Sketching 'trinions' and 'heptanions'

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Abstract

Attempting to abstract exterior derivative and Hodge star operator, we discuss two number systems sketchily.

1 Introduction

Trying to abstract exterior derivative (d) and Hodge star operator ($\star$), we deal with them as if they were mere mathematical symbols. In other words, we intentionally forget the well-known and/or minute roles they play in the field of physics for the moment. We frequently use d and $\star$ -inspired [1] symbol $\diamond$. [2] These symbols are *usually* considered to be noninterchangeable. [3] In the meantime, we come up with two number systems, which we tentatively call 'trinion (t_r)' and 'heptanion (h_e)'.

2 Taking a cursory look at t_r's and h_e's

2.1 t_r's

At the outset, we make some definitions. [4]

Definition 2.1.1. $(d \cdot d = dd =) d^2 := 0$ [2, 3].

Definition 2.1.2. $(\diamond \cdot \diamond = \diamond\diamond =) \diamond^2 := \pm 1$, and $(\diamond \cdot \diamond \cdot \diamond \cdot \diamond = \diamond\diamond\diamond\diamond =) \diamond^4 := 1$.

Definition 2.1.3. $i := \diamond\diamond\diamond d \diamond$, whereas $j := \diamond d \diamond\diamond\diamond$.

t_r's are a number system whose basis elements are 1, i, j. In addition,

Definition 2.1.4. t_r (resp. $\bar{t_r}$) $:= a + bi + cj$ (resp. $a - bi - cj$), where a, b, c belong to the set

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[1] We were inspired by the relation $\star\star\omega = (-1)^{k(n-k)}\omega$. See, *e.g.,* [1], in which the author employs the symbol $*$ instead of $\star$, however.

[2] See **5.1** for details about the origin of $\diamond$.

[3] For instance, $\diamond d$ is regarded as distinct from $d\diamond$. But what if we accepted interchangeability? See **5.2**.

[4] In what follows, $\cdot$ denotes multiplication and is often omitted.

of real numbers ($\mathbb{R}$).

The set of t_r's is denoted by $\mathbb{T}_r$. Sc (t_r) and Vec (t_r) stand for a and $bi + cj$, respectively. And we immediately get the following.

$$\begin{aligned} i \cdot i &= {}^{5}\ \Diamond\Diamond\Diamond d\Diamond \cdot \Diamond\Diamond\Diamond d\Diamond = \Diamond\Diamond\Diamond d\Diamond\Diamond\Diamond\Diamond d\Diamond = \Diamond\Diamond\Diamond d \cdot \Diamond\Diamond\Diamond\Diamond \cdot d\Diamond = \Diamond\Diamond\Diamond d \cdot \Diamond^4 \cdot d\Diamond \\ &= {}^{6}\ \Diamond\Diamond\Diamond d \cdot 1 \cdot d\Diamond = \Diamond\Diamond\Diamond dd\Diamond = \Diamond\Diamond\Diamond \cdot d^2 \cdot \Diamond = {}^{7}\ \Diamond\Diamond\Diamond \cdot 0 \cdot \Diamond = 0. \end{aligned}$$

Likewise, $j \cdot j = \Diamond d\Diamond\Diamond\Diamond \cdot \Diamond d\Diamond\Diamond\Diamond = 0$, $i \cdot j = \Diamond\Diamond\Diamond d\Diamond \cdot \Diamond d\Diamond\Diamond\Diamond = 0$, $j \cdot i = \Diamond d\Diamond\Diamond\Diamond \cdot \Diamond\Diamond\Diamond d\Diamond = 0$. We thus get the table below.

Table 1. Multiplication table of t_r's [8]

$\times$	1	i	j
1	1	i	j
i	i	0	0
j	j	0	0

Having managed to get the table above without explicit reference to the so-called physics, we wish to remember it and raise a question.

Question 2.1.5. Do t_r's have *physical* implications? [9]

2.2 h_e's

We forget physics again and consider a basis $e_0, \ldots, e_6$ corresponding to basis elements $1, i, \ldots, m$, n, respectively. Using the above-mentioned symbols d and $\Diamond$, we define the following.

Definition 2.2.1. $i := \Diamond$, $j := d$, $k := \Diamond d$, $\ell := d\Diamond$, $m := \Diamond d\Diamond$, and $n := d\Diamond d$.

h_e's are a number system whose basis elements are 1, i, j, k, ℓ, m, n. The set of h_e's is denoted by $\mathbb{H}_e$.

