Conformal, Parameter-Free Riemannian Gravity

William O. Straub
Pasadena, California 91104
November 11, 2017
(Revised November 28, 2017)

Abstract

The Einstein-Hilbert action of general relativity is not invariant with respect to *conformal transformations* — transformations in which the metric tensor is varied continuously via $g_{\mu\nu} \to \exp\pi(x)\, g_{\mu\nu}$, where $\pi(x)$ is an arbitrary scalar function. This basic type of *local scale invariance* was introduced in 1918 by the German mathematical physicist Hermann Weyl, who believed that Nature should incorporate this type of mathematical symmetry into her laws.

While Einsteinian general relativity is not scale invariant, there does exist an elementary quantity that is conformally invariant in Riemannian geometry. Based on a combination of the Riemann-Christoffel curvature tensor $R_{\mu\nu\alpha\beta}$ and its contracted variants $R_{\mu\nu}$ and R^2, the gravitational equations of motion associated with this quantity (usually referred to as *conformal gravity* are highly complicated, and their physical relevance remains open to interpretation. In this elementary paper we show that the conformal equations of motion can be obtained from the R^2 term alone, equations that are parameter-free yet fully consistent with the standard predictions of Einstein's theory. We demonstrate the formalism by deriving the classical Schwarzschild metrics for a point mass, the field of a charged particle, and the Tolman-Oppenheimer-Volkoff equation of state, all of which are essentially identical to their classical forms.

1. Introduction

Einstein's 1915 theory of general relativity geometrized space and time, and its success motivated others to see if the only other force of Nature known at the time—electromagnetism—might be geometrized as well. In 1918, the German mathematical physicist Hermann Weyl proposed a more general form of Riemannian geometry that indeed appeared to accomplish this feat based on the notion of what is known today as *scale-* or *conformal* invariance. In doing so, Weyl was forced to introduce a vector field that he subsequently identified with the four-vector of electromagnetism. In Weyl's theory, this vector was a purely geometrical quantity that he associated with the Ricci scalar R and its first derivatives. Weyl's theory is actually an example of what is considered today a conformally invariant theory, meaning that physics should not change when the lengths (or magnitudes) of vector quantities are allowed to vary arbitrarily from point to point in spacetime. Although Weyl's theory was quickly shown to be unphysical, it gave birth to a variant of conformal symmetry known as *gauge invariance*, which today is a cornerstone of modern theoretical physics.

For purposes of review, the Einstein-Hilbert action for conventional free-space gravity theory is given by

$$S_{EH} = \int \sqrt{-g}\, R\, d^4x \tag{1.1}$$

where g is the determinant of the metric tensor (assumed to have the signature $(1,-1,-1,-1)$ and $R = g^{\mu\nu}R_{\mu\nu} = R^\mu_\mu$ is the Ricci scalar. Upon variation with respect to the metric tensor $g^{\mu\nu}$, this goes over to

$$\delta S_{EH} = \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu}\, d^4x \tag{1.2}$$

Setting the integrand to zero then gives the Einstein equations for free space,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{1.3}$$

Classical Einstein gravity has proven to be a highly successful theory; its planetary and cosmological predictions typically agree with observation to a very high degree. However, the theory has nothing to say regarding the phenomena known as dark matter and dark energy, a problem has spurred the search for modified versions of the theory. While the addition of the cosmological constant (-2Λ) term to the action yields the revised Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}, \tag{1.4}$$

it is not known whether it relates in any meaningful way to the dark energy question. Many attempts (going by the general name *modified gravity*) have been made over the years to generalize (1.1) by imposing additional parameters into the action, including scalar, vector, tensor and spinor fields designed to make the action conformally invariant and to produce field equations that might explain the dark energy and dark matter problems. To date, most such attempts have been discarded. Whether Nature demands such invariance in its laws (including gravity) is open to conjecture, but in quantum mechanics we find that it is responsible for the conservation of electric charge (where it appears as gauge invariance), a highly desirable aspect of quantum theory. The conformal invariance of gravity thus remains a tantalizing challenge.

Like Weyl, we will assume in the following that conformal invariance is (or should be) a fundamental aspect of Nature. Unlike Weyl's non-Riemannian approach, however, we will see that conformal symmetry can easily be incorporated into ordinary, parameter-free Riemannian gravity.

2. Notation

Much of the notation and analytical approaches follow those of Adler et al. Greek indices run from 0 to 3 as usual. Ordinary partial differentiation is expressed by a single subscripted bar before the differentiation variable, as in

$$\xi_{\alpha|\beta} = \frac{\partial \xi_\alpha}{\partial x^\beta}$$

Covariant differentiation is expressed by a subscripted double bar, as in

$$F^\lambda_{\ \mu\nu||\alpha} = F^\lambda_{\ \mu\nu|\alpha} + F^\beta_{\ \mu\nu} \begin{Bmatrix} \lambda \\ \alpha\beta \end{Bmatrix} - F^\lambda_{\ \mu\beta} \begin{Bmatrix} \beta \\ \alpha\nu \end{Bmatrix} - F^\lambda_{\ \beta\nu} \begin{Bmatrix} \beta \\ \alpha\mu \end{Bmatrix}$$

The quantities in braces are the Levi-Civita connection terms, symmetric in their lower indices, defined by

$$\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta|\nu} + g_{\beta\nu|\mu} - g_{\mu\nu|\beta} \right)$$

The end points of all the integrals appearing in this paper are $[\infty, -\infty]$. It is further assumed that surface terms resulting from integration by parts vanish during the variation process.

