Faster than light signals and colliding gravitational waves

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Abstract

We show there are two colliding gravitational plane wave pulses that result in signals that travel faster than light.

1 Introduction

We will be using [1]. We begin with a gravitational plane wave pulse with metric $g_{\mu\nu}$ so that [2]

$$ds^2 = -dt^2 + dx^2 + [L(u)]^2[e^{2\beta(u)}dy^2 + e^{-2\beta(u)}dz^2] \tag{1}$$

where $u = t - x$ and $g_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$. The Einstein field equations give for this metric

$$\frac{d^2L}{du^2} + \left(\frac{d\beta}{du}\right)^2 L = 0 \tag{2}$$

Since $g_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$ we must have $\beta(u) = 0$ for $u < 0$. Choose $\beta(u)$ so that $\beta(u)$ is increasing for small $u > 0$. Consequently by (2) there is a small $u_1 > 0$ such that $g_{22}(u_1) > 1$ and $(dg_{22}/du)(u_1) \neq 0$.

Let $dx/dt, dy/dt, dz/dt$ be components of the velocity of a signal. If this signal does not travel faster than light then

$$g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \leq 0 \tag{3}$$

2 Lorentz transformation

Consider a coordinate transformation from t, x, y, z to t', x', y', z' coordinates that is a composition of a rotation by θ about the z axis followed by a boost by $2\cos\theta/(1+\cos^2\theta)$ in the x direction followed by a rotation by $\theta + \pi$ about the z axis. For θ/π not an integer this is a proper Lorentz transformation such that

$$t = t'(1 + 2\cot^2\theta) - 2x'\cot^2\theta + 2y'\cot\theta \tag{4}$$

$$x = 2t'\cot^2\theta + x'(1 - 2\cot^2\theta) + 2y'\cot\theta \tag{5}$$

$$y = 2t'\cot\theta - 2x'\cot\theta + y' \tag{6}$$

$$z = z' \tag{7}$$

By (4) and (5) we have $u = t - x = t' - x' = u'$. For (4)-(7) the metric transforms as

$$g'_{\mu\nu}(u) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(u) \tag{8}$$

Since $g_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$ and (4)-(7) is a Lorentz transformation we have $g'_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$. The metric $g'_{\mu\nu}(u)$ is then also the metric of a gravitational plane wave pulse.

For the metric (1) we have by (4)-(8) that

$$\begin{aligned} g'_{00}(u) &= -1 + 4[g_{22}(u) - 1]\cot^2\theta && (9)\\ g'_{01}(u) &= -4[g_{22}(u) - 1]\cot^2\theta && (10)\\ g'_{11}(u) &= 1 + 4[g_{22}(u) - 1]\cot^2\theta && (11)\\ g'_{02}(u) &= -g'_{12}(u) = 2[g_{22}(u) - 1]\cot\theta && (12)\\ g'_{22}(u) &= g_{22}(u) \qquad g'_{03}(u) = g'_{13}(u) = g'_{23}(u) = 0 \qquad g'_{33}(u) = g_{33}(u) && (13) \end{aligned}$$

3 Velocity quadric surface

Define the quadric surface

$$\begin{aligned} S'(u) &= \Big\{(v'_x, v'_y, v'_z) \in \mathbb{R}^3 : g'_{00}(u) + 2g'_{01}(u)v'_x + g'_{11}(u)v'^2_x + 2g'_{12}(u)v'_xv'_y + 2g'_{02}(u)v'_y \\ &+ g'_{22}(u)v'^2_y + 2g'_{03}(u)v'_z + 2g'_{13}(u)v'_xv'_z + 2g'_{23}(u)v'_yv'_z + g'_{33}(u)v'^2_z = 0\Big\} \qquad (14) \end{aligned}$$

formed from $g'_{\mu\nu}(u)\frac{dx'^\mu}{dt'}\frac{dx'^\nu}{dt'} = 0$ by setting $v'_x = dx'/dt', v'_y = dy'/dt', v'_z = dz'/dt'$. Now $S(u)$ is symmetric about the plane having $v_z = 0$ hence $S'(u)$ is symmetric about the plane having $v'_z = 0$. For the transformation (4)-(7)

$$\begin{aligned} v'_x &= \frac{2\cot^2\theta + v_x(1 - 2\cot^2\theta) - 2v_y\cot\theta}{1 + 2\cot^2\theta - 2v_x\cot^2\theta + 2v_y\cot\theta} && (15)\\ v'_y &= \frac{-2\cot\theta + 2v_x\cot\theta + v_y}{1 + 2\cot^2\theta - 2v_x\cot^2\theta + 2v_y\cot\theta} && (16)\\ v'_z &= \frac{v_z}{1 + 2\cot^2\theta - 2v_x\cot^2\theta + 2v_y\cot\theta} && (17) \end{aligned}$$

From the denominator of (15)-(17) construct the line of $\mathbb{R}^2$

$$\Big\{(v_x, v_y) \in \mathbb{R}^2 : 1 + 2\cot^2\theta - 2v_x\cot^2\theta + 2v_y\cot\theta = 0\Big\} \qquad (18)$$

and the curve of $\mathbb{R}^2$ formed by setting $v_z = 0$ in $S(u_1)$

$$\Big\{(v_x, v_y) \in \mathbb{R}^2 : -1 + v_x^2 + g_{22}(u_1)v_y^2 = 0\Big\} \qquad (19)$$

Solving for points of intersection of the line and curve gives

$$v_x = \frac{g_{22}(u_1)\cot\theta(1 + 2\cot^2\theta) \pm \sqrt{-4[g_{22}(u_1) - 1]\cot^2\theta - g_{22}(u_1)}}{2\cot\theta[1 + g_{22}(u_1)\cot^2\theta]} \qquad (20)$$