Definition 2.2.2. '$\star$-rule' is as follows: Consider the set $\{i, j, k, \ell, m, n\}$ and its subset which contains at most two elements. We then make some noninterchangeable products consisting of at most two elements chosen from its complement . Exponentiation of each element is acceptable. Such procedures are indicated by endowing the product coming from the subset we considered with subscript $\star$.

To p_{ab}'s, or products of e_a and e_b $(1 \le a, b \le 6)$ which are other than 0, ± 1, $\pm j$, $\pm k$, and $\pm \ell$, we apply the above '$\star$-rule' as needed. [10]

Example 2.2.3. We can derive ℓ^3, $m^2 n^5$, n, and so on from $ij_\star$, which comes from the subset

[5] See *Def.* 2.1.3.

[6] See *Def.* 2.1.2.

[7] See *Def.* 2.1.1.

[8] For a somewhat similar system of numbers, see [4], in which $i^2 = j^2 = -1$, and $ij = ji = 0$.

[9] By the way, $*d =$ curl [3].

[10] The reader is invited to compare footnote 13 with footnote 15.

$\{i,\, j\}$. [11]

Then, the relations below follow, to name a few.

$p_{01} = e_0 \cdot e_1 = 1 \cdot i = i,\ p_{12} = e_1 \cdot e_2 = i \cdot j =\ ^{12}\ \diamond \cdot d = \diamond d =\ ^{13}\ k,$

$p_{33} = e_3 \cdot e_3 = k \cdot k =\ ^{14}\ \diamond d \cdot \diamond d =\ \diamond d \diamond d =\ ^{15}\ \diamond \cdot d \diamond d$ (resp. $\diamond d \diamond \cdot d$) $=\ ^{16}\ in$ (resp. mj).

Computing the remainder of p_{ab}'s ($0 \le a,\, b \le 6$), we get the following.

Table 2. Multiplication table of h_e's

$\times$	1	i	j	k	ℓ	m	n
1	1	i	j	k	ℓ	m	n
i	i	± 1	k	$\pm j$	m	$\pm \ell$	k^2/mj
j	j	ℓ	0	n	0	ℓ^2/ni	0
k	k	m	0	in/mj	0	$i\ell^2$	0
ℓ	ℓ	$\pm j$	n	0	jm/ni	0	jk^2
m	m	$\pm k$	in/k^2	0	$k^2 i$	0	k^3
n	n	jm/ℓ^2	0	$\ell^2 j$	0	ℓ^3	0

After intermittent oblivion of physics, we wish to raise another question.

Question 2.2.4. Do h_e's have *physical* implications? [17]

3 Some (attempted) calculations

3.1 t_r's

First, we would like to know whether t_r's are commutative under multiplication . Let t_{r1}, t_{r2} $\in \mathbb{T}_r$ be given by

$$\begin{cases} t_{r1} = a_1 + b_1 i + c_1 j, \\ t_{r2} = a_2 + b_2 i + c_2 j, \end{cases} \qquad a_1,\ a_2,\ b_1,\ b_2,\ c_1,\ c_2\ \in\ \mathbb{R}\,.$$

Then,

[11] Untenable are iji, klm, and so forth, though for example, nmn can become $mnn = mn^2$, which is found to be derivable from $ij_\star$, if we accept interchangeability temporarily.

[12] See *Def.* 2.2.1.

[13] In this simple case, we virtually ignore '$\star$-rule', since we only need to reference *Def.* 2.2.1 to replace $\diamond d$ by k.

[14] See *Def.* 2.2.1.

[15] Things being a bit complicated in this case, products tantamount to $k \cdot k = k^2 = \diamond d \diamond d$ include $\diamond \cdot d \cdot \diamond \cdot d$, $\diamond d \cdot \diamond \cdot d$, $\diamond \cdot d \diamond \cdot d$, $\diamond \cdot d \cdot \diamond d$, $\diamond d \cdot \diamond d$, $\diamond d \diamond \cdot d$, and $\diamond \cdot d \diamond d$, which correspond to $i \cdot j \cdot i \cdot j$, $k \cdot i \cdot j$, $i \cdot \ell \cdot j$, $i \cdot j \cdot k$, $k \cdot k$, $m \cdot j$, and $i \cdot n$, respectively. So we try being amenable to $k_\star^2$, which comes from the subset $\{k\}$, to prune away the first five of these.