3. Conformal Transformations

By a conformal (or scale) transformation we mean a change in the metric given by $g'_{\mu\nu} = \exp[\pi(x)]\, g_{\mu\nu}$, where π is some arbitrary scalar field. Because the metric tensor accounts for vector length via $L^2 = g_{\mu\nu}\xi^\mu\xi^\nu$ while also defining the line element via $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, these quantities will naturally vary under a conformal transformation.

For simplicity, let us consider only infinitesimal transformations, $\exp(\pi) \to (1+\pi)$, so that

$$g'_{\mu\nu} = (1+\pi)\, g_{\mu\nu}, \quad g^{\mu\nu\prime} = (1-\pi)\, g^{\mu\nu}$$

or

$$\delta g_{\mu\nu} = \pi\, g_{\mu\nu}, \quad \delta g^{\mu\nu} = -\pi\, g^{\mu\nu} \tag{3.1}$$

where the infinitesimal variations we will be using are defined by $\delta g_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu}$ and $\delta g^{\mu\nu} = g^{\mu\nu\prime} - g^{\mu\nu}$. As for the metric determinant, it can easily be shown that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} \tag{3.2}$$

which, in four dimensions, is

$$\delta\sqrt{-g} = 2\,\pi\,\sqrt{-g} \tag{3.3}$$

In the following, we will also need the conformal variations of the Riemann-Christoffel curvature quantites $R_{\mu\nu\alpha\beta} = g_{\mu\lambda}R^{\lambda}{}_{\nu\alpha\beta}$, the contracted quantity $R_{\nu\beta} = R^{\alpha}{}_{\nu\alpha\beta}$ and the Ricci scalar $R = g^{\nu\beta}R_{\nu\beta}$, all of which are composed of the metric tensor and associated Levi-Civita connection terms. For review, we have

$$R^{\lambda}{}_{\nu\alpha\beta} = \begin{Bmatrix}\lambda\\ \nu\alpha\end{Bmatrix}_{|\beta} - \begin{Bmatrix}\lambda\\ \nu\beta\end{Bmatrix}_{|\alpha} + \begin{Bmatrix}\lambda\\ \sigma\beta\end{Bmatrix}\begin{Bmatrix}\sigma\\ \nu\alpha\end{Bmatrix} - \begin{Bmatrix}\lambda\\ \sigma\alpha\end{Bmatrix}\begin{Bmatrix}\sigma\\ \nu\beta\end{Bmatrix}$$

along with the contracted quantity $R_{\nu\beta} = R^{\alpha}{}_{\nu\alpha\beta}$, or

$$R_{\nu\beta} = \begin{Bmatrix}\alpha\\ \nu\alpha\end{Bmatrix}_{|\beta} - \begin{Bmatrix}\alpha\\ \nu\beta\end{Bmatrix}_{|\alpha} + \begin{Bmatrix}\alpha\\ \lambda\beta\end{Bmatrix}\begin{Bmatrix}\lambda\\ \nu\alpha\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \lambda\alpha\end{Bmatrix}\begin{Bmatrix}\lambda\\ \nu\beta\end{Bmatrix}$$

Lastly, we have the Ricci scalar

$$R = g^{\nu\beta}\,R_{\nu\beta}$$

The conformal variations of these quantities are surprisingly simple, and will be presented in the following sections.

4. Some Useful Identities

Any contravariant vector ξ^{μ} multiplied by $\sqrt{-g}$ is called a *vector density*. A commonly used identity in what follows relies heavily on the fact that the covariant divergence of a vector density is equivalent to its ordinary divergence:

$$\left(\sqrt{-g}\,\xi^{\mu}\right)_{||\mu} = \sqrt{-g}\,\xi^{\mu}{}_{||\mu} = \left(\sqrt{-g}\,\xi^{\mu}\right)_{|\mu} \tag{4.1}$$

as is easily verified by direct expansion using the contracted form of the Levi-Civita connection,

$$\begin{Bmatrix}\alpha\\ \mu\alpha\end{Bmatrix} = \left(\log\sqrt{-g}\right)_{|\mu}$$

The utility of (4.1) cannot be overemphasized, since ordinary divergences under an integral sign represent surface terms that go to zero at the boundaries of integration.

Another useful identity follows from the fact that an arbitrary variation of the Levi-Civita connection is a true tensor, whereas the Levi-Civita term by itself is not. Using this fact, it can be shown without difficulty that any arbitrary variation of the Riemann-Christoffel tensor collapses to the simple identity

$$\delta R^{\lambda}{}_{\nu\alpha\beta} = \left(\delta\begin{Bmatrix}\lambda\\ \nu\alpha\end{Bmatrix}\right)_{||\beta} - \left(\delta\begin{Bmatrix}\lambda\\ \nu\beta\end{Bmatrix}\right)_{||\alpha} \tag{4.2}$$

Similarly, we have

$$\delta R_{\nu\beta} = \delta R^{\lambda}{}_{\nu\lambda\beta} = \left(\delta\begin{Bmatrix}\lambda\\ \nu\lambda\end{Bmatrix}\right)_{||\beta} - \left(\delta\begin{Bmatrix}\lambda\\ \nu\beta\end{Bmatrix}\right)_{||\lambda}, \tag{4.3}$$

while the variation of the Ricci scalar is

$$\delta R = R_{\nu\beta}\,\delta g^{\nu\beta} + g^{\nu\beta}\left[\left(\delta\begin{Bmatrix}\lambda\\ \nu\lambda\end{Bmatrix}\right)_{||\beta} - \left(\delta\begin{Bmatrix}\lambda\\ \nu\beta\end{Bmatrix}\right)_{||\lambda}\right] \tag{4.4}$$

Because all the integral quantities we will encounter represent vector densities, we can integrate by parts using the above covariant derivatives and thus greatly simplify the calculations.

