpha n}\right)=\frac{1}{\sqrt{\alpha}}\cdot e^{-\frac{\pi}{12\alpha}}\prod_{n=1}^{\infty}\left(1-e^{-2\pi n/\alpha}\right). \tag{4}$$

The aim of this paper is to generalize these transformation formulas and obtain finite products that reduce to the above formulas in the infinite limit. It should be noted that the infinite products in (1),(3),(4) are modular forms, but the generalized products in the subsequent sections are not.

It is assumed throughout this paper that $\alpha>0$. However by analytic continuation all formulas are valid when Re $\alpha>0$.

2. Expanding cosech into partial fractions and interchanging the order of summation we get

$$\begin{aligned}f(\alpha,\beta)&=\sum_{n=-\infty}^{\infty}\frac{\alpha(-1)^n}{\sqrt{\alpha^2n^2+\alpha\beta^2}\sinh(\pi\sqrt{\alpha^2n^2+\alpha\beta^2})}\\&=\sum_{n=-\infty}^{\infty}\pi\alpha(-1)^n\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{\pi^2(\alpha^2n^2+\alpha\beta^2)+\pi^2m^2}\\&=\frac{1}{\pi}\sum_{m,n=-\infty}^{\infty}\frac{(-1)^{m+n}}{\alpha n^2+\alpha^{-1}m^2+\beta^2}=f(1/\alpha,\beta).\end{aligned} \tag{5}$$

An identity equivalent to $f(\alpha,\beta)=f(1/\alpha,\beta)$ is given in 3 as equations (1.5.2) and (1.5.3) with $l=0$.

Now we multiply $f(\alpha,\beta)$ by $\pi\beta$ and integrate termwise with respect to β. Using the integral

$$\int \frac{\pi\alpha\beta\, d\beta}{\sqrt{\alpha^2n^2+\alpha\beta^2}\,\sinh(\pi\sqrt{\alpha^2n^2+\alpha\beta^2})}$$
$$=\int \frac{d(\pi\sqrt{\alpha^2n^2+\alpha\beta^2})}{\sinh(\pi\sqrt{\alpha^2n^2+\alpha\beta^2})}=\ln\left(\tanh\frac{\pi\sqrt{\alpha^2n^2+\alpha\beta^2}}{2}\right)$$

one can see that the sum

$$\sum_{n=-\infty}^{\infty}(-1)^n\ln\left(\frac{\tanh(\pi\sqrt{\alpha^2n^2+\alpha\beta^2}/2)}{\tanh(\pi\sqrt{\alpha^{-2}n^2+\alpha^{-1}\beta^2}/2)}\right) \tag{6}$$

doesn't depend on β. Then it follows from the limit $\beta\to\infty$ that (6) is 0. This means that

$$\prod_{n=-\infty}^{\infty}\left(\frac{\tanh(\pi\sqrt{\alpha^2n^2+\alpha\beta^2}/2)}{\tanh(\pi\sqrt{\alpha^{-2}n^2+\alpha^{-1}\beta^2}/2)}\right)^{(-1)^n}=1, \tag{7}$$

or in the form easier to compare with (1)

$$\prod_{n=1}^{\infty}\left(\frac{1-e^{-\pi\alpha\sqrt{n^2+\beta^2}}}{1+e^{-\pi\alpha\sqrt{n^2+\beta^2}}}\right)^{(-1)^n}=\sqrt{\frac{\tanh\frac{\pi\beta}{2}}{\tanh\frac{\pi\alpha\beta}{2}}}\prod_{n=1}^{\infty}\left(\frac{1-e^{-\pi\sqrt{n^2/\alpha^2+\beta^2}}}{1+e^{-\pi\sqrt{n^2/\alpha^2+\beta^2}}}\right)^{(-1)^n}. \tag{8}$$

3. There is another way to write (5) in symmetric form. We start with

$$f(\alpha,\beta)=\sum_{n=-\infty}^{\infty}\frac{\alpha(-1)^n}{\sqrt{\alpha^2n^2+\alpha\beta^2}}\cdot 2\sum_{m=0}^{\infty}e^{-\pi(2m+1)\sqrt{\alpha^2n^2+\alpha\beta^2}}$$
$$=2\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\sqrt{n^2+\beta^2/\alpha}}e^{-\pi(2m+1)\alpha\sqrt{n^2+\beta^2/\alpha}}.$$

and then apply Poisson summation formula to the sum over n in the following form

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\sqrt{n^2+y^2}}e^{-x\sqrt{n^2+y^2}}=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\pi it}}{\sqrt{t^2+y^2}}e^{-x\sqrt{t^2+y^2}}\cdot e^{-2\pi int}dt.$$

