REAL-VALUED DIRAC EQUATION AND THREE-DIMENSIONAL DIFFERENTIABLE STRUCTURES OF QUANTUM PARTICLES

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Abstract: Having shown in our previous works that the real-valued Schrödinger wave equation can be used to find mathematical functions to construct spacetime structures of quantum particles, in this work, we will discuss the possibility to formulate a real-valued Dirac equation in which all physical objects and all differential operators that are used to describe the dynamics of a particle are real quantities and, furthermore, since solutions to the Dirac equation are wavefunctions that have four components, it is possible to suggest that solutions to the real-valued Dirac equation should be interpreted as a parameterisation of 3-dimensional differentiable manifolds which are embedded submanifolds of the Euclidean space R^4.

In our previous works on the quantum structures of elementary particles [1,2,3], we suggested that instead of viewing elementary particles as point-particles we consider elementary particles as three-dimensional differentiable manifolds, therefore we will need to extend the description of the dynamics of elementary particles in classical physics as point-particles to the dynamics of elementary particles as three-dimensional differentiable manifolds in an ambient space. Being viewed as three-dimensional differentiable manifolds, elementary particles are assumed to possess internal geometrical and topological structures that in turns possess internal symmetries that give rise to intrinsic dynamics. In this work, we will show that it is possible to formulate a real-valued Dirac equation in which all physical objects and all differential operators that are used to describe their dynamics are real quantities and solutions to the real-valued Dirac equation can be interpreted as a parameterisation of 3-dimensional differentiable manifolds that represent quantum particles as embedded submanifolds of the Euclidean space R^4. Also shown in our works on spacetime structures of quantum particles [4,5], in the time-independent Schrödinger wave mechanics, because the wavefunctions ψ are real-valued scalar functions therefore even though they themselves cannot be used to represent three-dimensional differentiable manifolds they can be employed as mathematical objects to construct the three-dimensional manifold structures of quantum particles via the Ricci scalar. For example, the spacetime structures of quantum particles can be constructed by applying the equation

$$-\frac{3}{c^2 DA}\frac{\partial^2 A}{\partial t^2}+\frac{2}{A^2}\nabla^2 A+\frac{3}{2A^3}(\boldsymbol{\nabla} A)^2=k\left(m\sum_{\mu=1}^{3}\left(\frac{dx^\mu}{dt}\right)^2-\hbar\frac{\partial_t\psi+\sum_{\mu=1}^{3}\partial_\mu\psi\left(\frac{dx^\mu}{dt}\right)}{\psi}\right) \qquad (1)$$

with the quantities D and A are defined according to a line element of the form

$$ds^2 = D(cdt)^2 - A(x, y, z, t)((dx)^2 + (dy)^2 + (dx)^2) \tag{2}$$

Unlike the Schrödinger wave equation, the Dirac equation is a system of four coupled first order partial differential equations whose solutions are wavefunctions that have four components. With this representation, the solutions of the Dirac equation can be used to described 3-dimensional submanifolds that are embedded into the four-dimensional Euclidean space R^4.

REAL VALUED DIRAC EQUATION IN THE PSEUDO-EUCLIDEAN SPACETIME

In the Minkowski spacetime of Einstein special relativity with pseudo-Euclidean metric, the energy-momentum relationship is given as

$$E^2 = (mc^2)^2 + (pc)^2 \tag{3}$$

From this relationship, the Dirac relativistic first order partial differential equation can be formulated by proposing that it is of the form [6]

$$E\psi = \left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m\right)\psi \tag{4}$$

where the unknown operators α_i and β are assumed to be independent of the momentum p and the mass m. From Equation (4), we obtain

$$E^2\psi = \left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m\right)^2\psi \tag{5}$$

