Ellipsoid Method for Linear Programming made simple

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Abstract

In this paper, ellipsoid method for linear programming is derived using only minimal knowledge of algebra and matrices. Unfortunately, most authors first describe the algorithm, then later prove its correctness, which requires a good knowledge of linear algebra.

1 Introduction

Ellipsoid method was perhaps the first polynomial time method for linear programming[4]. However, it is hardly ever covered in Computer Science courses. In fact, many of the existing descriptions[1, 6, 3] first describe the algorithm, then later prove its correctness. Moreover, to understand one require a good knowledge of linear algebra (like properties of semi-definite matrices and Jacobean)[1, 6, 3, 2]. In this paper, ellipsoid method for linear programming is derived using only minimal knowledge of algebra and matrices.

We are given a set of linear equations

$$Ax \geq B$$

and have to find a feasible point. Ellipsoid method can check whether the system $Ax \geq B$ has a solution or not, and find a solution if one is present.

The algorithm generates a sequence of ellipsoids[1], $E_0, E_1, \ldots$, with centres $x_0, x_1, \ldots$ such that the solution space (if there is a feasible solution) is is inside each of these ellipsoids. If the centre

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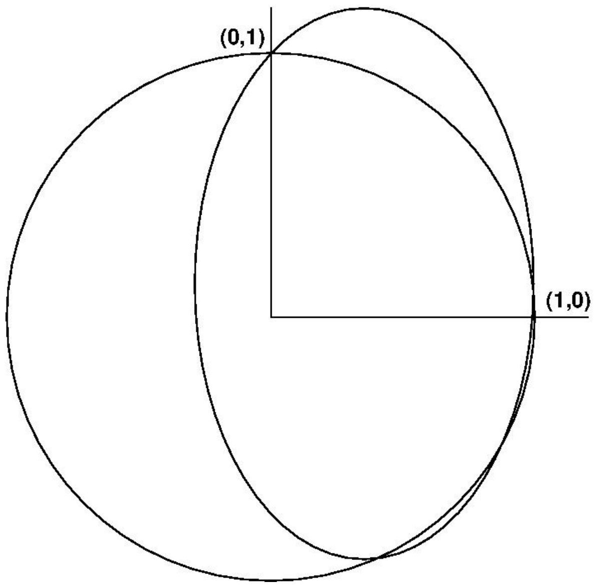

x_i of the (current) ellipsoid is not feasible, then some constraint (say) $a^T x \geq b$ is violated (for some row a of A), i.e., $a^T x_i < b$. As all points of solution space satisfy the constraint $a^T x \geq b$, we may add a new constraint $a^T x > a^T x_i$, without changing the solution space. *Observe that this "added" half-plane will also pass through x_i, the centre of the ellipsoid.* Thus, the solution space is contained in half-ellipsoid (intersection of Ellipsoid E_t with the half-plane). The next ellipsoid E_{i+1} will cover this half-ellipsoid and its volume will be a fraction of the volume of ellipsoid E_i.

The process is repeated, until we find a centre x_k for which $Ax_k \geq B$ or until the volume of ellipsoid becomes so small that we can conclude that there is no feasible solution.

If the set has a solution, then there is a number U [1, 3, 5, 7], such that each $x_i < U$, as a result, $\sum x_i^2 < nU^2$. If we scale each x_i, $x_i' = x_i/\sqrt{nU^2}$, then we know that feasible point x' will be inside unit sphere (centred at origin) $\sum x_i^2 \leq 1$.

Thus, we can take the initial "ellipsoid" to be unit sphere centred at origin.

2 Special Case: $x_1 \geq 0$

Let us first assume that the added constraint is just $x_1 \geq 0$.

