Lepton mass phases and the CKM matrix

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Abstract

The Brannen neutrino mass triplet extends Koide's rule for the charged leptons, which was used to correctly predict the τ mass. Assuming that Koide's rule is exact, we consider the fundamental 2/9 lepton phase, noting connections to the CKM matrix and arithmetic information. An estimate for the fine structure constant α is included.

Neutrino oscillations [1][2] indicate a triplet of non zero masses for active neutrinos. Any low energy relation between the masses in a triplet is a valuable clue to a theory beyond the Standard Model. Decades ago, Koide [3][4] found a relation for the charged lepton masses, that was later extended by Brannen [5][6] to the neutrino masses using density matrices. Both charged lepton and neutrino triplets are given as the eigenvalues of a circulant Hermitian mass matrix M. Let

$$\sqrt{M} = \frac{\sqrt{\mu}}{r} \begin{pmatrix} r & e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & r & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} & r \end{pmatrix}, \tag{1}$$

where μ is a scale parameter. For all leptons, the Koide rule corresponds to $r = \sqrt{2}$. For the charged leptons, 3μ happens to equal the proton mass m_p. For neutrinos, a global fit to data gives $\mu = 0.01\text{eV}$. Mass eigenvalues take the form

$$m_k = \mu(1 + \sqrt{2}\cos(\theta + \frac{2\pi k}{3}))^2 \tag{2}$$

for $k = 1, 2, 3$. Phenomenologically, the charged lepton phase θ_l is close to $2/9$ [5] while the neutrino phase is $2/9 + \pi/12$, as if the neutrino masses are governed by a phase triplet

$$\theta_\nu = \frac{2}{9} + \epsilon + \frac{\pi}{12} \tag{3}$$

for $\epsilon \sim 10^{-7}$. The $\pi/12$ is a fundamental arithmetic geometric phase. The $2/9$ phase is associated with electric charge, since the phases $2/27$ and $4/27$ give the Koide triplets for the quarks [7]. However, since the neutrinos also carry the $2/9$, the rational charge should be viewed as a derivation of charge from quantum gravity. One possibility is

$$\frac{2}{9} = \frac{2\pi}{27} \cdot \frac{3}{\pi} \tag{4}$$

where 27 is the dimension of the exceptional Jordan circulant algebra, and $3/\pi$ is the *apparent* dark energy/matter density fraction by the black hole pair production argument of Riofrio [8]. Then $4/27$ is $2/3$ ($= 18/27$) of $2/9$. As pointed out by the author many years ago, the charged lepton braids should be considered fundamental, since they correspond to the Cayley (adjoint) representation for B_3. Then the neutrinos lose charge because the left and right (mirror) factors of $2/9$ cancel, leaving a net phase of $\pi/6$. The diagonalisation of M

uses the 3×3 quantum Fourier transform [9], that is conjugation by

$$F = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix}, \tag{5}$$

where ω is the primitive cubed root of unity. The connection between F and the tribimaximal mixing matrix [10] is discussed in [11]. One expects the square root $\sqrt{M}$ to be the fundamental degree of freedom by color gravity duality in a motivic formulation for nonperturbative amplitudes. Neutrino masses are now well constrained by observation [12].

For any prime power dimension p, the Fourier transform matrix is viewed as a column set of eigenvectors for one Pauli operator, as is the identity. In dimension 2, the three Pauli matrices give

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{6}$$

There always exists [13][14] a circulant R that generates p out of the $p+1$ mutually unbiased bases, all but F. For $p = 3$,

$$R = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{pmatrix}. \tag{7}$$

Since $R^3 = iI$, R is a matrix representation of the phase $\delta = \pi/6$. In a six dimensional system that combines one qubit ($p = 2$) and one qutrit ($p = 3$), the tensor product $R \otimes R_2$ stands for the arithmetic phase $\pi/12$ [5].