[16] See *Def.* 2.2.1.

[17] Incidentally, $*d* =$ div [3].

$$
\begin{aligned}
t_{r1}\cdot t_{r2} &= (a_1+b_1i+c_1j)\cdot(a_2+b_2i+c_2j)\\
&= a_1\cdot(a_2+b_2i+c_2j)+b_1i\cdot(a_2+b_2i+c_2j)+c_1j\cdot(a_2+b_2i+c_2j)\\
&= a_1a_2+a_1b_2i+a_1c_2j+b_1ia_2+b_1ib_2i+b_1ic_2j+c_1ja_2+c_1jb_2i+c_1jc_2j\\
&= a_1a_2+a_1b_2i+a_1c_2j+a_2b_1i+b_1b_2ii+b_1c_2ij+a_2c_1j+b_2c_1ji+c_1c_2jj\\
&=^{18}\ a_1a_2+a_1b_2i+a_1c_2j+a_2b_1i+b_1b_2\cdot 0+b_1c_2\cdot 0+a_2c_1j+b_2c_1\cdot 0+c_1c_2\cdot 0\\
&= a_1a_2+a_1b_2i+a_1c_2j+a_2b_1i+a_2c_1j\\
&= a_1a_2+(a_1b_2+a_2b_1)i+(a_1c_2+a_2c_1)j, \qquad (1)
\end{aligned}
$$

and

$$
\begin{aligned}
t_{r2}\cdot t_{r1} &= (a_2+b_2i+c_2j)\cdot(a_1+b_1i+c_1j)\\
&= a_2\cdot(a_1+b_1i+c_1j)+b_2i\cdot(a_1+b_1i+c_1j)+c_2j\cdot(a_1+b_1i+c_1j)\\
&= a_2a_1+a_2b_1i+a_2c_1j+b_2ia_1+b_2ib_1i+b_2ic_1j+c_2ja_1+c_2jb_1i+c_2jc_1j\\
&= a_1a_2+a_2b_1i+a_2c_1j+a_1b_2i+b_1b_2ii+b_2c_1ij+a_1c_2j+b_1c_2ji+c_1c_2jj\\
&=^{19}\ a_1a_2+a_2b_1i+a_2c_1j+a_1b_2i+b_1b_2\cdot 0+b_2c_1\cdot 0+a_1c_2j+b_1c_2\cdot 0+c_1c_2\cdot 0\\
&= a_1a_2+a_2b_1i+a_2c_1j+a_1b_2i+a_1c_2j\\
&= a_1a_2+(a_1b_2+a_2b_1)i+(a_1c_2+a_2c_1)j. \qquad (2)
\end{aligned}
$$

Since (1) = (2), t_r's are commutative under multiplication . Next, what about $|t_r|^2$? We recall the square of $|z|$, the modulus of complex number , which equals $z\cdot\bar{z}=\bar{z}\cdot z$, where z (resp. $\bar{z}$) $=a+b\ i$ (resp. $a-b\ i$), $a, b\in\mathbb{R}$, and $i^2=-1$. In the case of t_r's,

$$
\begin{aligned}
t_r\cdot\bar{t_r} &=^{20}\ (a+bi+cj)\cdot(a-bi-cj)=a\cdot(a-bi-cj)+bi\cdot(a-bi-cj)+cj\cdot(a-bi-cj)\\
&= a^2-abi-acj+bia-bibi-bicj+cja-cjbi-cjcj\\
&= a^2-abi-acj+abi-bbii-bcij+acj-bcji-ccjj\\
&=^{21}\ a^2-abi-acj+abi-bb\cdot 0-bc\cdot 0+acj-bc\cdot 0-cc\cdot 0\\
&= a^2. \qquad (3)
\end{aligned}
$$

We also compute $\bar{t_r}\cdot t_r$, though it should equal $t_r\cdot\bar{t_r}$, since t_r's have been shown to be commutative under multiplication . Sure enough,

$$
\begin{aligned}
\bar{t_r}\cdot t_r &=^{22}\ (a-bi-cj)\cdot(a+bi+cj)=a\cdot(a+bi+cj)-bi\cdot(a+bi+cj)-cj\cdot(a+bi+cj)\\
&= a^2+abi+acj-bia-bibi-bicj-cja-cjbi-cjcj\\
&= a^2+abi+acj-abi-bbii-bcij-acj-bcji-ccjj\\
&=^{23}\ a^2+abi+acj-abi-bb\cdot 0-bc\cdot 0-acj-bc\cdot 0-cc\cdot 0\\
&= a^2, \qquad (4)
\end{aligned}
$$

which amounts to (3). Hence, $|t_r|^2=t_r\cdot\bar{t_r}=\bar{t_r}\cdot t_r=a^2$. What about multiplicative inverse $\frac{1}{t_r}$, then? We would like to be so careful again that we compute it in two ways and expect both to coincide.