The ellipsoid (see figure), will pass through the point $(1, 0, \ldots, 0, 0)$ and also points on (lower dimensional) sphere $x_2^2 + x_3^2 + \ldots + x_n^2 = 1$ (intersection of unit sphere with hyper-plane $x_1 = 0$). By symmetry, the equation of the ellipsoid should be:

$$\alpha(x_1 - c)^2 + \beta(x_2^2 + x_3^2 + \ldots + x_n^2) \leq 1$$

As $(1, 0, \ldots, 0)$ lies on the ellipsoid, $\alpha(1-c)^2 = 1$ or $\alpha = \frac{1}{(1-c)^2}$. Again, the ellipsoid should also pass through lower dimensional sphere $\sum_{i=2}^{n} x_i^2 = 1$ (with $x_1 = 1$)[1], we have $\alpha c^2 + \beta = 1$ or $\beta = 1 - \alpha c^2 = 1 - \frac{c^2}{(1-c)^2} = \frac{1-2c}{(1-c)^2}$. Thus, the equation becomes

$$\frac{1}{(1-c)^2}(x_1 - c)^2 + \frac{1-2c}{(1-c)^2}(x_2^2 + x_3^2 + \ldots + x_n^2) \leq 1$$

or equivalently,

$$\frac{1-2c}{(1-c)^2}\sum_{i=1}^{n} x_i^2 + \frac{c^2 + 2cx_1(x_1 - 1)}{(1-c)^2} \leq 1$$

For this to be ellipsoid, $0 \leq c < \frac{1}{2}$.

We want our ellipsoid to contain half-sphere. If point is inside the half-sphere then, $\sum x_i'^2 \leq 1$. Moreover, the term $(x_1'^2 - x_1') = x_1'(x_1' - 1)$ will be negative, hence,

$$\begin{aligned}
\frac{1-2c}{(1-c)^2}\sum_{i=1}^{n} x_i'^2 + \frac{c^2 + 2cx_1'(x_1' - 1)}{(1-c)^2} &\leq \frac{1-2c}{(1-c)^2} + \frac{c^2 + 2cx_1'(x_1' - 1)}{(1-c)^2} \\
&\leq \frac{1-2c}{(1-c)^2} + \frac{c^2}{(1-c)^2} = 1
\end{aligned}$$

Or x' is also on the ellipsoid. Thus, entire half-sphere is inside our ellipsoid.

We have certain freedom in choosing the ellipsoid (unit sphere is also an ellipsoid which satisfies these conditions with $c = 0$). We will choose c so as to reduce volume by at least a constant factor.

If we scale the coordinates as follows:

$$x_1' = \frac{x_1}{1-c} \text{ and } x_i' = \frac{\sqrt{1-2c}}{1-c} x_i \text{ for } i \geq 2$$

,

Then the coordinates x_i's will lie on a sphere of unit radius. Thus, the volume of our ellipsoid will be a fraction $\frac{1}{1-c}\left(\frac{\sqrt{1-2c}}{1-c}\right)^{n-1}$ of the volume of a unit sphere. Or the ratio of volumes

$$\frac{V_0}{V_1} = \frac{1}{1-c}\left(\frac{\sqrt{1-2c}}{1-c}\right)^{n-1} \frac{1}{1-c}\left(1 - \frac{c^2}{(1-c)^2}\right)^{(n-1)/2}$$

[1] e.g. the point $(0, 1, 0, \ldots, 0)$ lies on ellipsoid

We want $\frac{V_0}{V_1} > \alpha > 1$ (for some constant α) or equivalently,

$$
\begin{aligned}
1 < \alpha &\approx \frac{1}{1-c}\left(1 - \frac{c^2}{(1-c)^2}\right)^{(n-1)/2} \quad \text{using binomial theorem} \\
&\approx \frac{1}{1-c}\left(1 - \frac{n-1}{2}\frac{c^2}{(1-c)^2}\right) \approx \frac{1}{1-c}\left(1 - \frac{n}{2}c^2\right) \approx (1+c)\left(1 - \frac{nc^2}{2}\right) \approx 1 + c - \frac{nc^2}{2}
\end{aligned}
$$

We should have $\frac{nc^2}{2} < c$ or $c = \Theta(1/n)$. We choose $c = 1/(n+1)$, as this maximises the ratio.[2]

With this choice,

$$
\frac{V_0}{V_1} = \frac{1}{1-c}\left(1 - \frac{c^2}{(1-c)^2}\right)^{(n-1)/2} = \frac{n+1}{n}\left(1 - \frac{1}{n^2}\right)^{(n-1)/2}
$$

Or equivalently, using $1 - \alpha < e^{-\alpha}$ and $1 + \alpha < e^{\alpha}$

$$
\begin{aligned}
\frac{V_1}{V_0} &= \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2} \\
&= \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^{(n-1)/2} \\
&< \exp\left(-\frac{1}{n+1}\right)\exp\left(\frac{1}{n^2-1} \times \frac{n-1}{2}\right) \\
&= \exp\left(-\frac{1}{2(n+1)}\right)
\end{aligned}
$$

Thus, volume decreases by a constant factor.