By expanding Equation (5), and due the fact that all linear momentum operators commute mutually, in order to reduce to the form of the relationship given in Equation (3), the operators α_i and β must satisfy the following relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \text{for } i \neq j \tag{6}$$

$$\beta\alpha_i + \alpha_i\beta = 0 \tag{7}$$

$$\alpha_i^2 = 1 \tag{8}$$

$$\beta^2 = 1 \tag{9}$$

As shown in Appendix 1, to satisfy the conditions given in Equations (6-9), the operators α_i and β can be represented as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{10}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

where σ_i are Pauli matrices given by $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If we multiply Equation (4) by the operator β on the left and by applying the method of quantisation in quantum mechanics in which the energy E, the momentum p and the mass m are replaced by operators

$$E \to i\frac{\partial}{\partial t}, \quad p_x \to -i\frac{\partial}{\partial x}, \quad p_y \to -i\frac{\partial}{\partial y}, \quad p_z \to -i\frac{\partial}{\partial z} \quad \text{and} \quad m \to m \tag{12}$$

then with the mathematical units in which $\hbar = c = 1$, the Dirac equation can be rewritten in a covariant form as

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \tag{13}$$

where $\partial_\mu = \left(\partial_t, \partial_x, \partial_y, \partial_z\right)$, $\gamma^i = \beta\alpha_i$ and $\gamma^0 = \beta$. In the original equation formulated by Dirac, the order of the σ_i operators are as follows

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

therefore the order of the α_i operators are given accordingly as

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{15}$$

together with the operator β defined by

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{16}$$

Using $\gamma^i = \beta\alpha_i$ and $\gamma^0 = \beta$, the γ^i operators are found in the following order

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{17}$$

With the γ^i operators given in Equation (17), the Dirac equation given in Equation (13) can be written out in the full form as below

$$\begin{pmatrix} i\frac{\partial}{\partial t} & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t} & i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} & 0 \\ -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t} \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}=m\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \tag{18}$$

Now we will show that the Dirac equation can be formulated to take real values rather than complex values as given in Equation (13). Instead of the order of the α_i operators given in the original Dirac equation as in Equation (14), we use a new order by swapping the operators α_2 and β. Accordingly, we now have the following order of the α_i operators and β

$$\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\quad \alpha_2=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\quad \alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix} \tag{19}$$

together with the operator β defined by

$$\beta=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}=i\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}=i\alpha_0 \tag{20}$$

If we still apply the canonical method of quantisation in quantum mechanics given in Equation (12), we obtain the new form of the Dirac equation

$$i\frac{\partial\psi}{\partial t}=-i\alpha_1\frac{\partial\psi}{\partial x}-i\alpha_2\frac{\partial\psi}{\partial y}-i\alpha_3\frac{\partial\psi}{\partial z}+i\alpha_0 m\psi \tag{21}$$

It is seen from Equation (21) that since all the α_i operators now take real values, the new form of the Dirac equation is real and can be rewritten as

$$\frac{\partial\psi}{\partial t}=-\alpha_1\frac{\partial\psi}{\partial x}-\alpha_2\frac{\partial\psi}{\partial y}-\alpha_3\frac{\partial\psi}{\partial z}+\alpha_0 m\psi \tag{22}$$

Multiplying Equation (22) by α_0 on the left and since $\alpha_0^2=-1$, we obtain

$$\left(\gamma^\mu\partial_\mu+m\right)\psi=0 \tag{23}$$

where $\gamma^i=\alpha_0\alpha_i$ and $\gamma^0=\alpha_0$. The γ^μ operators can be calculated as

$$\gamma^0=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix},\qquad \gamma^1=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{24}$$

With the γ^i operators given in Equation (24), the real valued Dirac equation given in Equation (23) can also be written out as

$$\begin{pmatrix} -\frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial t}+\frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial t}-\frac{\partial}{\partial y} & 0 \\ 0 & -\frac{\partial}{\partial t}-\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t}+\frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = -m \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{25}$$