The integral is calculated via formula 3.961.2 from [4] and equals $2K_0\left(y\sqrt{\pi^2(2n+1)^2+x^2}\right)$. So

$$f(\alpha,\beta)=4\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}K_0\left(\frac{\beta}{\sqrt{\alpha}}\sqrt{\pi^2(2n+1)^2+\pi^2(2m+1)^2\alpha^2}\right)$$
$$=\sum_{m,n=-\infty}^{\infty}2K_0\left(\pi\beta\sqrt{\frac{(2n+1)^2}{\alpha}+(2m+1)^2\alpha}\right).$$

4. The function

$$h(z)=\frac{1}{\sqrt{z^2+\beta^2}\left(1+e^{\pi\alpha\sqrt{z^2+\beta^2}}\right)}\frac{1}{1+e^{-\pi iz}} \tag{9}$$

is analytic in the complex plane with a cut $[i\beta,+i\infty)$. It has simple poles on the real line at $z_n=2n+1$, and on the imaginary line at $\zeta_n=\sqrt{(2n+1)^2\alpha^{-2}+\beta^2}$, $n\in\mathbb{Z}$. Consider contour C in Fig.1, with small circles

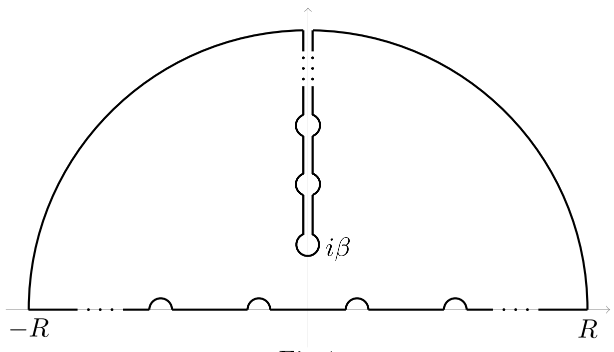

Fig.1

and semicircles of radii ε around poles of $h(z)$. According to residue theorem $\int_C h(z)dz = 0$. Integrals along large arcs are 0 in the limit $R \to \infty$. Sum of integrals along straight horizontal segments in the limit $R \to \infty$, $\varepsilon \to 0$ is

$$I_h = 2\int_0^\infty \frac{dx}{\sqrt{x^2+\beta^2}\left(1+e^{\pi\alpha\sqrt{x^2+\beta^2}}\right)}.$$

Similarly, sum of integrals along straight vertical segments in the limit $R \to \infty$, $\varepsilon \to 0$ is

$$I_v = 2\int_{i\beta}^{+i\infty} \frac{dz}{\sqrt{z^2+\beta^2}\,(1+e^{-\pi i z})} = -2\int_0^\infty \frac{dy}{\sqrt{y^2+\beta^2}\left(1+e^{\pi\sqrt{y^2+\beta^2}}\right)}.$$

Therefore in the limit $R \to \infty$, $\varepsilon \to 0$

$$I_h + I_v - \pi i \sum_{n=-\infty}^{\infty} \operatorname*{res}_{z=z_n} h(z) - 2\pi i \sum_{n=0}^{\infty} \operatorname*{res}_{z=\zeta_n} h(z) = 0.$$

Putting it altogether we arrive at

$$\int_0^\infty \frac{dx}{\sqrt{x^2+\beta^2}\left(1+e^{\pi\alpha\sqrt{x^2+\beta^2}}\right)} - \int_0^\infty \frac{dy}{\sqrt{y^2+\beta^2}\left(1+e^{\pi\sqrt{y^2+\beta^2}}\right)}$$
$$-\sum_{n=0}^{\infty} \frac{1}{\sqrt{(2n+1)^2+\beta^2}\left(1+e^{\pi\alpha\sqrt{(2n+1)^2+\beta^2}}\right)} + \sum_{n=0}^{\infty} \frac{1/\alpha}{\sqrt{(2n+1)^2/\alpha^2+\beta^2}\left(1+e^{\pi\sqrt{(2n+1)^2/\alpha^2+\beta^2}}\right)} = 0.$$