3 General Case

Let us choose our coordinate system such that the ellipsoid is aligned with our coordinate axes. This involves rotation of axes (and possibly translation of origin). Next, we scale the axes, such that the scaled ellipsoid is a unit sphere (with centre as origin). Constraint $a^T x > a^T x_i$ (which will be modified as axes are rotated) can again be transformed by rotation to $x_1 > 0$.

Under the new scales, ratio of the two volumes, will again be $\exp\left(-\frac{1}{2(n+1)}\right)$. But, as both V_0 and V_1 will scale by the same amount, the ratio will be independent of the scale.

[2] Let $f(c) = \frac{1}{(1-c)^n}\left(\sqrt{1-2c}\right)^{n-1} = (1-c)^{-n}(1-2c)^{(n-1)/2}$
Then $f'(c) = (-1)(-n)(1-c)^{-(n+1)}(1-2c)^{(n-1)/2} + (1-c)^{-n}(-2)((n-1)/2)(1-2c)^{(n-3)/2}$
$= (1-c)^{-(n+1)}(1-2c)^{(n-3)/2}\left(n(1-2c) - (n-1)(1-c)\right)$, putting $f'(c) = 0$, we get
$n(1-2c) = (n-1)(1-c) = n(1-c) - (1-c)$ or $nc = 1 - c$ or $c = 1/(n+1)$

Thus after k iterations, the ratio of final ellipsoid to initial sphere will be at most $\exp\left(-\frac{k}{2(n+1)}\right)$. If we are to stop as soon as volume of ellipsoid becomes less than ϵ, then we want $\exp\left(-\frac{k}{2(n+1)}\right) = \frac{\epsilon}{V_0}$, or taking logs, $\ln \frac{V_0}{\epsilon} = \frac{k}{2(n+1)}$ or $k = O\left(n \ln \frac{V_0}{\epsilon}\right)$.

4 Details and Algorithm

General axes-aligned ellipsoid with centre as $(c_1, \ldots, c_n)$ is described by

$$\frac{(x_1 - c_1)^2}{a_1^2} + \frac{(x_2 - c_2)^2}{a_2^2} + \ldots + \frac{(x_n - c_n)^2}{a_n^2} = 1$$

This can be written as $(x-c)^T D^{-2}(x-c) = 1$ where D is a diagonal matrix with $d_{ii} = a_i$. If we change the origin to c, (using the transformation $x' = x - c$), the equation becomes, $x'^T D^{-2} x' = 1$. If we rotate the axes, and if R is the rotation matrix $x' = Rx''$, we get $x''^T R^T D^{-2} R x'' = 1$. This is the equation of general ellipsoid with centre as origin. If we now wish to further rotate the axes (say to make a particular direction as x_1-axis), say with rotation matrix S, $x'' = Sx'''$, the equation becomes $x'''^T S^T R^T D^{-2} RSx''' = 1$. If we want the centre to be $c' = (c_1', c_2', \ldots, c_n')$, (using the transform $x'''' - c' = x'''$), then the equation becomes (dropping all primes) $(x-c)^T (SR)^T D^{-2} (SR)(x-c) = 1$. This is the the general equation of ellipsoid with arbitrary centre and arbitrary direction as x_1-axis.

Conversely, if we are given a general ellipsoid $(x-c)^T K^{-1}(x-c) = 1$ with $K^{-1} = R^T S^T D^{-2} SR$ having centre c, we can transform it to unit sphere centred at origin by proceeding in reverse direction. In other words, we first translate the coordinate system (change the origin) using transform $x' = x - c$ to make the centre as the origin. Now the ellipsoid is of the form $x'^T (R^T S^T D^{-2} SR) x' = 1$. Next, we rotate the coordinates twice, $x'' = SRx'$, not just to make it axes-aligned, but also to make a particular "direction" as x_1-axis. The equation becomes, $x''^T D^{-2} x'' = 1$. We then scale the coordinates, $x''' = x_1''/a_i$ (or equivalently, $x''' = D^{-1}x''$) and we are left with unit sphere $x'''^T x''' = 1$ with centre as origin. Thus to summarise, $x''' = D^{-1}x'' = D^{-1}SRx' = D^{-1}SR(x-c)$

