Pi and Phi

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ABSTRACT. This note presents some formulas for pi.

1. Introduction

The number pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.141592... \tag{1}$$

The number phi:

$$\phi = \frac{1+\sqrt{5}}{2} = 1.6180... \tag{2}$$

2. Formulas

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$$\frac{\pi}{5}\sqrt{2-\frac{2}{\sqrt{5}}} = \sum_{n=0}^{\infty} 2^{-n}\phi^{-2n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-2)^{-k}}{n+k+1} \tag{3}$$

$$\frac{\pi}{5}\sqrt{2-\frac{2}{\sqrt{5}}} = \sum_{n=0}^{\infty} \phi^{-2n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}}{n+k+1} \tag{4}$$

$$\frac{\pi}{5}\sqrt{2+\frac{2}{\sqrt{5}}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}\phi^{-2k-1}}{n+k+1} \tag{5}$$

$$\frac{3\pi}{5}\sqrt{2+\frac{2}{\sqrt{5}}} = \sum_{n=0}^{\infty} 2^{-n}\phi^{2n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-2)^{-k}}{n+k+1} \tag{6}$$

$$\frac{\pi}{10}\sqrt{\frac{2}{5+\sqrt{5}}} = \phi\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{7-4\phi}{5}\right)^{n+1} \tag{7}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}(-1)^n(2\phi)^{-2n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-2)^k}{2n-k+1} \tag{8}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}(-1)^n(1+\phi)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2n-k+1}\left(1-\left(1+\frac{1}{2\phi}\right)^{k-2n-1}\right) \tag{9}$$

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$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}c_n\left(\frac{1}{2\phi}\right)^{n+1}\left(\frac{1}{n+1}+\frac{1}{2(n+2)}+\frac{1}{4\phi^2(n+3)}\right) \tag{10}$$

$$c_{n+4}=-(c_{n+3}+c_{n+2}+c_{n+1}+c_n),\ c_0=1,c_1=-1,c_2=0,c_3=0 \tag{11}$$

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$$\frac{2\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}\frac{2^{-n}\phi^{-2n-1}}{n+1}c_n \tag{12}$$

$$\frac{\pi}{5}\sqrt{2-\frac{2}{\sqrt{5}}}=\sum_{n=0}^{\infty}\frac{\phi^{-2n-1}}{n+1}c_n \tag{13}$$

$$c_{n+2}=c_{n+1}-\phi^2c_n \quad ,c_0=c_1=1 \tag{14}$$

$$c_n=a_n+b_n\phi \quad ,n=0,1,2,3,\ldots \tag{15}$$

$$\begin{cases}a_{n+2}=a_{n+1}-a_n-b_n \\ b_{n+2}=b_{n+1}-2b_n-a_n\end{cases},\ \begin{matrix}a_0=1,b_0=0\\ a_1=1,b_1=0\end{matrix} \tag{16}$$

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$$\frac{\pi}{5}\sqrt{2+\frac{2}{\sqrt{5}}}=\sum_{n=0}^{\infty}\frac{\phi^{-n-1}}{n+1}c_n \tag{17}$$

$$\frac{3\pi}{5}\sqrt{\frac{5+\sqrt{5}}{10}}=\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{\phi}{2}\right)^{n+1}c_n \tag{18}$$

$$c_{n+2}=\phi c_{n+1}-c_n \quad ,c_0=1,c_1=\phi \tag{19}$$

$$c_n=a_n+b_n\phi \quad ,n=0,1,2,3,\ldots \tag{20}$$

$$\begin{cases}a_{n+2}=b_{n+1}-a_n \\ b_{n+2}=a_{n+1}+b_{n+1}-b_n\end{cases},\ \begin{matrix}a_0=1,b_0=0\\ a_1=0,b_1=1\end{matrix} \tag{21}$$

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$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}\left(-1\right)^{n}\phi^{2n+1}\sum_{k=0}^{2n}\left(-1\right)^{k}\binom{2n}{k}f\left(n,k\right) \tag{22}$$

$$f\left(n,k\right)=\begin{cases}\ln\left(2/\phi\right) & ,n=k\\ \dfrac{1-\left(\phi/2\right)^{k-n}}{k-n} & ,n\neq k\end{cases} \tag{23}$$

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$$\frac{\pi}{2}\sqrt{\frac{2}{5+\sqrt{5}}}=3\phi-4-\phi\int_{0}^{7-4\phi}\frac{x}{5+x+\phi\sqrt{50-25\phi-10\phi x-\left(2+\phi\right)x^{2}}}dx \tag{24}$$

$$\frac{\pi}{5}\sqrt{2+\frac{2}{\sqrt{5}}}=\phi-2\int_{0}^{\phi}\frac{x}{\phi+\phi x+\sqrt{1+\phi+2\left(\phi-1\right)x-\left(3-\phi\right)x^{2}}}dx \tag{25}$$

