A DERIVATION OF DIRAC EQUATION FROM A GENERAL SYSTEM OF LINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Vu B Ho

Advanced Study, 9 Adela Court, Mulgrave, Victoria 3170, Australia

Email: vubho@bigpond.net.au

Abstract: In this work, we discuss a method to derive Dirac equation and other equations, such as the Cauchy-Riemann equations, from a general system of linear first order partial differential equations, with the hope that when studied more thoroughly the general system may provide deeper insights into geometrical and topological structures of quantum particles and fields.

Even though Dirac highly influential relativistic wave equation, which brought together Einstein special theory of relativity and quantum mechanics, has played a significant role in the development of quantum physics, the way in which the equation was derived seems unusual, except for the suggestive relativistic relationship between the energy and the momentum of a particle. In this work we will discuss a method to derive Dirac equation and other first order equations in physics from a general system of linear first order partial differential equations. For example, if ψ is the concentration of a fluid flowing at a constant rate c along the positive direction x then ψ satisfies the first order transport equation

$$\frac{\partial\psi}{\partial t}+c\frac{\partial\psi}{\partial x}=0 \tag{1}$$

In this case it is seen that the concentration ψ is a function of $(x-ct)$ only and the substance is transported at a constant speed c. The Cauchy-Riemann equations with two real-valued functions ψ_1 and ψ_2 of two real variables x and y are given [1]

$$\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_2}{\partial y}=0 \tag{2}$$

$$\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x}=0 \tag{3}$$

In the Cauchy-Riemann equations, the function ψ_1 can be interpreted as the velocity potential of an incompressible steady fluid flow and the function ψ_2 is the stream function of ψ_1. From Equations (2) and (3) it is shown that the velocity potential ψ_1 satisfies Laplace equation $\nabla^2\psi_1=0$. On the other hand, we have shown in our previous works that different forms of physical interactions that follow Laplace equation can also be derived from the massless Dirac equation [2]. Dirac equation is written out in full form in terms of the components of the wavefunction $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^T$ as [3]

$$i\frac{\partial\psi_1}{\partial t}+i\frac{\partial\psi_4}{\partial x}+\frac{\partial\psi_4}{\partial y}+i\frac{\partial\psi_3}{\partial z}=m\psi_1 \tag{4}$$

$$i\frac{\partial\psi_2}{\partial t}+i\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_3}{\partial y}-i\frac{\partial\psi_4}{\partial z}=m\psi_2 \tag{4}$$

$$-i\frac{\partial\psi_3}{\partial t}-i\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_2}{\partial y}-i\frac{\partial\psi_1}{\partial z}=m\psi_3 \tag{6}$$

$$-i\frac{\partial\psi_4}{\partial t}-i\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_1}{\partial y}+i\frac{\partial\psi_2}{\partial z}=m\psi_4 \tag{7}$$

Now, it is observed that the above-mentioned systems of linear first order partial differential equations can be considered as particular cases of a general system of linear first order partial differential equation written as follows [4]

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^{r}\frac{\partial\psi_i}{\partial x_j}=k\sum_{l=1}^{n}b_l^r\,\psi_l,\qquad r=1,2,\dots,n \tag{8}$$

The system of equations given in Equation (8) can be rewritten in a matrix form as

$$\left(\sum_{i=1}^{n}A_i\frac{\partial}{\partial x_i}\right)\psi=k\sigma\psi \tag{9}$$

where $\psi=(\psi_1,\psi_2,\dots,\psi_n)^T$, $\partial\psi/\partial x_i=(\partial\psi_1/\partial x_i\,,\partial\psi_2/\partial x_i\,,\dots,\partial\psi_n/\partial x_i)^T$, A_i and σ are matrices representing the coefficients a_{ij}^k and b_l^r, and k is an undetermined constant. Now if we apply the operator defined by $\sum_{i=1}^{n}A_i\frac{\partial}{\partial x_i}$ on the left of Equation (9) then we have

$$\left(\sum_{i=1}^{n}A_i\frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n}A_j\frac{\partial}{\partial x_j}\right)\psi=\left(\sum_{i=1}^{n}A_i\frac{\partial}{\partial x_i}\right)(k\sigma\psi) \tag{10}$$

If we assume further that the coefficients a_{ij}^k and b_l^r are constants and $A_i\sigma=\sigma A_i$, then Equation (10) can be rewritten in the following form

$$\left(\sum_{i=1}^{n}{A_i}^2\frac{\partial^2}{\partial x_i^2}+\sum_{i=1}^{n}\sum_{j>i}^{n}\left(A_iA_j+A_jA_i\right)\frac{\partial^2}{\partial x_i\partial x_j}\right)\psi=k^2\sigma^2\psi \tag{11}$$

In the following we will show that, depending on the dimension of space under consideration, Equation (11) can be reduced to different forms that can be applied to different physical problems and the requirement imposed for the reduction will specify the form of the corresponding system of linear first order partial differential equations. For simplicity we will also assume σ to be a unit matrix.