For the case of a particle at rest with $\mathbf{p} = 0$, the system of first-order partial differential equations given in Equation (25) reduces to following system of equations

$$\frac{\partial \psi_4}{\partial t} = m\psi_1 \tag{26}$$

$$\frac{\partial \psi_1}{\partial t} = -m\psi_4 \tag{27}$$

$$\frac{\partial \psi_3}{\partial t} = -m\psi_2 \tag{28}$$

$$\frac{\partial \psi_2}{\partial t} = m\psi_3 \tag{29}$$

It is seen from Equations (26-29) that the component ψ_1 is coupled with the component ψ_4, the component ψ_2 is coupled with the component ψ_3 and all the components ψ_μ execute a simple harmonic motion with respect to time described by the following equation

$$\frac{\partial^2 \psi_\mu}{\partial t^2} + m^2 \psi_\mu = 0 \tag{30}$$

It is interesting to observe that the frequency of the harmonic motion is determined by the mass of an elementary particle. Equation (30) can be solved to give solutions to the real valued Dirac equation for a particle at rest, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ as follows

$$\psi = \begin{pmatrix} A\cos(mt+\varepsilon_1) \\ B\sin(mt+\varepsilon_2) \\ B\cos(mt+\varepsilon_2) \\ A\sin(mt+\varepsilon_1) \end{pmatrix} \tag{31}$$

$$\psi = \begin{pmatrix} A\sin(mt+\varepsilon_1) \\ B\sin(mt+\varepsilon_2) \\ B\cos(mt+\varepsilon_2) \\ -A\cos(mt+\varepsilon_1) \end{pmatrix} \tag{32}$$

$$\psi = \begin{pmatrix} A\cos(mt+\varepsilon_1) \\ B\cos(mt+\varepsilon_2) \\ -B\sin(mt+\varepsilon_2) \\ A\sin(mt+\varepsilon_1) \end{pmatrix} \tag{33}$$

$$\psi = \begin{pmatrix} A\sin(mt+\varepsilon_1) \\ B\cos(mt+\varepsilon_2) \\ -B\sin(mt+\varepsilon_2) \\ -A\cos(mt+\varepsilon_1) \end{pmatrix} \tag{34}$$

The undetermined quantities ε_1 and ε_2 are constants but while the quantities A and B are time-independent, they may depend on the spatial coordinates (x, y, z). From the solutions given in Equations (31-34), we obtain the relations $\psi_1^2 + \psi_4^2 = A^2$, $\psi_2^2 + \psi_3^2 = B^2$ and $\psi^2 = r^2$ with $r^2 = A^2 + B^2$. While two pairs of components of the solutions form two separate circles, the solutions themselves form three-dimensional manifolds embedded in the four-dimensional space of states. These results suggest that the Dirac solutions with four components can be reinterpreted as a parameterisation of three-dimensional differentiable manifolds embedded in the four-dimensional Euclidean space R^4. Similar to the case when the circle $x_1^2 + x_2^2 = r^2$ is embedded in the plane R^2 with coordinates (x_1, x_2) given by the parameterisation $(x_1, x_2) = (rcost, r\text{sin}t)$, solutions to the Dirac equation given in Equations (31) can be considered as a parameterisation of a 3-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2$ embedded in the Euclidean space R^4 with coordinates (x_1, x_2, x_3, x_4) by the parameterisation $(x_1, x_2, x_3, x_4) = \big(A\cos(mt+\varepsilon_1), B\sin(mt+\varepsilon_2), B\cos(mt+\varepsilon_2), A\sin(mt+\varepsilon_1)\big)$, with the condition $\varepsilon_1 = \varepsilon_2$. In this case, as shown in Appendix 2, circles forming by two pairs of coordinates can be visualised as the spinning rotor of a gyroscope whose intrinsic angular momentum can take half-integral values which are determined by the Schrödinger wave equation in two-dimensional space.