Since $g_{22}(u_1) > 1$ we have

$$-4[g_{22}(u_1) - 1]\cot^2\theta - g_{22}(u_1) < 0 \qquad (21)$$

so at u_1 the line and curve have no points of intersection. Consequently the denominators of (15)-(17) are not zero for all $(v_x, v_y, v_z) \in S(u_1\ )$ hence v'_x, v'_y, v'_z are finite. We can conclude $S'(u_1)$ is an ellipsoid of $\mathbb{R}^3$. Thre is then a v'_- and a v'_+ with $v'_- < v'_+$ such that planes

$$\Big\{(v'_x, v'_y, v'_z) \in \mathbb{R}^3 : v'_x = v'_-\Big\} \qquad \Big\{(v'_x, v'_y, v'_z) \in \mathbb{R}^3 : v'_x = v'_+\Big\} \qquad (22)$$

are tangent to $S'(u_1)$. A point of $S'(u_1)$ will be on or between these planes. Now $S'(u_1)$ is symmetric about $v'_z = 0$ so the values $v'_\pm$ can be determined by taking the derivative of (14) and setting $v'_z = dv'_x/dv'_y = 0$. We obtain

$$2g'_{12}(u_1)v'_x + 2g'_{02}(u_1) + 2g'_{22}(u_1)v'_y = 0 \qquad (23)$$

Substituting v'_y from this equation into (14) with $v'_z = 0$ and solving the resulting quadratic equation for v'_x gives

$$\begin{aligned} v'_{\pm} &= \frac{-[g'_{01}(u_1)g'_{22}(u_1) - g'_{02}(u_1)g'_{12}(u_1)]}{g'_{11}(u_1)g'_{22}(u_1) - g'_{12}(u_1)^2} \\ &\pm \frac{\sqrt{[g'_{01}(u_1)g'_{22}(u_1) - g'_{02}(u_1)g'_{12}(u_1)]^2 - [g'_{11}(u_1)g'_{22}(u_1) - g'_{12}(u_1)^2][g'_{00}(u_1)g'_{22}(u_1) - g'_{02}(u_1)^2]}}{g'_{11}(u_1)g'_{22}(u_1) - g'_{12}(u_1)^2} \end{aligned} \tag{24}$$

Substituting (9)-(13) in (24) gives

$$v'_{-} = \frac{4[g_{22}(u_1) - 1]\cot^2\theta - g_{22}(u_1)}{4[g_{22}(u_1) - 1]\cot^2\theta + g_{22}(u_1)} \qquad v'_{+} = 1 \tag{25}$$

Letting $\theta \to 0$ we have since $g_{22}(u_1) > 1$ that $v'_{-} \to 1$ at u_1. Consequently if $g_{22}(u_1) > 1$ and θ close to zero then any point of $S'(u_1)$ has v'_x close to $+1$.

4 Colliding gravitational plane wave pulses

Define the function $u(t')$ by

$$\frac{du}{dt'}(t') = 1 - f(u) \qquad u(0) = 0 \tag{26}$$

where

$$f(u) = \min\left\{v'_x \in \mathbb{R} : (v'_x, v'_y, v'_z) \in S'(u)\right\} < v'_{+} = 1 \tag{27}$$

By (26) we can define t'_1 by

$$t'_1 = \int_0^{u_1} \frac{dw}{1 - f(w)} \tag{28}$$

As we saw in the previous section for θ close to zero any point of $S'(u_1)$ has v'_x close to $+1$ hence $f(u_1)$ is aproximately $+1$. Consequently we can choose θ so that $t'_1 > u_1$.

Let W'_{+} be the gravitational plane wave pulse $g'_{\mu\nu}(u)$ and let W'_{-} be the reflection of W'_{+} about the x' plane. For a system of two gravitational plane wave pulses approaching each other and colliding such that for $t' < 0$ the two waves are W'_{+} and W'_{-} we have at any time the metric to the right of the x' plane will be a reflection of the metric to the left of the x' plane.

Define $x'(t') = t - u(t')$. Define $P'(t') \subset \mathbb{R}^3$ to be the plane with normal the x' axis and $(x'(t'), 0, 0) \in P'(t')$. Since $(dx'/dt')(t')$ is the minimum v'_x of all points of $S'(t' - x'(t'))$ we have, on assuming signals cannot travel faster than light, that no signal originating to the right of $P'(t')$ can in time be to the left of $P'(t')$. The wave coming from the right can be viewed as a signal. The metric to the left of $P'(t')$ is then just the metric of the wave coming from the left with no interference from the wave coming from the right.

Since $t'_1 > u_1$ we have the origin is to the left of $(x'(u_1), 0, 0)$. Consequently the origin is to the left of $P'(u_1)$. The metric at the point $(u_1, 0, 0, 0)$ of $\mathbb{R}^4$ is then that due solely to the wave coming from the left. We began with $g_{\mu\nu}(u)$ having $(dg_{22}/du)(u_1) \neq 0$ hence $\partial g'_{22}/\partial x' \neq 0$ at the point $(u_1, 0, 0, 0)$. Consequently the metric to the right of the x' plane will not be a reflection of the metric to the left of the x' plane. This is a contradiction. There are signals that travel faster than light.

5 Conclusion

Starting with a gravitational plane wave pulse with metric so that (1) we can construct a system of two colliding gravitational plane wave pulses where signals travel faster than light.

References

[1] K. De Paepe, Physics Essays, December 2012

[2] C. M. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Fransico, CA, 1973), p. 957.