On the general solution for the nonlinear differential equation from Troesch boundary value problem

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Abstract

This paper shows, for the first time, that the explicit and exact solution to the Troesch nonlinear two-point boundary value problem may be computed in a direct and straightforward fashion from the general solution obtained by a generalized Sundman transformation for the related differential equation, which appeared to be a special case of a more general equation. As a result, various initial and boundary value problems may be solved explicitly and exactly.

Theory

The Troesch nonlinear two-point boundary value problem is well known to be of the form [1]

$$u''(x) + a\sinh au(x) = 0 \tag{1}$$

where

$$u(0) = 0, \quad u(1) = 1 \tag{2}$$

and a is a positive constant. The purpose is now to establish that the equation (1) is a limiting case of a more general equation.

1. Generalized equation

According to [2-4], the general class of equations for the application of the generalized Sundman linearization theory developed by Akande et al. [2] may be written as

$$u''(x) + a^2 e^{\varphi(u)} \int e^{\varphi(u)} du = 0 \tag{3}$$

where $\gamma = 1$, under the conditions that

$$y(\tau) = \int e^{\varphi(u)} du, \quad d\tau = e^{\varphi(u)} dx \tag{4}$$

and $y(\tau)$ satisfies

$$\ddot{y}(\tau) + a^2 y(\tau) = 0 \tag{5}$$

that is

$$y(\tau) = A_0 \sin(a\tau + \alpha) \tag{6}$$

Inserting

$$\varphi(u) = \ln(\sinh qu) \tag{7}$$

into (3) yields

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$$u''(x) + \frac{a^2}{2q}\sinh(2qu) = 0 \qquad (8)$$

where $q \neq 0$, is an arbitrary parameter. The equation (8) is the desired generalized differential equation. Substituting $2q = a$ into (8), leads to the differential equation (1) of the Troesch nonlinear two-point boundary value problem under the condition that $a \succ 0$. In such a situation the general solutions to (1) and (8) may be explicitly and exactly established.

2. General solutions

The application of (4), taking into consideration the equations (6) and (7), may lead to

$$qA_0 \sin(a\tau + \alpha) = \cosh(qu) \qquad (9)$$

that is

$$u(x) = \frac{1}{q}\cosh^{-1}\left[qA_0 \sin(a\tau + \alpha)\right] \qquad (10)$$

such that

$$\frac{d\tau}{\sqrt{q^2 A_0^2 \sin^2(a\tau + \alpha) - 1}} = dx \qquad (11)$$

The integration of (11) yields after a few algebraic manipulations [5]

$$\cos(a\tau + \alpha) = \varepsilon\, p\, sn\left[aqA_0(x + C), p\right] \qquad (12)$$

where $\varepsilon = \pm 1$, $p = \sqrt{1 - \left(\frac{1}{qA_0}\right)^2}$, and C is a constant of integration, so that the general solution (10) to the generalized equation (8) becomes [5,6]

$$u(x) = \frac{1}{q}\cosh^{-1}\left\{qA_0 dn\left[aqA_0(x + C), p\right]\right\} \qquad (13)$$

Making $2q = a$, yields the general solution to the differential equation (1) of the Troesch nonlinear two-point boundary value problem as

$$u(x) = \frac{2}{a}\cosh^{-1}\left\{\frac{a}{2}A_0 dn\left[\frac{a^2}{2}A_0(x + C), p\right]\right\} \qquad (14)$$

where $p = \sqrt{1 - \left(\frac{2}{aA_0}\right)^2}$. Therefore, the exact solution to the Troesch nonlinear two-point boundary value problem may be computed by the determination of the two integration constants by applying the boundary conditions (2).

3. Exact solution of Troesch problem

The application of (2) leads, for the constants of integration A_0 and C, from the general solution (13), to the two transcendental equations

$$dn(qaA_0C, p) = \frac{1}{aqA_0} \qquad (15)$$

$$dn[aqA_0(1+C), p] = \frac{\cosh(q)}{aqA_0} \qquad (16)$$

Therefore, the exact solution to the Troesch nonlinear two-point boundary value problem, for $q = \frac{a}{2}$, is given by the solution (14) under the conditions.

$$dn\left(\frac{a^2}{2}A_0C, p\right) = \frac{2}{a^2A_0} \qquad (17)$$

$$dn\left[\frac{a^2}{2}A_0(1+C), p\right] = \frac{2\cosh(\frac{a}{2})}{a^2A_0} \qquad (18)$$

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