Lastly, for the conformal variation defined by (3.1) the variation of the Levi-Civita terms reduces to

$$\delta\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} = \frac{1}{2}\left(\delta^{\alpha}_{\mu}\,\pi_{|\nu} + \delta^{\alpha}_{\nu}\,\pi_{|\mu} - g_{\mu\nu}\,g^{\alpha\beta}\,\pi_{|\beta}\right) \tag{4.5}$$

while

$$\delta \begin{Bmatrix} \alpha \\ \mu\alpha \end{Bmatrix} = 2\sqrt{-g}\,\pi_{|\mu} \tag{4.6}$$

5. Standard Conformal Gravity

Because of the factor 2 in (4.6), conformal invariance necessarily requires that the action Lagrangian include two terms involving the upper-index metric tensor $g^{\mu\nu}$. A little thought shows that only the quadratic forms $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, R_{\mu\nu}R^{\mu\nu}$ and R^2 can be used to develop a conformally invariant Lagrangian in parameter-free Riemannian geometry. While this involves terms that are of fourth order with respect to the metric tensor and its derivatives (an undesirable aspect), we will discover that the action we end up with is only of second order, like the Einstein-Hilbert action itself.

To begin, we note that there exists a unique, conformally invariant action in Riemannian geometry that Weyl himself proposed in 1918. It is

$$S = \int \sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta}\, d^4x \tag{5.1}$$

where $C_{\mu\nu\alpha\beta}$ is known as the *Weyl conformal tensor*. In four dimensions, this quantity is defined via the purely conformally invariant form

$$C^{\lambda}{}_{\nu\alpha\beta} = R^{\lambda}{}_{\nu\alpha\beta} + \frac{1}{2}\left(\delta^{\lambda}_{\beta} R_{\nu\alpha} - \delta^{\lambda}_{\alpha} R_{\nu\beta} + g_{\nu\alpha} R^{\lambda}_{\beta} - g_{\nu\beta} R^{\lambda}_{\alpha}\right) + \frac{1}{6}\left(\delta^{\lambda}_{\alpha}\, g_{\beta\nu} - \delta^{\lambda}_{\beta}\, g_{\alpha\nu}\right) R \tag{5.2}$$

By laborious expansion, (5.1) can be shown to be equal to

$$C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \tag{5.3}$$

so that our action now appears as

$$S = \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) d^4x \tag{5.4}$$

Might (5.4) be suitable for deriving scale-invariant equations of motion? Perhaps, but the resulting equations are complicated by both the fourth-order issue and the presence of the $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ term in the Lagrangian, which makes calculating equations of motion exceedingly difficult. However, that term can be effectively eliminated using a clever approach first suggested by Lanczos in 1938, which will now detail following a more straightforward approach that is of interest in its own right.

Let us assume that the general conformally invariant action

$$S = \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} + BR^2\right) d^4x \tag{5.5}$$

exists, where A, B are constants. If we can find constants that maintain the conformal invariance of (5.4), then we can then subtract the resulting expression from (5.4) and thus eliminate the curvature term, leaving an invariant Lagrangian consisting of just two terms in $R_{\mu\nu}R^{\mu\nu}$ and R^2. As we will see, this approach in fact produces two solutions, with one involving the Bianchi identities.

We now proceed to pass the conformal variation operator δ through the integral in (5.5). For instructional purposes, we will demonstrate the procedure for the $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ term in detail, from which the variations of $\sqrt{-g}R_{\mu\nu}R^{\mu\nu}$ and $\sqrt{-g}R^2$ should then be transparent. Using the identities from (3.3) and (4.2), it is easy to show that

$$\delta S = 2\int \sqrt{-g}\, g_{\mu\lambda} R^{\lambda\nu\alpha\beta}\, \delta R^{\mu}{}_{\nu\alpha\beta}\, d^4x \tag{5.6}$$

or

$$\delta S = 2\int \sqrt{-g}\, g_{\mu\lambda} R^{\lambda\nu\alpha\beta}\left[\left(\delta \begin{Bmatrix}\mu \\ \nu\alpha\end{Bmatrix}\right)_{||\beta} - \left(\delta \begin{Bmatrix}\mu \\ \nu\beta\end{Bmatrix}\right)_{||\alpha}\right] d^4x$$

Because the curvature tensor is antisymmetric in α, β, this simplifies to

$$\delta S = 4\int \sqrt{-g}\, g_{\mu\lambda} R^{\lambda\nu\alpha\beta}\left(\delta \begin{Bmatrix}\mu \\ \nu\alpha\end{Bmatrix}\right)_{||\beta} d^4x$$

In view of (4.1), we now use integration by parts to reduce this to

$$\delta S = -4\int \sqrt{-g}\, g_{\mu\lambda} R^{\lambda\nu\alpha\beta}{}_{||\beta}\, \delta \begin{Bmatrix}\mu \\ \nu\alpha\end{Bmatrix} d^4x$$

Using (4.5), we then have

$$\delta S = -2\int \sqrt{-g}\, g_{\mu\lambda} R^{\lambda\nu\alpha\beta}{}_{||\beta}\left(\delta^{\mu}_{\ \nu}\pi_{|\alpha} + \delta^{\mu}_{\ \alpha}\pi_{|\nu} - g_{\nu\alpha}\, g^{\mu\kappa}\, \pi_{|\kappa}\right) d^4x$$