1. For $n = 1$

In this case we have

$$\frac{d\psi}{dx} = k\psi \tag{12}$$

$$\frac{d^2\psi}{dx^2} - k^2\psi = 0 \tag{13}$$

If $k = i\omega$, then Equation (13) describes a harmonic motion.

2. For $n = 2$

The matrices A_i can be written in a general form as

$$A_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{14}$$

If the matrices A_i satisfy the condition ${A_i}^2 = 1$, then, as shown in the appendix, they can take the following forms

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{15}$$

It can be checked that all matrices given in Equation (15) satisfy the following relations

$${A_i}^2 = 1 \tag{16}$$

$$A_iA_j + A_jA_i = 0 \quad \text{for} \quad i \neq j \tag{17}$$

Therefore all components of the function $\psi = (\psi_1, \psi_2)^T$ satisfy the equation

$$\nabla^2\psi_i = k^2\psi_i \quad \text{for} \quad i = 1, 2 \tag{18}$$

If we choose

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{19}$$

then we obtain the following system

$$\frac{\partial\psi_1}{\partial x_1} + \frac{\partial\psi_2}{\partial x_2} = k\psi_1 \tag{20}$$

$$-\frac{\partial\psi_2}{\partial x_1} + \frac{\partial\psi_1}{\partial x_2} = k\psi_2 \tag{21}$$

If $k = 0$, then Equations (20) and (21) reduce to a system similar to the Cauchy-Riemann equations.

If we choose

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{22}$$

then we obtain

$$\frac{\partial \psi_1}{\partial x_1} - i\frac{\partial \psi_2}{\partial x_2} = k\psi_1 \tag{23}$$

$$-\frac{\partial \psi_2}{\partial x_1} + i\frac{\partial \psi_1}{\partial x_2} = k\psi_2 \tag{24}$$

If we choose

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{25}$$

then we obtain

$$\frac{\partial \psi_2}{\partial x_1} - i\frac{\partial \psi_2}{\partial x_2} = k\psi_1 \tag{26}$$

$$\frac{\partial \psi_1}{\partial x_1} + i\frac{\partial \psi_1}{\partial x_2} = k\psi_2 \tag{27}$$

3. For $n = 3$

The matrices A_i can be written in a general form as

$$A_i = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \tag{28}$$

If the matrices A_i satisfy the condition ${A_i}^2 = 1$ and if we choose to have A_i of the form

$$A_i = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & k \end{pmatrix} \tag{29}$$

then, as shown in the appendix, we obtain

$$A_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{30}$$

If we choose to have A_i of the form

$$A_i = \begin{pmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{pmatrix} \tag{31}$$

then we obtain

$$A_i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{32}$$

For the case of $n = 3$, all matrices given in Equations (30) and (32) satisfy the following relations ${A_i}^2 = 1$, but they do not always satisfy either the relations $A_iA_j - A_jA_i = 0$ or the relations $A_iA_j + A_jA_i = 0$ for all $i \neq j$. For example, if we choose the matrices A_i to be of the following

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{33}$$

then the corresponding system of linear first order differential equations is found as

$$\frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_3}{\partial x_2} + i\frac{\partial \psi_3}{\partial x_3} = k\psi_1 \tag{34}$$

$$\frac{\partial \psi_2}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} + \frac{\partial \psi_2}{\partial x_3} = k\psi_2 \tag{35}$$

$$\frac{\partial \psi_3}{\partial x_1} + \frac{\partial \psi_1}{\partial x_2} - i\frac{\partial \psi_1}{\partial x_3} = k\psi_3 \tag{36}$$

Whether the system of linear first order partial differential equations given in Equations (34-36) and their corresponding second order partial differential equations can be realised and applied to physical situations required further investigations. However, as shown below, the particular case of $n = 4$ can be applied to Dirac equation in both pseudo-Euclidean and Euclidean metrics to describe the dynamics of quantum particles and fields.

