Proof that a Derivative is a Fraction, and the Chain Rule is the Product of Such Fractions

Carl Wigert, Princeton University
Quincy-Howard Xavier, Harvard University

December 16, 2017

Theorem 1. A derivative, denoted $\frac{dy}{dx}$, is a fraction with dy and dx as real numbers.

Proof. Assume without loss of generality that some function $f(x)$ is continuous over some closed interval $[a, b]$ and differentiable over some open interval (a, b).
From the definition of a derivative, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, where we have $x \in (a, b)$ and $x+h \in (a, b)$. Let us consider the definition of this limit. $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in (a, b)$, $|h-0| < \delta \Rightarrow |\frac{f(x+h)-f(x)}{h} - f'(x)| < \epsilon$. Since $h \in \mathbb{R}$, we know that $h = \{x \in \mathbb{Q} | x < h\}$. Similarly, because $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$, we know that $x = \{y \in \mathbb{Q} | y < x\}$ and $f(x) = \{y \in \mathbb{Q} | y < f(x)\}$.

Definition A number α is **very small** if and only if it is in $\mathbb{R}$ and $0 \leq \alpha \leq f(x+h) - f(x-h)$. Furthermore, a number α is **very very small** if and only if it is in $\mathbb{R}$ and $0 \leq \alpha \leq f(x+h) - f(x)$.

As we can see, $f(x+h) - f(x) = \alpha$ is very very small. Similarly, for some $h > 0$, and for some $g(x) = x$, we can see that $0 < \beta \leq x + h - h = h$, so β is very very small. Therefore, $f'(x) = \frac{\alpha}{\beta}$, both of which are real numbers. $f'(x)$ is a ratio of very very small real numbers α and β. $\mathbb{R}$ is a field, closed under multiplication. Therefore, $f'(x) = \frac{df}{dx} = \frac{\alpha}{\beta} \in \mathbb{R}$. □

Theorem 2. As we know, the Chain Rule is defined as $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$ for some differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ and $x \mapsto g(x)$. This is a product of real numbers.

Proof. As we know, a derivative is just a ratio of very very small real numbers. Therefore, the chain rule $\frac{df}{dx} = \frac{\alpha}{\gamma} \cdot \frac{\gamma}{\beta}$ which is simply a product the ratio of very very small real numbers. □