To convert it to a product one can multiply by $\pi\alpha\beta$ and integrate with respect to β using

$$\int \frac{\pi\alpha\beta\, d\beta}{\sqrt{x^2+\beta^2}\left(1+e^{\pi\alpha\sqrt{x^2+\beta^2}}\right)} = -\ln\left(1+e^{-\pi\alpha\sqrt{x^2+\beta^2}}\right)$$

to get the following symmetric relation

$$\prod_{n=0}^{\infty} \frac{1+e^{-\pi\alpha\sqrt{(2n+1)^2+\beta^2}}}{1+e^{-\pi\sqrt{(2n+1)^2/\alpha^2+\beta^2}}} = \exp\left\{\frac{1}{2}\int_0^\infty \ln\frac{1+e^{-\pi\alpha\sqrt{x^2+\beta^2}}}{1+e^{-\pi\sqrt{x^2/\alpha^2+\beta^2}}}\,dx\right\}. \tag{10}$$

Note. Contour integrals similar to the one considered in this section has been investigated in ch. 1.5 of the book 3, for example in eq. (1.5.1), with the aim of application to lattice sums. We study a more general integral in section 7.

5. The same analysis as in the preceding section is applicable to the function

$$h(z)=\frac{1}{\sqrt{z^2+\beta^2}\left(1-e^{\pi\alpha\sqrt{z^2+\beta^2}}\right)}\frac{1}{1-e^{-\pi i z}} \tag{11}$$

with the difference that the poles are now $z_n=2n$ and $\zeta_n=\sqrt{4n^2\alpha^{-2}+\beta^2}$, $n\in\mathbb{Z}$. The result

$$\prod_{n=1}^{\infty}\frac{1-e^{-2\pi\alpha\sqrt{n^2+\beta^2}}}{1-e^{-2\pi\sqrt{n^2/\alpha^2+\beta^2}}}=\sqrt{\frac{1-e^{-2\pi\beta}}{1-e^{-2\pi\alpha\beta}}}\cdot\exp\left\{\int_0^{\infty}\ln\frac{1-e^{-2\pi\alpha\sqrt{x^2+\beta^2}}}{1-e^{-2\pi\sqrt{x^2/\alpha^2+\beta^2}}}\,dx\right\}, \tag{12}$$

gives a function for which sum equals integral:

$$\sum_{n=-\infty}^{\infty}\ln\frac{1-e^{-2\pi\alpha\sqrt{n^2+\beta^2}}}{1-e^{-2\pi\sqrt{n^2/\alpha^2+\beta^2}}}=\int\limits_{-\infty}^{\infty}\ln\frac{1-e^{-2\pi\alpha\sqrt{x^2+\beta^2}}}{1-e^{-2\pi\sqrt{x^2/\alpha^2+\beta^2}}}\,dx. \tag{13}$$

Analogous result is true for (10). In fact more generally

$$\begin{aligned}\sum_{n=-\infty}^{\infty}&\ln\left\{\frac{1-e^{-2\pi\alpha\sqrt{\beta^2+(n+\theta)^2}}}{1-e^{-2\pi\sqrt{\beta^2+\frac{n^2}{\alpha^2}}+2\pi i\theta}}\,\frac{1-e^{-2\pi\alpha\sqrt{\beta^2+(n-\theta)^2}}}{1-e^{-2\pi\sqrt{\beta^2+\frac{n^2}{\alpha^2}}-2\pi i\theta}}\right\}\\&=\int\limits_{-\infty}^{\infty}\ln\left\{\frac{1-e^{-2\pi\alpha\sqrt{\beta^2+(x+\theta)^2}}}{1-e^{-2\pi\sqrt{\beta^2+\frac{x^2}{\alpha^2}}+2\pi i\theta}}\,\frac{1-e^{-2\pi\alpha\sqrt{\beta^2+(x-\theta)^2}}}{1-e^{-2\pi\sqrt{\beta^2+\frac{x^2}{\alpha^2}}-2\pi i\theta}}\right\}dx.\end{aligned} \tag{14}$$

A list of functions for which sum equals integral along with references can be found in [10]. Note that when $\beta\to 0$ eq. (12) reduces to the functional equation for the Dedekind eta function (4), while (14) reduces to the Jacobi imaginary transform for theta functions[5].