Now, using the fact the curvature tensor is antisymmetric in λ, ν, the identity $R^{\nu}{}_{\mu\nu\alpha} = R_{\mu\alpha}$ and some simple algebra, the student should have no difficulty showing that the variation of (5.6) reduces to

$$\delta S = -4\int \sqrt{-g}\, R^{\mu\nu}{}_{||\nu}\, \pi_{|\mu}\, d^4x \tag{5.7}$$

The above exercise can be used as a guide to show that the variations of the remaining terms in (5.5) are then

$$\delta \int \sqrt{-g}\, R_{\mu\nu} R^{\mu\nu}\, d^4x = -\int \sqrt{-g}\left(2R^{\mu\nu}{}_{||\nu} + g^{\mu\nu} R_{|\nu}\right)\pi_{|\mu}\, d^4x \tag{5.8}$$

and

$$\delta \int \sqrt{-g}\, R^2\, d^4x = -6\int \sqrt{-g}\, g^{\mu\nu} R_{|\nu}\, \pi_{|\mu}\, d^4x \tag{5.9}$$

(We will refrain from integrating by parts over the $\pi_{|\mu}$ terms because we do not need to.) Putting this all together, the conformal variation of (5.5) is then

$$\delta S = \int \sqrt{-g}\left[2(A+2)R^{\mu\nu}{}_{||\nu} + (A+6B)\, g^{\mu\nu} R_{|\nu}\right]\pi_{|\mu}\, d^4x \tag{5.10}$$

Setting the integrand to zero, we have the obvious solution $A = -2, B = 1/3$. However, this is just the expression we discovered in (5.4) by tedious calculation and so provides nothing new. On the other hand, if $A \neq -2$ then we can divide the $2(A+2)$ term out of (5.10), arriving at

$$\delta S = \int \sqrt{-g}\left[R^{\mu\nu}{}_{||\nu} + \frac{A+6B}{2(A+2)}\, g^{\mu\nu} R_{|\nu}\right]\pi_{|\mu}\, d^4x \tag{5.11}$$

If we now set

$$\frac{A+6B}{A+2} = -1, \tag{5.12}$$

then the integrand is just the set of Bianchi identities, which vanish automatically in Riemannian geometry:

$$\left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R\right)_{||\nu} = 0$$

With the exception of $A = -2$, we are free to choose any set of constants A, B that satisfies (5.12); the conventional choice (known as the Gauss-Bonnet solution) is $A = -4, B = 1$. The solution $A = -1, B = 0$ would seem to be a better choice (if only out of simplicity), but it really doesn't matter.

Using $A = -1, B = 0$ for convenience as our constants, (5.5) is now

$$S = \int \sqrt{-g} \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - R_{\mu\nu} R^{\mu\nu} \right) d^4x \tag{5.13}$$

We now need only subtract (5.4) from (5.13) to rid ourselves of the full Riemann-Christoffel curvature term. This results in the conformally invariant action

$$S = \int \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) d^4x \tag{5.14}$$

Equation (5.14) is recognized as the formal action of modern conformal gravity theory.

By varying (5.14) with respect to $g^{\mu\nu}$, in 1989 Mannheim and Kazanas managed to derive an exact solution for a spherically symmetric gravitational field. The solution they obtained was very similar to the Schwarzschild metric, along with several additional terms that the researchers believed might be relevant to the cosmological dark matter and dark energy problems.

6. Toward a Simpler Approach to Conformal Gravity

While the Mannheim-Kazanas solution is exact, the associated equations of motion are highly complicated and unwieldy, and questions regarding their applicability and relevance to gravitational physics and cosmology remain. This prompts one to step back and ask why so much complication should be associated with a conformally invariant version of gravity, particularly if one considers such an invariance to be a basic law of Nature. It must be remembered that the goal after all is to derive a scale-invariant action. The simplification of (5.4) to (5.14) is noteworthy, but it is still complicated, and it also depends on the validity of the Biachi identities, which generally do not hold in non-Riemannian geometries (such as the one Weyl proposed).

In looking back at Weyl's effort, one may ask if the action in (5.14) might be simplified even further. A glance at (5.9), which was derived for the R^2 term alone, indicates that it is itself conformally invariant if we impose the condition that the Ricci scalar R be a non-zero constant. Although Weyl had to assume a non-Riemannian geometry in his 1918 effort, we will now show that the R^2 term represents a perfectly valid basis for conformal gravity in Riemannian space when the Ricci scalar is considered a non-zero constant. This is not a new approach, as Einstein's equations long ago showed that the Ricci scalar can be associated with the cosmological constant Λ in Riemannian geometry. In that case, it is easily shown that for $\Lambda = R/4$, Einstein's equations for free space become

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = 0$$

This form of Einstein's equations has the characteristic of being *traceless*, a convenient characteristic that coincides with the tracelessness of the energy-momentum tensor associated with electromagnetism. Indeed, the expression

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = -\frac{8\pi G}{c^2} \left(F_{\alpha\mu} F^{\alpha}_{\ \nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)$$

(where $F_{\mu\nu}$ is the antisymmetric electromagnetic tensor) is completely traceless, and represents a quantity that Einstein himself believed might correctly represent gravity in the presence of an electromagnetic field. But unlike Weyl, who sought to geometrize electromagnetism in his effort to unify the equations of Maxwell with general relativity, we will restrict ourselves to Riemannian geometry, and focus on the possibility that the simple Lagrangian R^2 can provide a conformally invariant gravity theory.