Remark: Observe that as R and S are rotation matrices[3] $R^T = R^{-1}$ and $S^T = S^{-1}$

$$K = (K^{-1})^{-1} = (R^T S^T D^{-2} RS)^{-1} = R^{-1} S^{-1} D^2 (S^T)^{-1} (R^T)^{-1} = R^T S^T D^2 (SR)$$

Let us now look at our constraint $a^T x > a^T x_i$. As $x_i = c$ the constraint is actually, $a^T(x-c) > 0$. Let $e_1 = (1, 0, \ldots, 0)^T$. Then, in the new coordinate system, the constraint $a^T(x-c) > 0$ becomes $e_1^T x''' > 0$ or $e_1^T (D^{-1} SR(x-c)) = 0$. Thus, we choose the direction (or rotation matrix S)

[3] If S is a rotation matrix, then as rotation preserves distance, $(x-y)(x-y)^T = ((x-y)S)((x-y)S)^T = ((x-y)S)(S^T(x-y)^T) = (x-y)(SS^T)(x-y)^T$, it follows that $SS^T = I$.

so that $a^T = \alpha e_1^T(D^{-1}SR)$ for some constant α. Or equivalently, $\alpha e_1^T = a^T R^T S^T D$, or $\alpha e_1 = DSRa$. To determine the constant α, observe that $\alpha^2 = (\alpha e_1)^T(\alpha e_1) = (a^T R^T S^T D)(DSRa) = a^T(R^T S^T D^2 SR)a = a^T Ka$, thus $\alpha = \sqrt{a^T Ka}$

We know that the next ellipsoid in the new coordinate system will be (dropping all primes) $\frac{1}{(1-c)^2}(x_1 - c)^2 + \frac{1-2c}{(1-c)^2}(x_2^2 + x_3^2 + \ldots + x_n^2) = 1$, or putting $c = 1/(n+1)$, and simplifying,

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(x_2^2 + x_3^2 + \ldots + x_n^2) = 1$$

Thus, the

$$K_{\text{new}}^{-1} = \text{diag}\left(\left(\frac{n+1}{n}\right)^2, \frac{n^2-1}{n^2}, \ldots, \frac{n^2-1}{n^2}\right)$$

And the new ellipsoid (in new coordinate system) can be written as (with primes)

$$\left(x''' - \frac{e_1}{n+1}\right)^T K_{\text{new}}^{-1} \left(x''' - \frac{e_1}{n+1}\right) \leq 1$$

Now, as $x''' = D^{-1}SR(x-c)$

$$\begin{aligned} x''' - \frac{e_1}{n+1} &= D^{-1}SR(x-c) - \frac{e_1}{n+1} \\ &= D^{-1}SR\left(x - c - R^{-1}S^{-1}D\frac{e_1}{n+1}\right) \end{aligned}$$

But, as $DSRa = \alpha e_1$, $e_1 = \frac{1}{\alpha}DSRa$, and recalling that R and S are rotation matrices, we have

$$\begin{aligned} x''' - \frac{e_1}{n+1} &= D^{-1}SR\left(x - c - R^{-1}S^{-1}D\frac{e_1}{n+1}\right) \\ &= D^{-1}SR\left(x - c - R^{-1}S^{-1}D\frac{1}{n+1}\frac{1}{\alpha}DSRa\right) \\ &= D^{-1}SR\left(x - c - \frac{1}{\alpha(n+1)}(R^T S^T D^2 SR)a\right) \\ &= D^{-1}SR\left(x - c - \frac{1}{\alpha(n+1)}Ka\right) \end{aligned}$$