In the eigenvalues of (2), $\theta = 0$ gives a degenerate pair of masses while the $\theta = \delta$ triplet includes the mass scale μ. We therefore define a basic dimensionless mass splitting parameter

$$\Delta \equiv m_{\theta=0} - m_{\theta=\delta} = \frac{1}{2} + 2\sqrt{2} - \sqrt{6}. \tag{8}$$

If this mass comes from a Koide triplet, the phase satisfies a relation

$$\Delta = (1 + \sqrt{2}\cos(\overline{\omega} + \theta + \frac{\pi}{12}))^2, \tag{9}$$

which has a solution

$$\theta = 0.21760 \text{ rad} \equiv \frac{2}{9} - \kappa. \tag{10}$$

TABLE I. phase differences (°)

	0	2/9	5/27
$2/9+\kappa$	13.0	0.26	2.4
	$\pi/4-5/27$	4/27	$\pi/4-2/27$
0	34.4	8.5	40.8

Alternatively, we may define

$$\kappa = \frac{1}{216} = \frac{1}{24} - \frac{1}{27}, \tag{11}$$

where 24 is the number of SM states and the 27 includes the mirror neutrinos, as noted below. Now compare κ to the lepton phases. Phase differences are given in Table 1. Observe the striking similarity to the three Euler angles of the CKM matrix [15][16] using *only* rational phases. An analogous possibility for the PMNS matrix is also shown in Table 1. If these quark and lepton mass phases are exact, this leaves at most four parameters for *all* masses and mixing phases: an r value for the quarks, two mass scale ratios, and one small unknown correction. It is likely that this set can be reduced further, leaving only the two mass scale ratios μ_l/μ_ν and μ_q/μ_ν, which should also be derived eventually.

Koide et al [17] develop a Yukawaon model that accurately recovers mixing matrices and quark and lepton masses with only six parameters. In the ribbon scheme [11] for leptons and quarks, one expects some form of quark lepton complementarity, since leptons are easily transformed into quarks with ribbon twist operators.

Now consider the normalised determinant of (1),

$$D(\theta) \equiv -1 + \sqrt{2}\cos(3\theta), \tag{12}$$

arising as a cubic in r. Since D is invariant under the Fourier transform, it is a product $\pm\sqrt{m_1 m_2 m_3}$ of eigenvalues. The range of $D+1$ is centred around zero precisely when $r=\sqrt{2}$, and this $\sqrt{2}$ defines the tribimaximal mixing matrix [10] as a derivative of F [11]. The diagonal element r for the circulant M is

$$r_M = \frac{4}{\sqrt{5-4D}}. \tag{13}$$

The condition $D=0$ is obtained for $\theta = \pm\pi/12$. Allowing for a $-\pi/12$ in (3), we obtain a set of mirror masses, used in [18][19] to study the mirror neutrino CMB cosmology [19]. At

$D+1=0$ (ie. the phase δ) and $r=\sqrt{2}$, the Brannen rule for neutrinos gives

$$\sqrt{m_1 m_2 m_3} = \frac{1}{27}(\sqrt{m_1} + \sqrt{m_2} - \sqrt{m_3})^3. \tag{14}$$

Allowing the mass differences between θ and $(\theta+\delta)$ to define

$$\Delta(\theta) = \frac{1}{2}\cos^2\theta + \sqrt{2}\sin\theta + (2\sqrt{2}-\sqrt{6})\cos\theta + \frac{\sqrt{3}}{4}\sin 2\theta, \tag{15}$$

we obtain the Weierstrass cubic for $\mathbf{D}^2 = 4m_1 m_2 m_3$,

$$\mathbf{D}^2 = 4\Delta^3 + 24\Delta - 14. \tag{16}$$

Neutrinos are neutral because their ribbon diagrams have no charge twists [11]. In the categorical ribbon scheme, one full twist stands for a unit of charge. To estimate the fine structure constant using this charge, consider quantum data for Jones invariants from Chern-Simons field theory [20]. Empirically, we take the Hopf link invariant at $5\pi/6$, which traces over a twist at the minimal mass in the δ triplet, to obtain [21]

$$\sqrt{\alpha} = 4\cosh(\frac{2\pi}{5\pi/6+1}), \tag{17}$$

roughly giving $\alpha = 137.096$.

These results motivate a further study of ribbon graphs in color gravity theories, where the obviously arithmetic nature of neutrino triplets plays an important role.

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