As noted earlier, the action for the $\sqrt{-g}\, R^2$ Lagrangian is conformally invariant when the Ricci scalar R is a constant. We will now show that, with the added constraint that R be a *non-zero* constant, the associated equations of motion reproduce all of the traditional predictions of Einsteinian gravity.

We therefore focus solely on the Lagrangian quantity $\sqrt{-g}\, R^2$ in Riemannian space. As we have shown, its action is scale invariant, but we now move to its more general variation, tha twith respect to the metric tensor $g^{\mu\nu}$. This is straightforward, since the Lagrangian consists only of the metric tesnor and its first derivatives. We therefore consider the variation of

$$\delta S = \delta \int \sqrt{-g}\, R^2\, d^4x$$

with respect to $g^{\mu\nu}$ which, following the identities presented earlier, can easily be seen to be

$$\delta S = \int \sqrt{-g} \left[R \left(R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R \right) + R_{|\mu||\nu} - g_{\mu\nu}\, g^{\alpha\beta}\, R_{|\alpha||\beta} \right] \delta g^{\mu\nu}\, d^4x \tag{6.1}$$

Given the constraint that R be a constant, this reduces to

$$\delta S = \int \sqrt{-g}\, R \left(R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R \right) \delta g^{\mu\nu}\, d^4x \tag{6.2}$$

Before proceeding any further, let us note that the dimensionality of the metric tensor is identically zero, while that of the Riemann-Christoffel tensor and its variants—in view of the derivatives appearing in any variant—are of length^{-2}. Consequently, the dimensionality of the integrand of (6.2) is length^{-4}. By comparison, that of the conventional Einstein equations is just length^{-2}. This may be considered a virtue, given that the overall dimensionality of the quantity $\sqrt{-g}\, R^2\, d^4x$ is zero, but it will present a problem later on when matter fields are considered for the action. For example, if the density ρ and pressure P/c^2 of some matter distribution were given as kg/m^3, then the dimensionality of the combined quantity $8\pi G/c^2$ is just length^{-2} (if matter is specified by *energy density*, then we use $8\pi G/c^4$). Consequently, we'll need to find a solution to the dimensionality issue associated with (6.2).

We first consider the free-space equations

$$R\,(R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R) = 0$$

which strongly resemble the Einstein field equations for free space. We can divide out R, since we must have $R =$ constant in order to preserve scale invariance. Therefore, our field equations are given by

$$R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R = 0 \quad (R \neq 0) \tag{6.3}$$

For a spherically symmetric, time-independent point mass, we assume the usual Schwarzschild metric

$$ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2 \right) \tag{6.4}$$

applies, where ν, λ are functions of the radial coordinate r alone. From any text on general relativity, we find the only non-zero terms are

$$R_{00} = -\frac{1}{2}\, e^{\nu-\lambda} \left(\nu'' + \frac{1}{2}\, \nu'^{\,2} - \frac{1}{2}\, \nu'\lambda' \right) - \frac{1}{r}\, e^{\nu-\lambda}\, \nu' \tag{6.5}$$

$$R_{11} = \frac{1}{2} \left(\nu'' + \frac{1}{2}\, \nu'^{\,2} - \frac{1}{2}\, \nu'\lambda' \right) - \frac{1}{r}\, \lambda' \tag{6.6}$$

$$R_{22} = \frac{1}{2}\, r\, e^{-\lambda} \left(\nu' - \lambda' \right) + e^{-\lambda} - 1 \tag{6.7}$$

$$R_{33} = \sin^2\theta\, R_{22} \tag{6.8}$$

and

$$R = -e^{-\lambda} \left(\nu'' + \frac{1}{2}\, \nu'^{\,2} - \frac{1}{2}\, \nu'\lambda' \right) - \frac{2}{r}\, e^{-\lambda} \left(\nu' - \lambda' \right) - \frac{2}{r^2} \left(e^{-\lambda} - 1 \right) \tag{6.9}$$

where the primes indicate differentiation with respect to r. From (6.3), we then have the three equations

$$-\frac{1}{2}\, e^{\nu-\lambda} \left(\nu'' + \frac{1}{2}\, \nu'^{\,2} - \frac{1}{2}\, \nu'\, \lambda' \right) - \frac{1}{r}\, e^{\nu-\lambda}\, \nu' - \frac{1}{4}\, e^{\nu} R = 0 \tag{6.10}$$

$$\frac{1}{2}\left(\nu'' + \frac{1}{2}\,\nu'^{\,2} - \frac{1}{2}\,\nu'\,\lambda'\right) - \frac{1}{r}\,\lambda' + \frac{1}{4}\,e^{\lambda} R = 0 \tag{6.11}$$

and

$$\frac{1}{2}\,r\,e^{-\lambda}\left(\nu' - \lambda'\right) + e^{-\lambda} - 1 + \frac{1}{4}\,r^2 R = 0 \tag{6.12}$$

Like the standard Schwarzschild solution for general relativity, it is easy to see that $\nu' = -\lambda'$ and that the general solution is

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} - ar^2$$

where $2m, a$ are constants. We will assume that m is the geometrical radius GM/c^2 as usual. The constant a can be related to RR by plugging the terms for e^{ν}, e^{λ} into (6.9), where we find that $a = R/12$. Therefore,

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} - \frac{1}{12}\,R\,r^2 \tag{6.13}$$

As noted earlier, the Ricci scalar R has dimension length^{-2}, so that e^{ν}, e^{λ} are dimensionless, as expected. Note also that the free-space expression (6.3) is of second order in the metric. Furthermore, note while that R in the free-space Einstein equations is identically zero, we cannot make such an assumption for R in the present case. However, we are free to make it as negligibly small as we like. Consequently, all the traditional predictions of free-space Einsteinian gravity (advance of Mercury's perihelion, the deflection of light, gravitational redshift, etc.) are preserved in our conformal approach.