REAL VALUED DIRAC EQUATION IN THE EUCLIDEAN SPACETIME

As shown in our works on the Euclidean relativity [7], the energy-momentum relationship of a particle is given as

$$E^2 = (mc^2)^2 - (pc)^2 \tag{35}$$

The energy-momentum relationship given in Equation (35) differs from that in the pseudo-Euclidean Minkowski spacetime by the negative sign of the momentum term. We also showed that it is still possible to formulate a relativistic Dirac wave equation by proposing a first order relativistic partial differential equation in the form given in Equation (4). However, in order to reduce to the form of the relationship given in Equation (35), the operators α_i and β must now satisfy the following relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \text{for } i \neq j \tag{36}$$

$$\beta\alpha_i + \alpha_i\beta = 0 \tag{37}$$

$$\alpha_i^2 = -1 \tag{38}$$

$$\beta^2 = 1 \tag{39}$$

As shown in Appendix 1, to satisfy the conditions given in Equations (36-39), the operators α_i and β can be represented as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{40}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{41}$$

where σ_i are Pauli matrices given by $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If we multiply Equation (4) by the operator β, then by applying the canonical method of quantisation in quantum mechanics given in Equation (12), the Dirac equation can be rewritten in a covariant form as

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{42}$$

where $\partial_\mu = \left(\partial_t, \partial_x, \partial_y, \partial_z\right)$, $\gamma^i = \beta\alpha_i$ and $\gamma^0 = \beta$. If we follow the order of the σ_i operators as given in the original Dirac equation in Equation (14) then the order of the α_i operators are as follows

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{43}$$

together with the operator β defined by

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{44}$$

Using the relations $\gamma^i = \beta\alpha_i$ and $\gamma^0 = \beta$, the γ^i operators can be calculated and given in the following order

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{46}$$

Then Dirac equation given in Equation (42) takes the form

$$\begin{pmatrix} i\frac{\partial}{\partial t} & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t} & i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} & 0 \\ i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = m \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{47}$$

Now instead of the order of the α_i operators given in the original Dirac equation, we change to a new order by swapping the operator α_2 and β according to the following

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{48}$$

with the operator β now is defined by

$$\beta = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = i\alpha_0 \tag{49}$$

If we also apply the canonical method of quantisation in quantum mechanics given in Equation (12), then we obtain the following equation

$$i\frac{\partial \psi}{\partial t} = -i\alpha_1\frac{\partial \psi}{\partial x} - i\alpha_2\frac{\partial \psi}{\partial y} - i\alpha_3\frac{\partial \psi}{\partial z} + i\alpha_0 m\psi \tag{50}$$

The Dirac equation given in Equation (50) is rewritten as

$$\frac{\partial \psi}{\partial t} = -\alpha_1\frac{\partial \psi}{\partial x} - \alpha_2\frac{\partial \psi}{\partial y} - \alpha_3\frac{\partial \psi}{\partial z} + \alpha_0 m\psi \tag{51}$$

Multiplying Equation (51) by α_0 on the left and since $\alpha_0^2 = 1$, we obtain

$$\left(\gamma^\mu \partial_\mu - m\right)\psi = 0 \tag{52}$$

where $\gamma^i = \alpha_0\alpha_i$ and $\gamma^0 = \alpha_0$ are given as

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{53}$$

With the γ^i operators given in Equation (53), the real valued Dirac equation given in Equation (52) can also be written out as

$$\begin{pmatrix} \frac{\partial}{\partial x} & -\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial t}+\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} & \frac{\partial}{\partial t}-\frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial t}+\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial t}-\frac{\partial}{\partial y} & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = m \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{54}$$

For the case of particle at rest in which $\mathbf{p} = 0$, the system of first-order partial differential equations given in Equation (54) reduces to following system

$$\frac{\partial \psi_4}{\partial t} = -m\psi_1 \tag{55}$$

$$\frac{\partial \psi_1}{\partial t} = -m\psi_4 \tag{56}$$

$$\frac{\partial \psi_3}{\partial t} = m\psi_2 \tag{57}$$

$$\frac{\partial \psi_2}{\partial t} = m\psi_3 \tag{58}$$

It is seen from Equations (55-58) that all components ψ_μ execute a motion described by the following equation

$$\frac{\partial^2 \psi_\mu}{\partial t^2} - m^2\psi_\mu = 0 \tag{59}$$

Equation (59) can be solved to give solutions to the real valued Dirac equation for a particle at rest, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ as follows

$$\psi = \begin{pmatrix} Ae^{-mt} \\ Be^{mt} \\ Be^{mt} \\ Ae^{-mt} \end{pmatrix} \tag{60}$$