4. For $n = 4$

The matrices A_i can be written in a general form as

$$A_i = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{pmatrix} \tag{37}$$

As shown in the appendix, the requirement $A_iA_j + A_jA_i = 0$ for $i \neq j$ with either the condition ${A_i}^2 = 1$ or the condition ${A_i}^2 = -1$ can be satisfied by various matrices for the case of $n = 4$. In particular, if we select in those matrices the following forms of the matrices A_i with the conditions ${A_1}^2 = 1$ or ${A_i}^2 = -1$ for $i = 2, 3, 4$ then Dirac equation can be obtained from the general system of linear first order partial differential equations given in Equation (9)

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{38}$$

The corresponding second order partial differential equations for the components of the function $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ found from Equation (11) are

$$\frac{\partial^2 \psi_\mu}{\partial x_1^2} - \frac{\partial^2 \psi_\mu}{\partial x_2^2} - \frac{\partial^2 \psi_\mu}{\partial x_3^2} - \frac{\partial^2 \psi_\mu}{\partial x_4^2} = k^2 \psi_\mu \qquad for \quad \mu = 1,2,3,4 \tag{39}$$

Equation (39) has the form of the Klein-Gordon equation if we set $(x_\mu = t, x, y, z)$ as spacetime coordinates and $k = im$. However, in general, in order to select a suitable set of the matrices A_i we would need a suggestive relationship between physical objects, such as the relativistic relationship between the energy and the momentum of a particle for the Dirac equation. For example, for the Euclidean relativistic relationship between the energy and the momentum of a particle given by the relation $E^2 = -p^2 + m^2$, the suitable set of the matrices A_i is selected as [5]

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{40}$$

In this case the corresponding second order partial differential equations for the components of the function $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ found from Equation (11) are

$$\frac{\partial^2 \psi_\mu}{\partial x_1^2} + \frac{\partial^2 \psi_\mu}{\partial x_2^2} + \frac{\partial^2 \psi_\mu}{\partial x_3^2} + \frac{\partial^2 \psi_\mu}{\partial x_4^2} = k^2 \psi_\mu \qquad for \quad \mu = 1,2,3,4 \tag{41}$$

Finally, it is noted that we can also extend our discussions for spaces with dimensions $n > 4$, in particular, for the case of $n = 6$ in which we discuss possible spacetime dynamics of quantum particles in terms of isometric embedding in the six-dimensional Euclidean continuum [6,7].

Appendix

For the case of $n = 2$, the matrices A_i are written in the general form as

$$A_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

From the requirement ${A_i}^2 = 1$, we arrive at the following system of equations for the unknown quantities a, b, c and d

$$a^2 + bc = 1 \tag{2}$$

$$b(a + d) = 0 \tag{3}$$

$$c(a + d) = 0 \tag{4}$$

$$d^2 + bc = 1 \tag{5}$$

From the simultaneous equations given in Equations (2-5), we obtain the following forms for the matrices A_i

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{6}$$

It can be checked that all matrices given in Equation (6) satisfy the following relations

$${A_i}^2 = 1 \tag{7}$$

$$A_i A_j + A_j A_i = 0 \quad \text{for} \quad i \neq j \tag{8}$$

For the case of $n = 3$, the matrices A_i are written in the general form as

$$A_i = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \tag{9}$$

From the requirement ${A_i}^2 = 1$, we arrive at the following system of equations for the unknown quantities $a, b, c, d, e, f, g, h, k$

$$a^2 + bd + cg = 1 \tag{10}$$

$$e^2 + bd + fh = 1 \tag{11}$$

$$k^2 + cg + fh \tag{12}$$

$$ab + be + ch = 0 \tag{13}$$

$$ac + bf + ck = 0 \tag{14}$$

$$ad + de + fg = 0 \tag{15}$$

$$cd + ef + fk = 0 \tag{16}$$

$$ag + dh + gk = 0 \tag{17}$$

$$bg + eh + hk = 0 \tag{18}$$

If we assume the matrices A_i to be of the form

$$A_i = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & k \end{pmatrix} \tag{19}$$

then we obtain the following forms for the matrices A_i

$$A_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{20}$$

If we assume the matrices A_i to be of the form

$$A_i = \begin{pmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{pmatrix} \tag{21}$$

then we obtain the following forms for the matrices A_i

$$A_i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{22}$$

It can be checked that all matrices given in Equations (20) and (22) satisfy the following relations ${A_i}^2 = 1$ but they do not always satisfy either the relations $A_iA_j - A_jA_i = 0$ or the relations $A_iA_j + A_jA_i = 0$ for $i \neq j$.