6. The choice

$$h(z)=\frac{\cos\frac{3\pi z}{5}-\cos\frac{\pi z}{5}}{\sin\pi z}\,\frac{\cosh\frac{3\pi\alpha\sqrt{z^2+\beta^2}}{5}-\cosh\frac{\pi\alpha\sqrt{z^2+\beta^2}}{5}}{\sqrt{z^2+\beta^2}\,\sinh(\pi\alpha\sqrt{z^2+\beta^2})}$$

with the help of formulas

$$\cos\frac{3\pi n}{5}-\cos\frac{\pi n}{5}=(-1)^n\frac{\sqrt{5}}{2}\left(\frac{n}{5}\right),$$

$$\int\frac{\cosh\frac{3\pi t}{5}-\cosh\frac{\pi t}{5}}{\sinh\pi t}dt=\frac{\sqrt{5}}{2\pi}\ln\left(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh\frac{2\pi t}{5}}\right)$$

from [1], leads to

$$\prod_{n=1}^{\infty}\left(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh\frac{2\pi\alpha\sqrt{n^2+\beta^2}}{5}}\right)^{\left(\frac{n}{5}\right)}=\prod_{n=1}^{\infty}\left(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh\frac{2\pi\sqrt{n^2/\alpha^2+\beta^2}}{5}}\right)^{\left(\frac{n}{5}\right)}. \tag{15}$$

This is generalization of (3). From this it should be clear how to get generalized symmetric products from absolute invariants in section 5 of [1]. However not all absolute invariants seem to be amenable to such generalization (e.g. see Proposition 25 in [1]).

7. Note that in the previous section there were no integrals analogous to integrals I_v, I_h from section 3, thanks to the oddity of $h(z)$ they canceled out. With suitable function $h(z)$ this proof can be adapted to prove (8) as well.

In fact more is true:

$$f(\alpha)=\prod_{n=1}^{\infty}\left(\frac{\cosh\left(\pi\cos\theta\sqrt{n^2\alpha^2+\alpha y^2}\right)-\cos\left(\pi n\alpha\sin\theta\right)}{\cosh\left(\pi\cos\theta\sqrt{n^2\alpha^2+\alpha y^2}\right)+\cos\left(\pi n\alpha\sin\theta\right)}\right)^{(-1)^n}=f\left(\frac{1}{\alpha}\right)\frac{\tanh\frac{\pi y\cos\theta}{2\sqrt{\alpha}}}{\tanh\frac{\pi y\sqrt{\alpha}\cos\theta}{2}}. \tag{16}$$

Without presenting all the details of the proof we note the essential steps only. Take

$$h(z)=\frac{1}{\sqrt{z^2+\beta^2}}\frac{\sinh\left(\pi\alpha\cos\theta\sqrt{z^2+\beta^2}\right)\cos\left(\pi\alpha z\sin\theta\right)}{\cosh\left(2\pi\alpha\cos\theta\sqrt{z^2+\beta^2}\right)-\cos(2\pi\alpha z\sin\theta)}\cdot\frac{1}{\sin\pi z}.$$

Clearly $h(z)$ is odd, hence there will not be any integral contributions analogous to I_h and I_v from section 3. Besides $z_n=n$ this function has simple poles at

$$\zeta_n=\frac{n\sin\theta}{\alpha}+i\cos\theta\sqrt{n^2/\alpha^2+\beta^2},\quad n\in\mathbb{Z}.$$

ζ_n satisfies the relation

$$\zeta_n\sin\theta-n/\alpha=i\cos\theta\sqrt{\zeta_n^2+\beta^2}.$$

The poles z_n are outside the contour of integration, while the poles ζ_n are inside. Residues at $z_n=n$ are easily calculated and the sum over these residues can be converted to a product with the help of the integral

$$\pi\alpha\cos\theta\int\frac{\sinh\left(\pi\alpha t\cos\theta\right)\cos\left(\pi\alpha\sin\theta\right)}{\cosh\left(2\pi\alpha t\cos\theta\right)-\cos(2\pi\alpha z\sin\theta)}dt=\frac{1}{4}\ln\frac{\cosh\left(\pi\alpha t\cos\theta\right)-\cos\left(\pi\alpha\sin\theta\right)}{\cosh\left(\pi\alpha t\cos\theta\right)+\cos\left(\pi\alpha\sin\theta\right)}.$$