In the Euclidean plane R^2 with coordinates (x_1, x_2), a hyperbola of the form $x_1.x_2 = C$, where C is a constant, can be parameterised by the parameter equations $x_1 = Ae^{-kt}$ and $x_2 = Be^{kt}$, where A, B and k are constants. Similarly, if we consider the solution given in Equation (60) as a parameterisation of a 3-dimensional manifold embedded in the Euclidean R^4 with the parameter equations $(x_1, x_2, x_3, x_4) = (Ae^{-mt}, Be^{mt}, Be^{mt}, Ae^{-mt})$, then we obtain the relations $x_1.x_2 = AB$, $x_3.x_4 = AB$ and $x_1.x_2.x_3.x_4 = (AB)^2$. The geometrical, topological and physical properties of this type of three-dimensional manifolds which are embedded in the ambient Euclidean space R^4 require further investigation.

Appendix 1

Assume the operators α_i are represented in terms of the operators σ_i in the forms

$$\alpha_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{1}$$

then we obtain

$$\alpha_i^2 = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix} \tag{2}$$

If $\sigma_i^2 = 1$ then $\alpha_i^2 = 1$. On the other hand, if $\sigma_i^2 = -1$ then $\alpha_i^2 = -1$. Now, if the operators α_i are given in the forms

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{3}$$

then in this case we have

$$\alpha_i^2 = \begin{pmatrix} -\sigma_i^2 & 0 \\ 0 & -\sigma_i^2 \end{pmatrix} \tag{4}$$

If $\sigma_i^2 = 1$ then $\alpha_i^2 = -1$. On the other hand, if $\sigma_i^2 = -1$ and then $\alpha_i^2 = 1$.

Now, if we write the operator σ_i as a two by two matrix in the form

$$\sigma_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5}$$

then from the requirement $\sigma_i^2 = 1$, we arrive at the following system of equations for the unknown quantities a, b, c and d

$$a^2 + bc = 1 \tag{6}$$

$$b(a + d) = 0 \tag{7}$$

$$c(a + d) = 0 \tag{8}$$

$$d^2 + bc = 1 \tag{9}$$

From Equations (6) and (9) we require $d = \pm a$. If $d = a \neq 0$ then $b = c = 0$ and the operator σ_i can take the values $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. If $d = -a$ and if $b = c = 0$, then the operator σ_i can be as $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. If $d = -a$ but $b \neq 0$ and $c \neq 0$, then the operator σ_i can be written in the form $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\sigma_i = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. These are only a few standard representations of the operators σ_i. It is also seen from the representations of the operators α_i given in Equations (1) that there are many different combinations that can be chosen for the operators α_i and β to satisfy the following relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \qquad \text{for } i \neq j \tag{10}$$

$$\beta \alpha_i + \alpha_i \beta = 0 \tag{11}$$

$$\alpha_i^2 = 1 \tag{12}$$

$$\beta^2 = 1 \tag{13}$$

The most common use of the forms of the operators α_i is that they are defined in terms of Pauli matrices $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$. In this case the operators α_i are found as follows

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{14}$$

In addition, if the operator β is defined in terms of the operators σ_i as $\beta = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$ then with $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the operator β takes the form