For the case of $n = 4$, the matrices A_i are written in the general form as

$$A_i = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{pmatrix} \tag{23}$$

From the requirement ${A_i}^2 = 1$, we arrive at the following system of equations for the unknown quantities $a, b, c, d, e, f, g, h, k, l, m, n, p, q, r, s$

$$a^2 + be + ck + dp = 1 \tag{24}$$

$$f^2 + be + gl + hq = 1 \tag{25}$$

$$m^2 + ck + gl + nr = 1 \tag{26}$$

$$s^2 + dp + hq + nr = 1 \tag{27}$$

$$ab + bf + cl + dq = 0 \tag{28}$$

$$ac + bg + cm + dr = 0 \tag{29}$$

$$ad + bh + cn + ds = 0 \tag{30}$$

$$ae + ef + gk + hp = 0 \tag{31}$$

$$ce + fg + gm + hr = 0 \tag{32}$$

$$de + fh + gn + hs = 0 \tag{33}$$

$$ak + el + km + pn = 0 \tag{34}$$

$$bk + fl + lm + nq = 0 \tag{35}$$

$$dk + hl + mn + ns = 0 \tag{36}$$

$$ap + eq + kr + ps = 0 \tag{37}$$

$$bp + fq + lr + qs = 0 \tag{38}$$

$$cp + gq + mr + rs = 0 \tag{39}$$

If we assume the matrices A_i to be of the form

$$A_i = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & s \end{pmatrix} \tag{40}$$

then from the conditions given in Equations (24-39) we obtain the following forms for the matrices A_i

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{41}$$

If we assume the matrices A_i to be of the form

$$A_i = \begin{pmatrix} 0 & 0 & 0 & d \\ 0 & 0 & g & 0 \\ 0 & l & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix} \tag{42}$$

then from the conditions given in Equations (24-39) we obtain the following forms for the matrices A_i

$$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&1\\0&0&i&0\\0&-i&0&0\\1&0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&1\\0&0&-i&0\\0&i&0&0\\1&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&i\\0&0&1&0\\0&1&0&0\\-i&0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&-i\\0&0&1&0\\0&1&0&0\\i&0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix} \tag{43}$$

If we assume the matrices A_i to be of the form

$$A_i=\begin{pmatrix}0&0&c&0\\0&0&0&h\\k&0&0&0\\0&q&0&0\end{pmatrix} \tag{44}$$

then from the conditions given in Equations (24-39) we obtain the following forms for the matrices A_i

$$\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix} \tag{45}$$

On the other hand, from the requirement ${A_i}^2=-1$ and if we assume the matrices A_i to be of the form

$$A_i=\begin{pmatrix}a&0&0&0\\0&f&0&0\\0&0&m&0\\0&0&0&s\end{pmatrix} \tag{46}$$

then we obtain the following forms for the matrices A_i

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}i\quad\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}i\quad\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}i\quad\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}i$$

$$\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}i\quad\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}i\quad\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}i\quad\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}i \tag{47}$$

If we assume the matrices A_i to be of the form

$$A_i = \begin{pmatrix} 0 & 0 & 0 & d \\ 0 & 0 & g & 0 \\ 0 & l & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix} \tag{48}$$

then we obtain the following forms for the matrices A_i

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} i \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} i \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} i \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} i$$

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} i \quad \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} i \quad \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} i \quad \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} i \tag{49}$$

If we assume the matrices A_i to be of the form

$$A_i = \begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & h \\ k & 0 & 0 & 0 \\ 0 & q & 0 & 0 \end{pmatrix} \tag{50}$$

then we obtain the following forms for the matrices A_i

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{51}$$

References

[1] Walter A. Strauss, *Partial Differential Equation* (John Wiley & Sons, Inc., New York, 1992).

[2] Vu B Ho, *Derivation of Interactions from Dirac Equation* (Preprint, ResearchGate, 2017), viXra 1712.0068v1.

[3] P. A. M. Dirac, *The Quantum Theory of the Electron*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, **117** (1928).

[4] S. V. Melshko, *Methods for Constructing Exact Solutions of Partial Differential Equations*, Springer Science & Business Media, Inc, 2005.

[5] Vu B Ho, *On the EPR paradox and Dirac equation in Euclidean relativity* (Preprint, ResearchGate, 2017), viXra 1711.0268v1.

[6] Vu B Ho, *A Temporal Dynamics: A Generalised Newtonian and Wave Mechanics* (Pre-print, ResearchGate, 2016), viXra 1708.0198v1.

[7] Vu B Ho, *On the motion of quantum particles and Euclidean relativity* (Preprint, ResearchGate, 2017), viXra 1710.0253v1.