7. The Field of a Charged Particle in Conformal Gravity

The gravitational field around a charged particle for our simple conformal approach follows almost exactly the associated Schwarzschild problem, and we demonstrate it here for completeness. For review purposes, the Einstein equations for gravity in an electromagnetic field are given by

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu} R = -\frac{8\pi G}{c^2}\,T_{\mu\nu}$$

where the energy-momentum tensor for electromagnetism is

$$T_{\mu\nu} = F_{\alpha\mu}\,F^{\alpha}{}_{\nu} - \frac{1}{4}\,g_{\mu\nu}\,F_{\alpha\beta}\,F^{\alpha\beta}$$

Note that the right-hand side is both traceless and conformally invariant, characteristics that are lacking on the left-hand side (Einstein's lament that the "marble" of geometry is not matched by the "wood" of energy-momentum is quite apparent here). Nevertheless, the equations are easily solved, giving

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} + \frac{GQ^2}{4\pi c^2 r^2} \tag{7.1}$$

where Q is the charge of the particle. The details of the solution can be found in any text on general relativity, so we will not expound on them here.

For our conformal approach, we wish to write

$$R\left(R_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu} R\right) = -\frac{8\pi G}{c^2}\left(F_{\alpha\mu}\,F^{\alpha}{}_{\nu} - \frac{1}{4}\,g_{\mu\nu}\,F_{\alpha\beta}\,F^{\alpha\beta}\right)$$

However, as we alluded to earlier, there is an issue with dimensions here, as the left side is clearly of dimension length^{-4} while the right is of length^{-2}. Since we will find that the Ricci scalar R is again undefinable in this case, we might as well append it to the gravitational constant G and divide it out from both sides, so that

$$R_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu} R = -\frac{8\pi G}{c^2}\left(F_{\alpha\mu}\,F^{\alpha}{}_{\nu} - \frac{1}{4}\,g_{\mu\nu}\,F_{\alpha\beta}\,F^{\alpha\beta}\right)$$

Using the same Schwarzschild approach as before, we find that the equations are exactly solvable, except in this case we have

$$e^{\nu}=e^{-\lambda}=1-\frac{2m}{r}+\frac{GQ^2}{4\pi c^2 r^2}-\frac{1}{12}Rr^2 \tag{7.2}$$

Again, the magnitude of R is undefined, so we can make the Ricci scalar as small as we want, and we recover the classical expression for the gravitational field of a point electromagnetic charge.

8. The Tolman-Oppenheimer-Volkoff Equation for Conformal Gravity

As a final example, we examine the applicability of our formalism to the interior of a massive uncharged body. We will examine this case in some detail, because while the classical Tolman-Oppenheimer-Volkoff (TOV) equation is again recovered, the solution for the metric time component e^{ν} differs substantially from the classical case involving a body (a star, say) having a constant density.

We consider the conformal field equations for the interior of a body of incoherent matter having a density $\rho(r)$ and pressure $P(r)$. The classical energy-momentum tensor for such a body is given as

$$T^{\mu\nu}=\rho\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}+\frac{P}{c^2}\left(\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}-g^{\mu\nu}\right)$$

where dx^{μ}/ds is the velocity four-vector. In lower-case matrix notation, we have

$$T_{\mu\nu}=\begin{bmatrix} e^{\nu}\rho & 0 & & 0 \\ 0 & e^{\lambda}P/c^2 & 0 & 0 \\ 0 & 0 & r^2P/c^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta\ P/c^2 \end{bmatrix}$$

With the density and pressure components expressed in terms of kg/m^3, the field equations (adjusted as before with R to preserve dimensionality) are

$$R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R=-\frac{8\pi G}{c^2}T_{\mu\nu}$$

At this point a problem arises: like the energy-momentum tensor for electromagnetism, the $T_{\mu\nu}$ for a matter field must be traceless. This requires that

$$g^{\mu\nu}T_{\mu\nu}=\rho-\frac{3P}{c^2}=0 \tag{8.1}$$

or $\rho=3P/c^2$. In cosmology, equations of state in which this relationship holds are valid only for bodies composed of perfect relativistic fluids, such as a photon gas. To avoid this restriction, let us instead consider the completely traceless expression

$$R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R=\kappa\left(T_{\mu\nu}-\frac{1}{4}g_{\mu\nu}T\right)$$

where the coefficient κ is to be determined. Given the above expression for the stress-energy tensor, it will prove convenient to use the equation of state for a perfect baryotropic cosmological fluid

$$\frac{P}{c^2}=\omega\rho \tag{8.2}$$

where ω is a dimensionless constant that depends on the fluid composition. Non-relativistic matter corresponds to $\omega\approx 0$, while that of a highly relativistic gas is $\omega=1/3$ (another example is $\omega=-1$ for the cosmological constant).