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

Appendix 2

In this appendix, we will show that if the topological structure of an elementary particle is assumed to possess the structure of a gyroscope then it is possible to obtain the half-integral values for its intrinsic spin angular momentum. In classical mechanics, expressed in plane polar coordinates, the Lagrangian of a particle of mass m under the influence of a conservative force with potential $V(r)$ is given as follows [8]

$$L = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) - V(r) \tag{1}$$

With this Lagrangian, the canonical momentum p_θ is found as

$$p_\theta = \frac{\partial L}{\partial(d\theta/dt)} = mr^2\frac{d\theta}{dt} \tag{2}$$

The canonical momentum given in Equation (2) is the angular momentum of the system. By applying the Lagrange equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial(dq_i/dt)} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \dots, n \tag{3}$$

where q_i are the generalised coordinates, we obtain

$$\frac{dp_\theta}{dt} = \frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = 0 \tag{4}$$

It is shown that the conservation of angular momentum of a particle moving in a plane given by Equation (4) is equivalent to the conservation of the areal velocity, which is the area swept out by the position vector of the particle per unit time. Assume at the time t, the particle is located at the position $\mathbf{r}(t)$ from an origin O and at the time $t + \Delta t$, the particle has moved to the position $\mathbf{r}(t + \Delta t)$, then the area $\mathbf{A}$ swept out by the particle is

$$\mathbf{A} = \frac{\mathbf{r}(t) \times \mathbf{r}(t + \Delta t)}{2} \tag{5}$$

From Equation (5), the areal velocity is found as

$$\frac{d\mathbf{A}}{dt} = \frac{\mathbf{r}(t) \times \mathbf{v}(t)}{2} \tag{6}$$

On the other hand, in classical dynamics, the angular momentum of the particle is defined by the relation

$$\mathbf{L} = \mathbf{r}(t) \times m\mathbf{v}(t) \tag{7}$$

From Equations (6) and (7), we obtain the following relationship between the angular momentum $\mathbf{L}$ of a particle and the areal velocity $d\mathbf{A}/dt$

$$\mathbf{L} = 2m\frac{d\mathbf{A}}{dt} \tag{8}$$

In magnitudes, we have

$$L = 2m\frac{dA}{dt} \tag{9}$$

It is seen from these results that the use of conservation of angular momentum for the description of the dynamics of a particle can be replaced by the conservation of areal velocity. For example, consider the circular motion of a particle under an inverse square field $F = kq^2/r^2$. Applying Newton's second law, we obtain

$$\frac{mv^2}{r} = \frac{kq^2}{r^2} \tag{10}$$

Using Equations (9) and (10) and the relation $L = mrv$, we obtain

$$r = \frac{4m}{kq^2}\left(\frac{dA}{dt}\right)^2 \tag{11}$$

The total energy E of the particle is

$$E = \frac{1}{2}mv^2 - \frac{kq^2}{r} = -\frac{kq^2}{2r} \tag{12}$$

Using Equations (11), the total energy can be re-written as

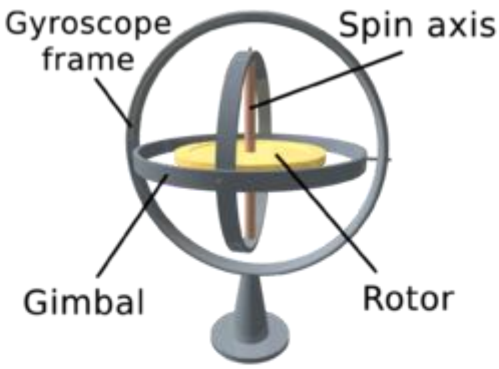
$$E = -\frac{k^2q^4}{8m\left(\frac{dA}{dt}\right)^2} \tag{13}$$

It is seen from Equation (13) that the total energy of the particle depends on the rate of change of the area dA/dt. In the case of Bohr's model of a hydrogen-like atom, from the quantisation condition $mrv = nh/2\pi$, we have

$$\frac{dA}{dt} = n\left(\frac{h}{4\pi m}\right) \tag{14}$$

Equation (14) shows that the rate of change of the area swept out by the electron is quantised in unit of $h/4\pi m$. These considerations suggest that the physical dynamics of an elementary particle may approximately be described in terms of a membrane.