Now we calculate the residues at ζ_n:

$$\begin{aligned}\operatorname*{res}_{z=\zeta_n}h(z)&=\frac{1}{2\pi\alpha\sin\pi\zeta_n}\frac{\sinh\left(\pi\alpha\cos\theta\sqrt{\zeta_n^2+\beta^2}\right)\cos\left(\pi\alpha\zeta_n\sin\theta\right)}{\zeta_n\cos\theta\sinh\left(2\pi\alpha\cos\theta\sqrt{\zeta_n^2+\beta^2}\right)+\sqrt{\zeta_n^2+\beta^2}\sin\theta\sin(2\pi\alpha\zeta_n\sin\theta)}\\&=\frac{1}{4\pi\alpha\sin\pi\zeta_n}\frac{(-1)^n}{\zeta_n\cos\theta+i\sin\theta\sqrt{\zeta_n^2+\beta^2}}\\&=\frac{1}{4\pi\alpha\sin\pi\zeta_n}\frac{(-1)^{n-1}i}{\sqrt{n^2/\alpha^2+\beta^2}}.\end{aligned}$$

From this it follows that

$$\operatorname*{res}_{z=\zeta_n}h(z)+\operatorname*{res}_{z=\zeta_{-n}}h(z)=\frac{(-1)^{n-1}}{\pi\alpha\sqrt{n^2/\alpha^2+\beta^2}}\frac{\sinh\left(\pi\cos\theta\sqrt{n^2/\alpha^2+\beta^2}\right)\cos\left(\pi z\alpha^{-1}\sin\theta\right)}{\cosh\left(2\pi\cos\theta\sqrt{n^2/\alpha^2+\beta^2}\right)-\cos(2\pi z\alpha^{-1}\sin\theta)}.$$

The RHS of this expression has the same form as $\operatorname*{res}_{z=z_n}h(z)$, and therefore can be converted to a product by integration.

8. The result of this section is quite similar to the previous section

$$f(\alpha)=\prod_{n=0}^{\infty}\left(\frac{\cosh\left(\pi\cos\theta\sqrt{\left(n+\frac{1}{2}\right)^2\alpha^2+\alpha y^2}\right)-\sin\left(\pi\left(n+\frac{1}{2}\right)\alpha\sin\theta\right)}{\cosh\left(\pi\cos\theta\sqrt{\left(n+\frac{1}{2}\right)^2\alpha^2+\alpha y^2}\right)+\sin\left(\pi\left(n+\frac{1}{2}\right)\alpha\sin\theta\right)}\right)^{(-1)^n}=f\left(\frac{1}{\alpha}\right). \tag{17}$$

In this case

$$h(z)=\frac{1}{\sqrt{z^2+\beta^2}}\frac{\sinh\left(\pi\alpha\cos\theta\sqrt{z^2+\beta^2}\right)\sin\left(\pi\alpha z\sin\theta\right)}{\cosh\left(2\pi\alpha\cos\theta\sqrt{z^2+\beta^2}\right)+\cos(2\pi\alpha z\sin\theta)}\cdot\frac{1}{\cos\pi z},$$

and

$$\pi\alpha\cos\theta\int\frac{\sinh\left(\pi\alpha t\cos\theta\right)\sin\left(\pi\alpha\sin\theta\right)}{\cosh\left(2\pi\alpha t\cos\theta\right)+\cos(2\pi\alpha z\sin\theta)}dt=\frac{1}{4}\ln\frac{\cosh\left(\pi\alpha t\cos\theta\right)-\sin\left(\pi\alpha\sin\theta\right)}{\cosh\left(\pi\alpha t\cos\theta\right)+\sin\left(\pi\alpha\sin\theta\right)}.$$

Note that $f\left(\frac{2N}{\sin\theta}\right)=1$ trivially for natural N. However $f\left(\frac{\sin\theta}{2N}\right)=1$ is quite non-trivial.

9. So far all products or series have been symmetric. But there are also identities for non-symmetric products as well. Here is one such example without proof:

$$\prod_{n=-\infty}^{\infty}\frac{\tanh\pi\sqrt{a^2n^2+\frac{1}{4}}}{\left(1-e^{-\frac{\pi}{a}\sqrt{n^2+1}}\right)^{(-1)^n}}=\exp\left\{\frac{1}{a}\int_{-\infty}^{\infty}\ln\left(\tanh\pi\sqrt{t^2+\tfrac{1}{4}}\right)dt\right\}. \tag{18}$$