As before, let us write down the terms corresponding to R_{00}, R_{11}, R_{22} (again, $R_{33}=\sin^2\theta\, R_{22}$) and R, respectively:

$$-\frac{1}{2}e^{\nu-\lambda}\left(\nu''+\frac{1}{2}\nu'^{\,2}-\frac{1}{2}\nu'\lambda'\right)-\frac{1}{r}e^{\nu-\lambda}\nu'-\frac{1}{4}e^{\nu}R=\kappa\left[e^{\nu}\rho-\frac{1}{4}e^{\nu}\left(\rho-\frac{P}{c^2}\right)\right]$$

which, in view of (8.2), is

$$-\frac{1}{2}\,e^{\nu-\lambda}\left(\nu''+\frac{1}{2}\,\nu'^{\,2}-\frac{1}{2}\,\nu'\lambda'\right)-\frac{1}{r}e^{\nu-\lambda}\,\nu'-\frac{1}{4}\,e^{\nu}R=\frac{3}{4}\,\kappa e^{\nu}\rho\,(\omega+1) \tag{8.3}$$

Similarly,

$$\frac{1}{2}\left(\nu''+\frac{1}{2}\,\nu'^{\,2}-\frac{1}{2}\,\nu'\lambda'\right)-\frac{1}{r}\,\lambda'+\frac{1}{4}\,e^{\lambda}R=\frac{1}{4}\,\kappa e^{\lambda}\rho\,(\omega+1) \tag{8.4}$$

along with

$$\frac{1}{2}\,r\,e^{-\lambda}\left(\nu'-\lambda'\right)+e^{-\lambda}-1+\frac{1}{4}\,r^2R=\frac{1}{4}\,\kappa r^2\rho\,(\omega+1) \tag{8.5}$$

and

$$R=-e^{-\lambda}\left(\nu''+\frac{1}{2}\,\nu'^{\,2}-\frac{1}{2}\,\nu'\lambda'\right)-\frac{2}{r}\,e^{-\lambda}\left(\nu'-\lambda'\right)-\frac{2}{r^2}\,\left(e^{-\lambda}-1\right) \tag{8.6}$$

Using (8.3) and (8.4), we can show that

$$\nu'+\lambda'=-\kappa r\rho\,(\omega+1)\,e^{\lambda} \tag{8.7}$$

while (8.5) is equivalent to

$$\nu'-\lambda'=\left[\frac{1}{2}\,\kappa r\rho\,(\omega+1)-\frac{2}{r}\,\left(e^{-\lambda}-1\right)-\frac{1}{2}\,Rr\right]e^{\lambda} \tag{8.8}$$

Subtracting (8.8) from (8.7) and reducing, we arrive at an expression involving only $e^{-\lambda}$ and its derivative:

$$e^{-\lambda}\lambda'=-\frac{3}{4}\,\kappa r\rho\,(\omega+1)+\frac{1}{r}\,\left(e^{-\lambda}-1\right)+\frac{1}{4}\,Rr \tag{8.9}$$

At this point, let us assume a solution for $e^{-\lambda}$ like that in (6.13), but with the provision that the geometric mass m is now a function of the interior radial coordinate r:

$$e^{-\lambda}=1-\frac{2m(r)}{r}-\frac{1}{12}\,Rr^2 \tag{8.10}$$

so that

$$e^{-\lambda}\lambda'=\frac{2m'}{r}-\frac{2m}{r^2}+\frac{1}{6}\,Rr \tag{8.11}$$

and (8.9) reduces to

$$e^{-\lambda}\lambda'=-\frac{3}{4}\,\kappa r\rho\,(\omega+1)-\frac{2m}{r^2}+\frac{1}{6}\,Rr \tag{8.12}$$

Equating (8.11) and (8.12), we see that

$$m'=-\frac{3}{8}\,\kappa r^2\rho\,(\omega+1) \tag{8.13}$$

Now, the geometric mass $m(r)=GM(r)/c^2$, and we know that

$$M'(r)=4\pi\rho r^2$$

where the fluid density ρ in general is some function of r. This is the classical expression for the mass of a body as a function of the density and the radius. For constant density, we have simply

$$M=\frac{4\pi\rho r^3}{3}$$

as expected. Comparing the above expressions for m' and M', we see that the coefficient κ is then

$$\kappa=-\frac{32\pi G}{3\,c^2(\omega+1)} \tag{8.14}$$

For a relativistic fluid, $\omega = 1/3$ and (8.14) reduces to the more familiar

$$\kappa = -\frac{8\pi G}{c^2}$$

In order to solve for R, we need to have identities for $e^{-\lambda}\nu''$, $e^{-\lambda}\nu'^{\,2}$ and $e^{-\lambda}\nu'\lambda'$. We can accomplish this by using (8.7) and (8.12) to solve for $e^{-\lambda}\nu'$, which comes out to be

$$e^{-\lambda}\nu' = -\frac{1}{4}\kappa r\rho\,(\omega+1) + \frac{2m}{r^2} - \frac{1}{6}Rr \tag{8.15}$$

from which all the other terms we'll need to compute R can be determined. After some tedious (and messy) algebra, these identities are

$$e^{-\lambda}\nu'' = e^{-\lambda}\nu'\lambda' - \kappa\rho\,(\omega+1) - \frac{1}{4}\kappa r\rho'\,(\omega+1) - \frac{4m}{r^3} - \frac{1}{6}R \tag{8.16}$$

$$e^{-\lambda}\nu'^{\,2} = \left[\frac{1}{16}\kappa^2 r^2\rho^2\,(\omega+1)^2 + \frac{4m^2}{r^4} + \frac{1}{36}R^2r^2 - \frac{1}{r}\kappa m\rho\,(\omega+1) + \frac{1}{12}\kappa r^2 R\rho\,(\omega+1) - \frac{2mR}{3r}\right]e^{\lambda} \tag{8.17}$$

and

$$e^{-\lambda}\nu'\lambda' = \left[\frac{3}{16}\kappa^2 r^2\rho^2\,(\omega+1)^2 - \frac{1}{r}\kappa m\rho\,(\omega+1) + \frac{1}{12}\kappa r^2 R\rho\,(\omega+1) - \frac{4m^2}{r^4} + \frac{2mR}{3r} - \frac{1}{36}R^2r^2\right]e^{\lambda} \tag{8.18}$$