[18] See **Table 1**.
[19] Ditto.
[20] See *Def.* 2.1.4.
[21] See **Table 1**.
[22] See *Def.* 2.1.4.
[23] See **Table 1**.

$$\frac{1}{t_r}=\frac{1}{a+bi+cj}=\frac{1\cdot(a-bi-cj)}{(a+bi+cj)\cdot(a-bi-cj)}=\frac{a-bi-cj}{(a+bi+cj)\cdot(a-bi-cj)}=\frac{a-bi-cj}{t_r\cdot\bar{t_r}}\overset{24}{=}\frac{a-bi-cj}{a^2}\overset{25}{=}\frac{\bar{t_r}}{a^2}. \qquad (5)$$

And

$$\frac{1}{t_r}=\frac{1}{a+bi+cj}=\frac{(a-bi-cj)\cdot 1}{(a-bi-cj)\cdot(a+bi+cj)}=\frac{a-bi-cj}{(a-bi-cj)\cdot(a+bi+cj)}=\frac{a-bi-cj}{\bar{t_r}\cdot t_r}\overset{26}{=}\frac{a-bi-cj}{a^2}\overset{27}{=}\frac{\bar{t_r}}{a^2}. \qquad (6)$$

As expected, (5) = (6). Hence, $\frac{1}{t_r}=\frac{\bar{t_r}}{a^2}$ $(a \in \mathbb{R}^*)$.

3.2 h_e's

We let two elements $h_{e1}, h_{e2} \in \mathbb{H}_e$ be given by

$$\begin{cases} h_{e1}=a_1+b_1i+c_1j+d_1k+e_1\ell+f_1m+g_1n, \\ h_{e2}=a_2+b_2i+c_2j+d_2k+e_2\ell+f_2m+g_2n, \\ \quad a_1,\ a_2,\ b_1,\ b_2,\ c_1,\ c_2,\ d_1,\ d_2,\ e_1,\ e_2,\ f_1,\ f_2,\ g_1,\ g_2 \in \mathbb{R}. \end{cases}$$

But since there are indefinite elements such as ± 1, $\pm j$, and so forth in **Table 2**, something (rather) subtle might turn out to be mandatory even for computing $h_{e1}\cdot h_{e2}$ (or $h_{e2}\cdot h_{e1}$). So for the moment, we would like to content ourselves with simplification of that table, or **Tables 4** and **5** in **5.2.2**.

4 Discussion

We have seen that t_r's are commutative under multiplication like $\mathbb{R}$ and that $|t_r|^2=a^2$, which coincides with the square of a real number a, irrespective of whether Vec $(t_r)=0$. Moreover, if Vec $(t_r)=0$, we have $\frac{1}{t_r}=\frac{1}{a}$, which also coincides with the multiplicative inverse of $a \in \mathbb{R}^*$. In these respects, t_r's are reminiscent of $\mathbb{R}$. However, complex numbers, whose set is denoted by $\mathbb{C}$, are commutative under multiplication, too. And computations of $|t_r|^2$ were performed in a way analogous to $|z|^2 = z\cdot\bar{z}$ (resp. $\bar{z}\cdot z$) $=(a+bi)\cdot(a-bi)$ (resp. $(a-bi)\cdot(a+bi)$) $=a^2+b^2$. Here we wonder if t_r's can 'outreach' $\mathbb{C}$ from a viewpoint of the square of norm and observe that the following interpretation on how to get $|z|^2$ is also possible.

Interpretation 4.1. $|z|^2$ is obtained by extracting real part and imaginary part from z or $\bar{z}$ and computing the sum of their square s. That is, we can obviate in-a-sense-naïve computations like $|z|^2=(a+bi)\cdot(a-bi)=a\cdot(a-bi)+bi\cdot(a-bi)=\cdots$, if we wish.