10. The coupled equations defining α_j and β_k

$$\cosh\alpha_j+\cos\frac{\pi(j-\frac{1}{2})}{n}=\cosh\beta_k+\cos\frac{\pi(k-\frac{1}{2})}{m}\qquad(1\le j\le n,\ 1\le k\le m) \tag{19}$$

arise in the solution of Dirichlet and Helmholtz equations on a lattice [6]. These set of α_j and β_k satisfy the reciprocal relation

$$\prod_{j=1}^{n}2\cosh m\alpha_j=\prod_{k=1}^{m}2\cosh n\beta_k. \tag{20}$$

The proof is very simple and uses the well known formula (see [7], formula 1026):

$$2^{m-1}\prod_{j=1}^{m}\left(\cosh\alpha-\cos\frac{\pi(j-1/2)}{m}\right)=\cosh m\alpha. \tag{21}$$

Indeed denoting by x the common value of the equations (19)

$$\begin{aligned}\prod_{j=1}^{n}2\cosh m\alpha_j&=\prod_{j=1}^{n}2\cdot2^{m-1}\prod_{k=1}^{m}\left(\cosh\alpha_j-\cos\frac{\pi(k-1/2)}{m}\right)\\&=2^{mn}\prod_{j=1}^{n}\prod_{k=1}^{m}\left(x-\cos\frac{\pi(j-1/2)}{n}-\cos\frac{\pi(k-1/2)}{m}\right).\end{aligned}$$

This expression is symmetric in m and n and therefore imply (20).

11. Similarly, if

$$\cosh\alpha_j+\cos\frac{\pi j}{n+1}=\cosh\beta_k+\cos\frac{\pi k}{m+1}\qquad(1\le j\le n,\ 1\le k\le m) \tag{22}$$

then

$$\prod_{j=1}^{n}\frac{\sinh(m+1)\alpha_j}{\sinh\alpha_j}=\prod_{k=1}^{m}\frac{\sinh(n+1)\beta_k}{\sinh\beta_k}. \tag{23}$$

The proof is similar to the one in previous section, this time using the well known formula

$$2^{m-1}\prod_{j=1}^{m-1}\left(\cosh\alpha-\cos\frac{\pi j}{m}\right)=\frac{\sinh m\alpha}{\sinh\alpha}. \tag{24}$$

12. If

$$\cosh\frac{\alpha_j}{2}\cos\frac{\pi(j-1/2)}{2n}=\cosh\frac{\beta_k}{2}\cos\frac{\pi(k-1/2)}{2m}\qquad(1\le j\le n,\ 1\le k\le m)\tag{25}$$

then

$$\prod_{j=1}^{n}\cosh m\alpha_j=\prod_{k=1}^{m}\cosh n\beta_k.\tag{26}$$

In this case, besides (21) we need

$$\prod_{j=1}^{n}\cos\frac{\pi(j-1/2)}{2n}=\frac{\sqrt{2}}{2^n}.\tag{27}$$

13. Here is a less trivial example inspired by the 'dispersion relation' $\cosh\frac{\omega}{2}=\frac{2}{\cos p}-\cos p$ on the triangular and hexagonal lattices [8]: If

$$\cosh\frac{\alpha_j}{2}=\frac{x}{\cos\frac{\pi(2j-1)}{4n}}-\cos\frac{\pi(2j-1)}{4n},\quad \cosh\frac{\beta_k}{2}=\frac{x}{\cos\frac{\pi(2k-1)}{4m}}-\cos\frac{\pi(2k-1)}{4m}\quad(1\le j\le n,\ 1\le k\le m),$$

then

$$\prod_{j=1}^{n}\left(\cosh m\alpha_j+\cos\frac{m\pi(2j-1)}{2n}\right)=\prod_{k=1}^{m}\left(\cosh n\beta_k+\cos\frac{n\pi(2k-1)}{2m}\right).\tag{28}$$

According to the following generalization of (21)

$$2^{m-1}\prod_{j=1}^{m}\left[\cosh\alpha-\cos\left(y+\frac{2\pi j}{m}\right)\right]=\cosh m\alpha-\cos my,\tag{29}$$

we write

$$\prod_{j=1}^{n}\left(\cosh m\alpha_j+\cos\frac{m\pi(2j-1)}{2n}\right)=\prod_{j=1}^{n}2^{m-1}\prod_{k=1}^{m}\left[\cosh\alpha_j-\cos\left(\frac{\pi(2j-1)}{2n}+\frac{\pi(2k-1)}{m}\right)\right]$$
$$=2^{n(2m-1)}\prod_{k=1}^{m}\prod_{j=1}^{n}\left[\cosh\frac{\alpha_j}{2}-\cos\left(\frac{\pi(2j-1)}{4n}+\frac{\pi(2k-1)}{2m}\right)\right]\left[\cosh\frac{\alpha_j}{2}+\cos\left(\frac{\pi(2j-1)}{4n}+\frac{\pi(2k-1)}{2m}\right)\right].$$