Or the ellipsoid is

$$\begin{aligned} 1 &\geq \left(x''' - \frac{e_1}{n+1}\right)^T K_{\text{new}}^{-1} \left(x''' - \frac{e_1}{n+1})\right) \\ &= \left(D^{-1}SR\left(x - c - \frac{1}{\alpha(n+1)}Ka\right)\right)^T K_{\text{new}}^{-1} \left(D^{-1}SR\left(x - c - \frac{1}{\alpha(n+1)}Ka\right)\right) \\ &= \left(x - c - \frac{1}{\alpha(n+1)}Ka\right)^T R^T S^T D^{-1} K_{\text{new}}^{-1} D^{-1}SR\left(x - c - \frac{1}{\alpha(n+1)}Ka\right) \end{aligned}$$

Thus, in the original coordinate system, the new centre is:

$$c + \frac{1}{\alpha(n+1)} Ka = c + \frac{1}{n+1} \frac{Ka}{\sqrt{a^T Ka}}$$

And the new matrix K^{-1} is: $K'^{-1} = R^T S^T D^{-1} K_{\text{new}}^{-1} D^{-1} SR$, or equivalently,

$$K' = \left(K'^{-1}\right)^{-1} = R^{-1} S^{-1} D K_{\text{new}} D (S^T)^{-1} (R^T)^{-1} = R^T S^T D K_{\text{new}} DSR$$

Now, matrix $K_{\text{new}} = \text{diag}\left(\frac{n^2}{(n+1)^2}, \frac{n^2}{n^2-1}, \ldots, \frac{n^2}{n^2-1}\right) = \frac{n^2}{n^2-1} I + \left(\frac{n^2}{(n+1)^2} - \frac{n^2}{n^2-1}\right) E_1$

Here I is an identity matrix, and $E_1 = \text{diag}(1, 0, \ldots, 0)$ is a square matrix with only first entry as 1 and rest as 0. Observe that

$$E_1 \quad = \quad e_1 e_1^T = \left(\frac{1}{\alpha} DSRa\right) \left(\frac{1}{\alpha} DSRa\right)^T \quad = \frac{1}{\alpha^2} DSRaa^T R^T S^T D$$

And $\frac{n^2}{(n+1)^2} - \frac{n^2}{n^2-1} = -2\frac{n^2}{(n+1)(n^2-1)}$

Thus,

$$\begin{aligned}
K' &= R^T S^T D \left(\frac{n^2}{n^2-1} I - 2\frac{n^2}{(n+1)(n^2-1)} \frac{1}{\alpha^2} DSRaa^T R^T S^T D \right) DSR \\
&= \frac{n^2}{(n^2-1)} R^T S^T DIDSR - 2\frac{n^2}{\alpha^2(n+1)(n^2-1)} R^T S^T D(DSRaa^T R^T S^T D)DSR \\
&= \frac{n^2}{(n^2-1)} K - 2\frac{n^2}{\alpha^2(n+1)(n^2-1)} (R^T S^T D^2 SR)(aa^T)(R^T S^T D^2 SR) \\
&= \frac{n^2}{n^2-1} K - 2\frac{n^2}{\alpha^2(n+1)(n^2-1)} K(aa^T)K \\
&= \frac{n^2}{n^2-1} \left(K - \frac{2}{\alpha^2(n+1)} (Kaa^T K^T)\right)
\end{aligned}$$

Or,

$$K' = \frac{n^2}{n^2-1} \left(K - \frac{2}{\alpha^2(n+1)} (Kaa^T K^T)\right) = \frac{n^2}{n^2-1} \left(K - \frac{2}{n+1} \frac{Kaa^T K^T}{a^T Ka}\right)$$

5 Formal Algorithm

Observe that computation of new centre and the new "K" matrix does not require knowledge of S, R or D matrices. In fact even the matrix K^{-1} is not required.

After possible scaling, we can assume that the solution, if present, is in the unit sphere centred at origin. Thus, we initialise $K = I$ and $c = 0$

We repeat following two steps, till we either get a solution, or volume of solution space becomes sufficiently small:

1. Let $ax \leq b$ be the first constraint for which $ac > b$. If there is no such constraint, then c is a feasible point, and we can return.
2. Make
$$c = c + \frac{1}{n+1} \frac{Ka}{\sqrt{a^T K a}}$$
and
$$K = \frac{n^2}{n^2-1} \left(K - \frac{2}{n+1} \frac{K a a^T K^T}{a^T K a} \right)$$

Clearly, each iteration takes $O(n^3)$ time. Note that we do *not* explicitly construct any ellipsoids.

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