There is a gravitational plane wave pulse incident on a finite mass where conservation of energy and momentum does not hold

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Abstract

We present an example of a gravitational plane wave pulse incident on a finite mass where conservation of energy and momentum does not hold.

1 Gravitational plane wave pulse metric

Define $u = t - x$ and let the metric $g_{\mu\nu}(u)$ be [1]

$$g_{00}(u) = -1 \qquad g_{11}(u) = 1 \qquad g_{22}(u) = [L(u)]^2 e^{2\beta(u)} \qquad g_{33}(u) = [L(u)]^2 e^{-2\beta(u)} \tag{1}$$

$$g_{01}(u) = g_{02}(u) = g_{03}(u) = g_{12}(u) = g_{13}(u) = g_{23}(u) = 0 \tag{2}$$

having $g_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$ and

$$\frac{d^2L}{du^2}(u) + \left[\frac{d\beta}{du}(u)\right]^2 L(u) = 0 \tag{3}$$

This metric will satisfy $R_{\mu\nu} = 0$. It is the metric of a gravitational plane wave pulse.

2 Proper Lorentz transformation

Consider a coordinate transformation from t, x, y, z to t', x', y', z' coordinates that is a composition of a rotation by θ about the z axis followed by a boost by $2\cos\theta/(1+\cos^2\theta)$ in the x direction followed by a rotation by $\theta + \pi$ about the z axis. For θ/π not an integer this is a proper Lorentz transformation such that

$$t = t'(1 + 2\cot^2\theta) - 2x'\cot^2\theta + 2y'\cot\theta \tag{4}$$

$$x = 2t'\cot^2\theta + x'(1 - 2\cot^2\theta) + 2y'\cot\theta \tag{5}$$

$$y = 2t'\cot\theta - 2x'\cot\theta + y' \tag{6}$$

$$z = z' \tag{7}$$

By (4) and (5) we have $u = t - x = t' - x' = u'$. For (4)-(7) define the metric $g'_{\mu\nu}(u)$ by

$$g'_{\mu\nu}(u) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(u) \tag{8}$$

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hence for the metric (1), (2), we get

$$
\begin{aligned}
g'_{00}(u) &= -1-4[1-g_{22}(u)]\cot^2\theta && (9)\\
g'_{01}(u) &= 4[1-g_{22}(u)]\cot^2\theta && (10)\\
g'_{11}(u) &= 1-4[1-g_{22}(u)]\cot^2\theta && (11)\\
g'_{02}(u) &= -g'_{12}(u) = -2[1-g_{22}(u)]\cot\theta && (12)\\
g'_{22}(u) &= g_{22}(u) \qquad g'_{03}(u) = g'_{13}(u) = g'_{23}(u) = 0 && (13)\\
g'_{33}(u) &= g_{33}(u) && (14)
\end{aligned}
$$

Since $g_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$ we have $g'_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$. The metric $g'_{\mu\nu}(u)$ satisfies $R_{\mu\nu} = 0$ and $g'_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$ is then also the metric of a gravitational plane wave pulse.

Let $t^{\mu\nu}$ be the energy-momentum tensor of the gravitational field. It is detemined by the metric and is zero for the Minkowski metric. Let $t^{\mu\nu}(u)$ be the energy-momentum tensor of the gravitational field determined by the metric $g_{\mu\nu}(u)$. We have

$$t^{00}(u) = t^{01}(u) = t^{11}(u) \qquad t^{02}(u) = t^{03}(u) = t^{12}(u) = t^{13}(u) = t^{22}(u) = t^{23}(u) = t^{33}(u) = 0 \quad (15)$$

Let $t'^{\mu\nu}(u)$ be the energy-momentum tensor of the gravitational field determined by the metric $g'_{\mu\nu}(u)$. For the transformation (4)-(7) and by (15) we have

$$t'^{\mu\nu}(u) = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} t^{\alpha\beta}(u) = t^{\mu\nu}(u) \quad (16)$$

3 Variable G

We will be letting G be a variable. Let G_N be Newton's constant and let $f(u)$ be a smooth function that is zero for $u < 0$ and increasing for $u > 0$. Define the metric $g_{\mu\nu}(G,u)$ by letting $\beta(u)$ be $(G/G_N)^2 f(u)$ in (1). Now choose units so that $G_N = 1$. We then have by (1)-(3) that there are functions $W_2(G,u)$ and $W_3(G,u)$ such that

$$
\begin{aligned}
g_{22}(G,u) &= 1 + 2G^2 f(u) + 2G^4 W_2(G,u) && (17)\\
g_{33}(G,u) &= 1 - 2G^2 f(u) + 2G^4 W_3(G,u) && (18)
\end{aligned}
$$

In (9)-(14) let

$$\cot\theta = G^{-1} \quad (19)$$

and replace $g_{\mu\nu}(u)$ by $g_{\mu\nu}(G,u)$ and $g'_{\mu\nu}(u)$ by $g'_{\mu\nu}(G,u)$ giving

$$
\begin{aligned}
g'_{00}(G,u) &= -1 + 8f(u) + 8G^2 W_2(G,u) && (20)\\
g'_{01}(G,u) &= -8f(u) - 8G^2 W_2(G,u) && (21)\\
g'_{11}(G,u) &= 1 + 8f(u) + 8G^2 W_2(G,u) && (22)\\
g'_{02}(G,u) &= -g'_{12}(G,u) = 4Gf(u) + 4G^3 W_2(G,u) && (23)\\
g'_{22}(G,u) &= 1 + 2G^2 f(u) + 2G^4 W_2(G,u) && (24)\\
g'_{33}(G,u) &= 1 - 2G^2 f(u) + 2G^4 W_3(G,u) && (25)\\
g'_{03}(G,u) &= g'_{13}(G,u) = g'_{23}(G,u) = 0 && (26)
\end{aligned}
$$

Define $\bar{g}_{\mu\nu}(u)$ to be $g'_{\mu\nu}(0,u)$.