Plugging all this into the definition of R in (8.6), we find that most terms cancel out, leaving

$$e^{-\lambda}\rho' = \frac{1}{2}\kappa r\rho^2\,(\omega+1) - \frac{4m\rho}{r^2} + \frac{1}{3}Rr\rho$$

We can now use the equation of state (8.2) to get the derivative of the fluid pressure. Using the identity for κ from (8.14), we have, after some simple reduction,

$$P' = -4\,\frac{\omega}{\omega+1}\left(\rho + \frac{P}{c^2}\right)\frac{\left(m + \frac{4\pi G r^3 P}{3\omega c^4} - \frac{1}{12}Rr^3\right)c^2}{r\left(r - 2m - \frac{1}{12}Rr^3\right)} \tag{8.19}$$

For a relativistic fluid, $\omega = 1/3$, and this reduces to

$$P' = -\left(\rho + \frac{P}{c^2}\right)\frac{\left(m + 4\pi G r^3 P/c^4 - \frac{1}{12}Rr^3\right)c^2}{r\left(r - 2m - \frac{1}{12}Rr^3\right)} \tag{8.20}$$

Note that the Ricci scalar R reappears, but the only restriction on its magnitude is $R \neq 0$. Again, we can set it to some small constant, and in doing so we recover the classical TOV equation.

While it is heartening that our conformal gravity approach appears capable of reproducing three quantities normally associated with ordinary Einsteinian gravity, it can be shown to fail when we consider the case of a relativistic fluid sphere having a constant gas density ρ. Unlike the classical TOV case, the various differential equations we have derived from the field equations lead to contradictory results when we consider such a system. With $\rho =$ constant, Equation (8.7) can be integrated immediately, yielding (with the appropriate constant of integration) the interior solution

$$e^{\nu} = \left(1 - r_0^2/\hat{R}^2\right)^2\left(1 - r^2/\hat{R}^2\right)^{-1} \tag{8.21}$$

Here, r is the interior distance from the fluid sphere's center, r_0 is the sphere's radius at the surface, and $\hat{R}$ is a convenient radial constant defined by

$$\hat{R}^2 = \frac{3c^2}{8\pi G\rho}$$

In (8.21) we set $\omega = 1/3$ and $R = 0$ for clarity, but even so it in no way resembles the classical TOV interior solution

$$e^{\nu} = \left(\frac{3}{2}\sqrt{1 - r_0^2/\hat{R}^2} - \frac{1}{2}\sqrt{1 - r^2/\hat{R}^2}\right)^2 \tag{8.22}$$

At the sphere's surface and beyond, both solutions revert to the usual $e^{\nu} = 1 - 2m/r$ expression, but the interior solution (8.21) differs significantly from (8.22) for typical values of $r/\hat{R}$ and $r_0/\hat{R}$. The most plausible reason for this is that our formalism simply does not hold in situations in which matter is represented by a traceless energy-momentum tensor.

9. Conclusions

Theories of modified gravity have multiplied in recent decades, mostly with the intent of obtaining a conformally invariant action that is linear in the Ricci scalar (like the Einstein-Hilbert action) coupled with various *ad hoc* parameters (scalar, vector, tensor and spinor fields) in the hope of providing a solution to the dark matter and dark energy problems. The work of Mannheim and others, based on their solutions to the full set of field equations derived from the action in (5.14), is particularly interesting.

The intent of this paper is far less ambitious. Certainly, application of the simplified action presented in Section 6 to the examples of a radial electromagnetic field and the interior of a baryotropic fluid is overly simplistic. But by merely requiring that the Ricci scalar in Riemannian geometry be a non-zero constant, the parameter-free action

$$S = \int \sqrt{-g}\, R^2\, d^4 x \tag{9.1}$$

is not only conformally invariant, but for free space yields a traceless set of field equations that are of only second order in the metric tensor and its derivatives, a distinct advantage over fourth-order approaches. The field equations in turn yield solutions for simple Schwarzschild problems that appear to be equivalent to those of classical general relativity. For finite values of R in the metric coefficients

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} - \frac{1}{12} R r^2$$

the Ricci scalar can be shown to behave like a cosmological acceleration term via

$$\Phi = -\frac{GM}{r} - \frac{1}{24} R c^2 r^2$$

where Φ is the classical gravitational potential. However, recent research by Mannheim, Mureika and others indicates that R is likely to be vanishingly small, eliminating the possibility that a Ricci scalar of finite magnitude can explain the observed acceleration of the expansion of the universe. Even if this is indeed the case, (9.1) is still equivalent to the Einstein-Hilbert action, and this is exactly what we set out to demonstrate.

References

1. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, McGraw-Hill, 2nd. Edition, 1975.
2. C. Lanczos, *A remarkable property of the Riemann-Christoffel tensor in four dimensions*, Ann. Math. **39** 4 (1938) pp. 842-850.
3. P. D. Mannheim and D. Kazanas, *Exact vacuum solution to conformal Weyl gravity and galactic rotation curves*, Astrophys. J. 342 (1989).
4. P. D. Mannheim, Prog. Part. Nucl. Phys. **56**, 340 (2006), astro-ph/0505266.
5. J. R. Mureika and G. U. Varieschi, *Black hole shadows in fourth-order conformal Weyl gravity*, arXiv:1611.00399 [gr-qc] 16, November 2017.
6. P. Payandeh and M. Fathi, *R^2 theory of gravity*, J. Phys. Conf. Series, **442** (2013).