Computing $|t_r|^2$ in this vein, we directly get $|t_r|^2=a^2+b^2+c^2$ [28], which we view as the diminution of $a^2+b^2+c^2+d^2$ $(a, b, c, d \in \mathbb{R})$, the square of the norm of a quaternion. Then, we notice that $i, j \in \mathbb{T}_r$ is to $ij+ji$ what $i, j \in \mathbb{H}$, the set of quaternions is to $ij+ji$, since $ij+ji \overset{29}{=} 0+0$

[24] See (3).
[25] See *Def.* 2.1.4.
[26] See (4).
[27] See *Def.* 2.1.4.
[28] Correspondingly, we have $\frac{1}{t_r}=\frac{a-bi-cj}{a^2+b^2+c^2}=\frac{\bar{t_r}}{a^2+b^2+c^2}$. *Cf.* (5) and (6).
[29] See **Table 1**.

(resp. $k+(-k)$) $=0$. Furthermore, *Def.* 2.1.4 seems to reflect $a \pm b\,i$ and/or $a \pm bi \pm cj \pm dk$ where double-signs correspond. Taken together, t_r's might play a role in bridging a 'gap' between $\mathbb{C}$ and $\mathbb{H}$, though as of writing, we have no specific answer to *Question* 2.1.5 with us.

What about h_e's? Provided that we forget ℓ's in **Table 4** to strike out its rightmost two columns and lowermost two rows, we get **Table 5**, where basis elements are 1, i, j, k, which turn our attention to $\mathbb{H}$. Likewise, at the time of writing, we have no specific answer to *Question* 2.2.4. Notwithstanding, we suspect that h_e's lie somewhere between $\mathbb{H}$ and octonions .

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5 Appendix

5.1 Where does the symbol $\diamond$ come from?

We abstract $\star\star = \pm 1$ from the relation in footnote 1. Dropping its minus sign and squaring both sides , we obtain $\star \cdot \star \cdot \star \cdot \star = 1$, the left-hand side of which we intuitively replace by $\diamond \cdot \diamond \cdot \diamond \cdot \diamond$. Then, we imagine the equation $x^4 - 1 = 0$, which we solve in the realm of $\mathbb{C}$ to obtain $x = \pm 1$ and $\pm\, i$ .We plot these solutions on the complex plane as shown in the following.

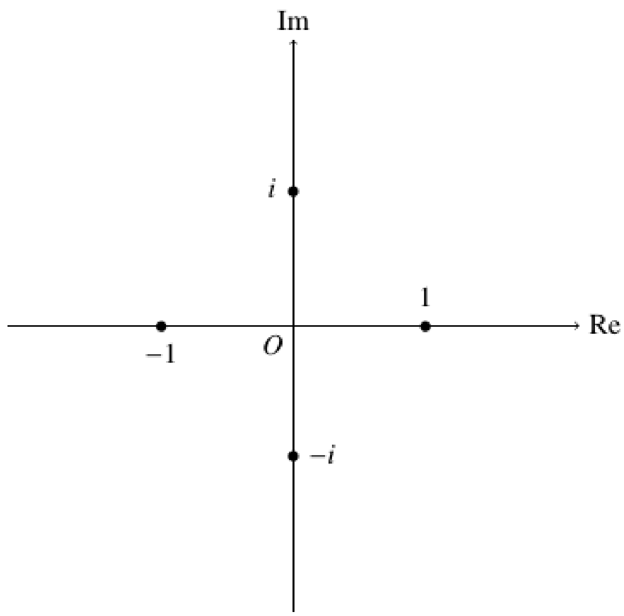

Fig. 1. Solutions to the equation $x^4 - 1 = 0$

We join the vertices 1, i, −1, and $-i$ in Fig. 1 to prepare the square below for the coming abstraction. [30]