Since

$$\prod_{j=1}^{n}\left[\cosh\frac{\alpha_j}{2}-\cos\left(\frac{\pi(2j-1)}{4n}+\frac{\pi(2k-1)}{2m}\right)\right]$$
$$=\prod_{j=1}^{n}\left[\frac{x}{\cos\frac{\pi(2j-1)}{4n}}-\cos\frac{\pi(2j-1)}{4n}-\cos\left(\frac{\pi(2j-1)}{4n}+\frac{\pi(2k-1)}{2m}\right)\right]$$
$$=\frac{2^n}{\sqrt{2}}\prod_{j=1}^{n}\left[x-2\cos\frac{\pi(2j-1)}{4n}\cos\frac{\pi(2k-1)}{4m}\cos\left(\frac{\pi(2j-1)}{4n}+\frac{\pi(2k-1)}{4m}\right)\right],$$

$$\prod_{j=1}^{n}\left[\cosh\frac{\alpha_j}{2}+\cos\left(\frac{\pi(2j-1)}{4n}+\frac{\pi(2k-1)}{2m}\right)\right]$$
$$=\prod_{j=1}^{n}\left[\frac{x}{\sin\frac{\pi(2j-1)}{4n}}-\sin\frac{\pi(2j-1)}{4n}+\sin\left(\frac{\pi(2j-1)}{4n}-\frac{\pi(2k-1)}{2m}\right)\right]$$

$$= \frac{2^n}{\sqrt{2}} \prod_{j=1}^{n} \left[x - 2\sin\frac{\pi(2j-1)}{4n} \sin\frac{\pi(2k-1)}{4m} \cos\left(\frac{\pi(2j-1)}{4n} - \frac{\pi(2k-1)}{4m}\right)\right],$$

the product on the LHS of (28) is symmetric in n and m. This completes the proof.

14. If

$$\cosh\alpha_j + \cos\frac{\pi j}{2n} = \cosh\beta_k + \cos\frac{\pi k}{2m}, \qquad (1 \le j \le 2n-1,\ 1 \le k \le 2m-1), \tag{30}$$

then

$$\prod_{j=1}^{2n-1} \left(\frac{\tanh m\alpha_j}{\sinh\alpha_j}\right)^{(-1)^j} = \prod_{k=1}^{2m-1} \left(\frac{\tanh n\beta_k}{\sinh\beta_k}\right)^{(-1)^k}. \tag{31}$$

Denote the common value of equations (30) by x. We use (21) and (24) to write (31) in symmetric form:

$$\prod_{j=1}^{2n-1} \left(\frac{\tanh m\alpha_j}{\sinh\alpha_j}\right)^{(-1)^j} = \frac{\sinh\alpha_{2n-1}}{\tanh m\alpha_{2n-1}} \prod_{j=1}^{n-1} \frac{\tanh m\alpha_{2j}}{\sinh\alpha_{2j}} \frac{\sinh\alpha_{2j-1}}{\tanh m\alpha_{2j-1}} \tag{32}$$

$$= \frac{\sinh\alpha_{2n-1}}{\tanh m\alpha_{2n-1}} \prod_{j=1}^{n-1} \frac{\cosh\alpha_{2j-1} - \cos\frac{\pi(m-1/2)}{m}}{\cosh\alpha_{2j} - \cos\frac{\pi(m-1/2)}{m}} \tag{33}$$

$$\times \prod_{j=1}^{n-1}\prod_{k=1}^{m-1} \frac{\cosh\alpha_{2j-1} - \cos\frac{\pi(k-1/2)}{m}}{\cosh\alpha_{2j} - \cos\frac{\pi(k-1/2)}{m}} \frac{\cosh\alpha_{2j} - \cos\frac{\pi k}{m}}{\cosh\alpha_{2j-1} - \cos\frac{\pi k}{m}} \tag{34}$$

$$= \frac{1}{\cosh\alpha_{2n-1} - \cos\frac{\pi(m-1/2)}{m}} \cdot \frac{\sinh\alpha_{2n-1}}{\tanh m\alpha_{2n-1}} \frac{\sinh\beta_{2m-1}}{\tanh n\beta_{2m-1}} \tag{35}$$

$$\times \prod_{j=1}^{n-1}\prod_{k=1}^{m-1} \frac{x - \cos\frac{\pi(j-1/2)}{n} - \cos\frac{\pi(k-1/2)}{m}}{x - \cos\frac{\pi j}{n} - \cos\frac{\pi(k-1/2)}{m}} \frac{x - \cos\frac{\pi j}{n} - \cos\frac{\pi k}{m}}{x - \cos\frac{\pi(j-1/2)}{n} - \cos\frac{\pi k}{m}}. \tag{36}$$

