**Abstract**

It is shown that an isobaric time independent perfect fluid equation mostly exists and that the state equation is highly likely of no relevance

# A proof of the inutility of the perfect fluid conjecture.

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## 1 Introduction

The perfect fluid conjecture is a rather vaguely formulated idea regarding the non existence of solutions to the Einstein equation for a perfect fluid satisfying some equation of state. We show in this short note that a very precise characterization is not possible by constructing an infinite dimensional manifold of rotational solutions with a nonzero uniform expansion rate.

## 2 Equations.

On $\Sigma$; $V^\alpha V_\alpha = 1$, $h^\alpha_\gamma h^\beta_\delta V_{(\alpha;\beta)} = a h_{\gamma\delta}$ (an example of the traceless case), equations $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = (\rho + p) V_\alpha V_\beta - p g_{\alpha\beta}$. Here, $h^{\alpha\beta} = V_\alpha V_\beta - g_{\alpha\beta}$ and raising and lowering of indices ocurs with the spacetime metric. These can be interpreted as

$$W^\beta Z^\alpha V_{(\alpha;\beta)} = a W^\alpha Z_\alpha$$

together with the normalization $V^\alpha V_\alpha = 1$ on $\Sigma$ and $Z^\alpha, W^\beta \in T\Sigma$. Now, it is convenient tot take for $V^\alpha$ a geodesic so that the normalization condition is satisfied. Integrability of the second condition then gives

$$V^\kappa \nabla_\kappa (W^\beta Z^\alpha V_{(\alpha;\beta)}) = 0$$

which together with

$$V^\kappa \nabla_\kappa W^\beta = 0$$

and likewise for $Z^\alpha$, gives

$$W^\beta Z^\alpha V^\kappa V_{(\alpha;\beta);\kappa} = 0.$$

The latter is equivalent to

$$R_{\alpha\kappa\beta\gamma} W^\beta Z^\alpha V^\kappa V^\gamma = 0$$

by means of the geodesic equation and first Bianchi identity. The Einstein equations lead to

$$R_{\alpha\beta} W^\beta Z^\alpha - (\frac{1}{2} R - p) W_\alpha Z^\alpha = 0$$

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which together with the conservation of energy momentum lead to

$$((\rho+p)V_\alpha V_\beta)^{;\beta} - p_{,\alpha} = 0.$$

The geodesic equation together with conservation of the fluid current

$$(\rho V^\alpha)_{;\alpha} = 0$$

leads to

$$(pV^\beta)_{;\beta}V_\alpha - p_{,\alpha} = 0.$$

Supposing now that $V^\alpha$ is hypersurface orthogonal, the equation is automatically satisfied given that $\Gamma^t_{tt} = 0$ in the coordinate system where $V = \partial_t$ and $g = dt^2 - h_{ij}(t,x)dx^i dx^j$. So, the constraint equations are

$$R_{ti} = 0, R_{tt} - \frac{1}{2}R = \rho, R_{ij} + \frac{1}{2}Rh_{ij} = ph_{ij}, R_{itjt} = 0$$

with initial condition

$$h^i_k h^j_l V_{(i;j)} = ah_{kl}$$

on the spatial part of the geodesic congruence and $\rho_{,t} = 0$. The Riemann conditions are compatible with the Einstein equations provided $R_{tt} = 0, R = R_{ij}h^{ij} = 2\rho$ and $R = R_{ij}h^{ij} = 2p$ and therefore $p = \rho$. The last condition can easily be destroyed by the introduction of a cosmological constant or instanton field and therefore, in nature, the energy density and pressure relation may be thought of as irrelevant although it mathematically appears here as a constraint due to the geodesic and hypersurface orthogonal character of the congruence. Dropping the last conditions would lead to different consistency conditions between the Riemann tensor, the fluid acceleration and vector which would induce consistency conditions on the fluid vector of a generically non integrable type leading to new relationships involving the pressure and energy density as well as the acceleration of the fluid lines which would intertwine with the previous integrability conditions. Therefore, mathematically, there will always be a relation between $p, \rho$ and the spatial and timelike components of the acceleration vector with respect to the fluid vector and it may be that some sectors are mathematically, but not effective physically, precluded. Physically, these relations are meaningless due to the introduction of an instanton or Higgs field. Specifically, in the geodesic case, the integrability on the pressure would be

$$\left((pV_\beta)^{;\beta}V_{[\alpha}\right)_{;\gamma]} = 0$$

which is equivalent to

$$(pV^\beta)_{;\beta[\gamma}V_{\alpha]} + (pV^\beta)_{;\beta}V_{[\alpha;\gamma]} = 0.$$

Using that $V_\alpha = ((pV^\beta)_{;\beta})^{-1}p_{,\alpha}$, this relation reduces to

$$(pV^\beta)_{;\beta[\gamma}p_{,\alpha]} + \left((pV^\beta)_{;\beta}\right)^2 V_{[\alpha;\gamma]} = 0.$$

This can be further reduced to

$$(V^\beta_{;[\gamma}p_{,\beta} + pV^\beta_{;\beta[\gamma} + V^\beta p_{,\beta[\gamma)}p_{,\alpha]} + \left((pV^\beta)_{;\beta}\right)^2 V_{[\alpha;\gamma]} = 0$$

and taking into account the geodesic property, this reduces to

$$(pV^\beta)_{;\beta[\gamma}p_{,\alpha]}V^\alpha = 0 = (pV^\beta_{;\beta} + p_{,t})_{[i}p_{,t]}$$

for the $t, i$ components. The $i \neq j$ components give further consistency conditions on the first time derivatives of $p$ which would need to be consistent with the previous three equations. The generic case is therefore $(pV^{\beta})_{;\beta} = 0$ which is the same as the mass conservation law but does not impose any restrictions on the rotation tensor. The Einstein equations remain the same and compatibility between the constraint equations and them needs to be investigated. In the more general case, where the acceleration can be nonzero, this would impose constraints on the acceleration and induce a new constraint problem on the initial data of the pressure versus energy density given that both generically satisfy different equations now. It would be interesting to see if submanifolds of generic acceleration exist which allow for solutions of the entire system. No such constraints were needed in the hypersurface orthogonal metric case at least what concerns the metric part.

Generically, the problem is rather a silly one; allowing for non-isobaric pressure tensors with the appropriate current conservation law, ie

$$g_{\alpha\beta} = V_{\alpha}V_{\beta} - h_{\alpha\beta}$$

with $V^{\alpha}h_{\alpha\beta} = 0$ and

$$(\rho V^{\alpha})_{;\alpha} = 0$$

and

$$T_{\alpha\beta} = \rho V_{\alpha}V_{\beta} + \sum_i p_i E_{i\alpha}E_{i\beta}$$

where $\sum_i E_i E_i = h$, it is to be expected that solutions exist for generic types of equations of state with functionally two free "pressure components" instead of three. This is to be expected given that balancing is required between energy and momentum and the gravitational field proved the on shell condition. In the hypersurface orhogonal free falling fluid no gravitationbal properties were really present given the absence of a Newtonian potential energy.