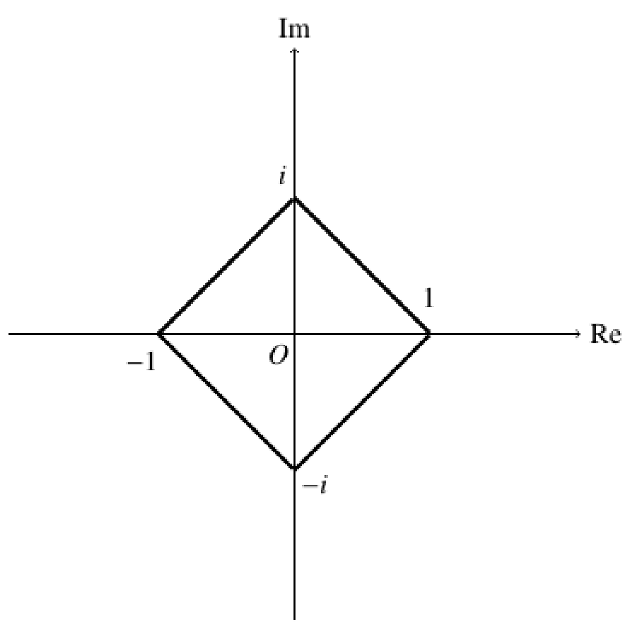

Fig. 2. Square before abstraction

Abstracting the square in Fig. 2 yields the symbol ◇, which suggests the aforementioned four solutions.

[30] *Cf.* here .

5.2 What if the symbols d and $\diamond$ were interchangeable?

5.2.1 t_r's

For example, we have $i =$ [31] $\diamond\diamond\diamond d\diamond =$ [32] $\diamond\diamond\diamond\diamond d = \diamond^4 \cdot d$ $=$ [33] $1 \cdot d = d$ and $j =$ [34] $\diamond d\diamond\diamond\diamond$ $=$ [35] $d\diamond\diamond\diamond\diamond = d \cdot \diamond^4 =$ [36] $d \cdot 1 = d$. Now that $i = j = d$, we get the following. [37]

Table 3. Slight simplification of **Table 1** [38]

$\times$	1	i
1	1	i
i	i	0

5.2.2 h_e's

We have, *e.g.*, $k =$ [39] $\diamond d =$ [40] $d\diamond =$ [41] ℓ, $m =$ [42] $\diamond d\diamond =$ [43] $\diamond\diamond d = \diamond\diamond \cdot d = \diamond^2 \cdot d =$ [44] $\pm 1 \cdot d$ $= \pm d =$ [45] $\pm j$ [46], and $n =$ [47] $d\diamond d =$ [48] $dd\diamond = d^2 \cdot \diamond =$ [49] $0 \cdot \diamond = 0$. [50] And we get the following.

[31] See *Def.* 2.1.3.
[32] This holds, because d and $\diamond$ are assumed to be interchangeable in this subsection.
[33] See *Def.* 2.1.2.
[34] See *Def.* 2.1.3.
[35] See footnote 32.
[36] See *Def.* 2.1.2.
[37] We have struck out the rightmost column and the lowermost row of **Table 1**.
[38] *Cf.* here .
[39] See *Def.* 2.2.1.
[40] See footnote 32.
[41] See *Def.* 2.2.1.
[42] Ditto.
[43] See footnote 32.
[44] See *Def.* 2.1.2.
[45] See *Def.* 2.2.1.
[46] Even if we calculate like $m = \diamond d\diamond = d\diamond\diamond = \cdots$, we can get $\pm j$.
[47] See *Def.* 2.2.1.
[48] See footnote 32.
[49] See *Def.* 2.1.1.
[50] Even if we calculate like $n = d\diamond d = \diamond dd = \cdots$, we are able to get 0. n can thus function as a 'null basis element'.

Table 4. Simplification of **Table 2** [51]

$\times$	1	i	j	$k\ (=\ell)$	$\pm j\ (=m)$	$0\ (=n)$
1	1	i	j	k	$\pm j$	0
i	i	± 1	k	$\pm j$	$\pm k$	0
j	j	k	0	0	0	0
$k\ (=\ell)$	k	$\pm j$	0	0	0	0
$\pm j\ (=m)$	$\pm j$	$\pm k$	0	0	0	0
$0\ (=n)$	0	0	0	0	0	0

The table above can be further simplified as follows:

Table 5. $\mathbb{H}$ -like simplification of **Table 4** [52]

$\times$	1	i	j	k
1	1	i	j	k
i	i	± 1	k	$\pm j$
j	j	k	0	0
k	k	$\pm j$	0	0

[51] In this table, we tried to decrease the number of letters containing ℓ, m, and n by using identifications such as $k = \ell$, $\pm j = m$, etc in order to make **Table 5** ensue without much difficulty.

[52] See arguments in **4**.