4 Wave and mass

Let the gravitational wave pulse with metric $g'_{\mu\nu}(G,u)$ having components (20)-(26) be incident on a finite mass A at rest at the origin. Let

$$\hat{g}_{\mu\nu}(G,t,\mathbf{x}) = \bar{g}_{\mu\nu}(u) + GQ_{\mu\nu}(G,t,\mathbf{x}) \quad (27)$$

be the metric of this system of wave and A where $GQ_{\mu\nu}(G,t,\mathbf{x})$ is the correction to $\bar{g}_{\mu\nu}(u)$ due to the G dependence of $g'_{\mu\nu}(G,u)$ and the G dependence of the gravitational field of A. When $G=0$ the metric of the wave is $\bar{g}_{\mu\nu}(u)$ and A has no gravitational field.

Let $t^{\mu\nu}(G,u)$ be the energy-momentum tensor of the gravitational field determined by the metric $g_{\mu\nu}(G,u)$ and $t'^{\mu\nu}(G,u)$ be the energy-momentum tensor of the gravitational field determined by the metric $g'_{\mu\nu}(G,u)$. Letting $G\to 0$ since $(G/G_N)^2 f(u)\to 0$ we have $g_{\mu\nu}(G,u)\to\eta_{\mu\nu}$ as $G\to 0$. Consequently $t^{\mu\nu}(G,u)\to 0$ as $G\to 0$ hence using (16) we have $t'^{\mu\nu}(G,u)=t^{\mu\nu}(G,u)\to 0$ as $G\to 0$. Since $g'_{\mu\nu}(G,u)\to\bar{g}_{\mu\nu}(u)$ and $t'^{\mu\nu}(G,u)\to 0$ as $G\to 0$ we can conclude the energy-momentum tensor $\bar{t}^{\mu\nu}(u)$ of the gravitational field determined by the metric $\bar{g}_{\mu\nu}(u)$ is zero.

Let $\hat{T}^{\mu\nu}(G,t,\mathbf{x})$ be the energy-momentum tensor of A and $\hat{t}^{\mu\nu}(G,t,\mathbf{x})$ the energy-momentum tensor of the gravitational field determined by the metric $\hat{g}_{\mu\nu}(G,t,\mathbf{x})$. Since $\hat{g}_{\mu\nu}(G,t,\mathbf{x})\to\bar{g}_{\mu\nu}(u)$ as $G\to 0$ we have $\hat{t}^{\mu\nu}(G,t,\mathbf{x})\to\bar{t}^{\mu\nu}(u)=0$. Consequently on assuming conservation of energy-momentum we have, as $G\to 0$, that

$$\frac{\partial\hat{T}^{\mu\alpha}}{\partial x^\alpha}=-\frac{\partial\hat{t}^{\mu\alpha}}{\partial x^\alpha}\to 0 \tag{28}$$

5 Contradiction

Let $\hat{\Gamma}^{\mu}_{\alpha\beta}(G,t,\mathbf{x})$ be the affine connection and $\hat{g}(G,t,\mathbf{x})$ the metric determinant both calculated using the metric $\hat{g}_{\mu\nu}(G,t,\mathbf{x})$. Assuming the coordinate free form of conservation of energy and monentum we also have

$$\hat{T}^{\mu\alpha}{}_{;\alpha}=\frac{1}{\sqrt{-\hat{g}}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-\hat{g}}\hat{T}^{\mu\alpha}\right)+\hat{\Gamma}^{\mu}_{\alpha\beta}\hat{T}^{\alpha\beta}=0 \tag{29}$$

Define $\bar{T}^{\mu\nu}(t,\mathbf{x})$ to be the limit of $\hat{T}^{\mu\nu}(G,t,\mathbf{x})$ as $G\to 0$. Taking the limit of (29) as $G\to 0$ we have by (28)-(29) and $\hat{g}_{\mu\nu}(G,t,\mathbf{x})\to\bar{g}_{\mu\nu}(u)$ that the first term of (29) goes to zero so from the second term

$$(\bar{T}^{00}-2\bar{T}^{01}+\bar{T}^{11})\frac{df}{du}=0 \tag{30}$$

Now $f(u)$ is an increasing function for $u>0$ so for $u>0$

$$\bar{T}^{00}-2\bar{T}^{01}+\bar{T}^{11}=0 \tag{31}$$

Before the wave comes in contact with A let A be a perfect fluid at rest with nonzero constant mass density. Position A so that at $t=0$ and $\mathbf{x}=0$ the wave first comes in contact with A. Since all of A is at rest at $t=0$ we have $\bar{T}^{01}(0)=0$. Also pressure is zero on the surface of A hence $\bar{T}^{11}(0)=0$. Consequently we must have by (31) that $\bar{T}^{00}(0)=0$ but with nonzero mass density $\bar{T}^{00}(0)\neq 0$ which is a contradiction.

6 Conclusion

We presented an example of a gravitational plane wave pulse incident on a finite mass where conservation of energy and momentum does not hold.

References

[1] C. Misner, K. Thorne, and J. Wheeler, Gravitation, p. 957 (Freeman & Co., San Fransico 1973)