Now, in this last formula the factor

$$\frac{1}{\cosh\alpha_{2n-1} - \cos\frac{\pi(m-1/2)}{m}} = \frac{1}{x + \cos\frac{\pi}{2n} + \cos\frac{\pi}{2m}}$$

is obviously symmetric when m and n are interchanged, and so is the factor $\frac{\sinh\alpha_{2n-1}}{\tanh m\alpha_{2n-1}} \frac{\sinh\beta_{2m-1}}{\tanh n\beta_{2m-1}}$ and the double product. So both sides in (4) are symmetric when m and n are interchanged, and hence they are equal.

15. What is the limit $n, m \to \infty$ of the reciprocal relation (31), if it exists? If the common value x of equations (30) is chosen to be close to 2, then α_j will be close to 0 for small j. Let $x = 2 + \frac{\pi^2\beta^2}{8n^2}$, then expanding cos and cosh we get approximately for small j and k

$$\alpha_j = \frac{\pi\sqrt{j^2+\beta^2}}{2n}, \quad \beta_k = \frac{1}{2n}\sqrt{\frac{n^2}{m^2}k^2 + \beta^2}.$$

Next assume that $\frac{m}{n} = \alpha$ is fixed. With these assumptions we have

$$\lim_{n,m\to\infty} \prod_{j=1}^{2n-1} (\tanh m\alpha_j)^{(-1)^j} = \prod_{j=1}^{\infty} \left(\tanh\frac{\pi\alpha\sqrt{j^2+\beta^2}}{2}\right)^{(-1)^j}, \tag{37}$$

$$\lim_{n,m\to\infty} \prod_{k=1}^{2m-1} (\tanh n\beta_k)^{(-1)^k} = \prod_{k=1}^{\infty} \left(\tanh\frac{\pi\sqrt{k^2/\alpha^2+\beta^2}}{2}\right)^{(-1)^k}. \tag{38}$$

The sinh factors are simplified as follows

$$\begin{aligned}\prod_{j=1}^{2n-1}(\sinh\alpha_j)^{(-1)^j} &= \prod_{j=1}^{2n-1}[(\cosh\alpha_j-1)(\cosh\alpha_j+1)]^{(-1)^j/2}\\ &= \prod_{j=1}^{2n-1}\left[\left(x-1-\cos\frac{\pi j}{2n}\right)\left(x+1-\cos\frac{\pi j}{2n}\right)\right]^{(-1)^j/2}\\ &= \left\{\frac{\tanh[n\operatorname{arccosh}(x-1)]}{\sinh[\operatorname{arccosh}(x-1)]}\frac{\tanh[n\operatorname{arccosh}(x+1)]}{\sinh[\operatorname{arccosh}(x+1)]}\right\}^{1/2}.\end{aligned}$$

Now it is easy to calculate the limit

$$\begin{aligned}&\lim_{n,m\to\infty}\frac{\prod_{j=1}^{2n-1}(\sinh\alpha_j)^{(-1)^j}}{\prod_{k=1}^{2m-1}(\sinh\beta_k)^{(-1)^k}}\\ &=\lim_{n,m\to\infty}\left\{\frac{\tanh[n\operatorname{arccosh}(x-1)]\tanh[n\operatorname{arccosh}(x+1)]}{\tanh[m\operatorname{arccosh}(x-1)]\tanh[m\operatorname{arccosh}(x+1)]}\right\}^{1/2}=\sqrt{\frac{\tanh\frac{\pi\beta}{2}}{\tanh\frac{\pi\alpha\beta}{2}}}.\end{aligned}\tag{39}$$

Combining equations (37)-(39) leads to (8). Thus we obtained an elementary proof of the modular transformation (1). Elementary proofs of modular transformations are known in the literature. For example Apostol [9] gives an elementary proof for $\sum_{n=1}^{\infty}\frac{n^{-p}x^p}{1-x^p}$, $|x|<1$ for an odd integer $p>1$.

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