It has been stated in quantum mechanics that spin is an intrinsic angular momentum that must be assigned to an elementary particle and the spin cannot be interpreted in terms of classical dynamics. In the following, however, we will show that if elementary particles are assumed to possess an internal structure that has the topological structure of a gyroscope, whose main dynamics is that of the rotor, as visualised in the figure below [9], then it is possible to show that elementary particles have an intrinsic angular momentum that can take half-integral values.



If the main component of the topologically gyroscopic structure of an elementary particle is the rotor then the elementary particle can be viewed as a planar system whose configuration space is multiply connected. With these assumptions, since the Schrödinger wave equation $-\frac{\hbar^2}{2\mu}\nabla^2\psi(r) - V(r)\psi(r) = E\psi(r)$ is invariant under rotations therefore we can invoke the Schrödinger wave equation for an analysis of the dynamics of the rotor. If we also assume that the overall potential $V(r)$ that holds the rotor together has the form $V(r) = A/r$, where A is a physical constant that is needed to be determined, then using the planar polar coordinates in two-dimensional space, the Schrödinger wave equation takes the form [10]

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right]\psi(r,\phi) - \frac{A}{r}\psi(r,\phi) = E\psi(r,\phi) \tag{15}$$

Solutions of the form $\psi(r,\phi) = R(r)\Phi(\phi)$ reduce Equation (15) to two separate equations for the functions $\Phi(\phi)$ and $R(r)$ as follows

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{16}$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{m^2}{r^2}R + \frac{2\mu}{\hbar^2}\left(\frac{A}{r} + E\right)R = 0 \tag{17}$$

where m is identified as the intrinsic angular momentum of the rotor. Equation (16) has solutions of the form

$$\Phi(\phi) = \exp(im\phi) \tag{18}$$

Normally, the intrinsic angular momentum m must take integral values for the single-valuedness condition to be satisfied. However, if we consider the configuration space of the rotor to be multiply connected and the polar coordinates have singularity at the origin then the use of multivalued wavefunctions is allowable. As shown below, in this case, the intrinsic angular momentum m can take half-integral values. If we define, for the case $E < 0$,

$$\rho = \left(\frac{8\mu(-E)}{\hbar^2}\right)^{1/2} r \tag{19}$$

$$\lambda = \left(\frac{A\mu}{2\hbar^2(-E)}\right)^{1/2} \tag{20}$$

then Equation (17) can be re-written in the following form

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{m^2}{\rho^2}R + \frac{\lambda}{\rho}R - \frac{1}{4}R = 0 \tag{21}$$

If we seek solutions for $R(\rho)$ in the form

$$R(\rho) = exp(-\rho/2)\rho^m S(\rho) \tag{22}$$

then we obtain the following differential equation for the function $S(\rho)$

$$\frac{d^2S}{d\rho^2} + \left(\frac{2m+1}{\rho} - 1\right)\frac{dS}{d\rho} + \left(\frac{\lambda - m - \frac{1}{2}}{\rho}\right)S = 0 \tag{23}$$

Equation (23) can be solved by a series expansion of $S(\rho)$

$$S(\rho) = \sum_{n=0}^{\infty} a_n \rho^n \tag{24}$$

with the coefficients a_n satisfying the recursion relation

$$a_{n+1} = \frac{n + m + \frac{1}{2} - \lambda}{(n+1)(n+2m+1)} a_n \tag{25}$$

The energy spectrum for E from Equation (20) can be written explicitly in the form

$$E = \frac{A^2\mu}{2\hbar^2\left(n + m + \frac{1}{2}\right)^2} \tag{26}$$

Even though it is not possible to specify the actual values of the intrinsic angular momentum m at the present state of development of physics, however, if the result given in Equation (26) can also be applied to the hydrogen-like atom, which is viewed as a two-dimensional physical system, then the intrinsic angular momentum m must take half-integral values.

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