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$$\pi=10\phi\sqrt{2+\phi}\int_{0}^{1}\int_{0}^{1}\frac{8\phi^{3}+4\phi xy-2x^{2}y^{2}-x^{3}y^{3}}{\left(\left(2\phi\right)^{5}-x^{5}y^{5}\right)\left(-\ln\left(xy\right)\right)}dxdy \tag{26}$$

$$\pi=\frac{5\phi\sqrt{2+\phi}}{2}\int_{0}^{1}\int_{0}^{1}\frac{\phi^{3}+\phi xy-x^{2}y^{2}-x^{3}y^{3}}{\left(\phi^{5}-x^{5}y^{5}\right)\left(-\ln\left(xy\right)\right)}dxdy \tag{27}$$

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$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{2n+1}\left(\frac{1}{2\phi}\right)^{2n+1}F\left(\left\{n+1,2n+1\right\},\left\{2n+2\right\},\frac{1}{2\phi^{2}}\right) \tag{28}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}\frac{\left(-2\right)^{-n}}{2n+1}\left(\frac{1}{\phi}\right)^{3n+2}F\left(\left\{1,n+1\right\},\left\{2n+2\right\},-\left(2\phi-3\right)\right) \tag{29}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{2n+1}\left(\frac{1}{\phi}\right)^{4n+2}F\left(\left\{n+1,2n+1\right\},\left\{2n+2\right\},-\left(2\phi-3\right)\right) \tag{30}$$

$$\frac{2\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}\frac{\left(-2\right)^{-n}}{2n+1}\left(\frac{1}{\phi}\right)^{3n+1}F\left(\left\{1,n+1\right\},\left\{2n+2\right\},\frac{1}{2\phi^{2}}\right) \tag{31}$$

$$\frac{2\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}\frac{2^{-n}}{n+1}\left(\frac{1}{\phi}\right)^{2n+1}F\left(\left\{n+1,\frac{n+1}{2}\right\},\left\{\frac{n+3}{2}\right\},-\frac{1}{4\phi^{2}}\right) \tag{32}$$

$$\frac{2\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\frac{1}{\phi}\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{2}{5+4\phi}\right)^{n}F\left(\left\{1,-\frac{n-1}{2}\right\},\left\{\frac{n+3}{2}\right\},-\frac{1}{4\phi^{2}}\right) \quad (33)$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\phi\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{2}{5+4\phi}\right)^{n+1}F\left(\{1,n+1\},\left\{\frac{n+3}{2}\right\},\frac{1}{5+4\phi}\right) \quad (34)$$

$$\frac{\pi}{5\sqrt{7-4\phi}}=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1}\left(\frac{13-8\phi}{2}\right)^{n}F\left(\left\{n+1,n+\frac{1}{2}\right\},\left\{n+\frac{3}{2}\right\},-\frac{1}{2}\right) \quad (35)$$

$$\frac{3\pi}{10\sqrt{7-4\phi}}=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1}\left(\frac{13-8\phi}{3}\right)^{n}F\left(\{1,n+1\},\left\{n+\frac{3}{2}\right\},\frac{1}{3}\right) \quad (36)$$

$$\frac{\pi\sqrt{3}}{5\sqrt{2}\sqrt{7-4\phi}}=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1}\left(\frac{13-8\phi}{2}\right)^{n}F\left(\left\{\frac{1}{2},n+\frac{1}{2}\right\},\left\{n+\frac{3}{2}\right\},\frac{1}{3}\right) \quad (37)$$

$$\frac{\pi}{5\sqrt{7-4\phi}}=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1}\left(\frac{13-8\phi}{3}\right)^{n}F\left(\left\{1,\frac{1}{2}\right\},\left\{n+\frac{3}{2}\right\},-\frac{1}{2}\right) \quad (38)$$

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$$\pi\left(\frac{2\phi}{\sqrt{2+\phi}}\right)=\frac{65}{12}-\sum_{n=0}^{\infty}\phi^{-5n-5}\left(\frac{1}{5n+2}+\frac{2}{5n+3}+\frac{2}{5n+4}-\frac{5}{5n+6}-\frac{2}{5n+7}+\frac{1}{5n+8}+\frac{1}{5n+9}\right) \quad (39)$$

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$$\begin{aligned}\frac{\pi}{5\sqrt{2+\phi}}=\sum_{n=0}^{\infty}\left(\frac{1}{2\phi}\right)^{15n}\Bigg(&\frac{1}{(2\phi)(15n+1)}+\frac{\phi^{-1}}{(2\phi)^{2}(15n+2)}-\frac{\phi^{-1}}{(2\phi)^{3}(15n+3)}\\&-\frac{1}{(2\phi)^{4}(15n+4)}+\frac{1}{(2\phi)^{6}(15n+6)}+\frac{\phi^{-1}}{(2\phi)^{7}(15n+7)}\\&-\frac{\phi^{-1}}{(2\phi)^{8}(15n+8)}-\frac{1}{(2\phi)^{9}(15n+9)}+\frac{1}{(2\phi)^{11}(15n+11)}\\&+\frac{\phi^{-1}}{(2\phi)^{12}(15n+12)}-\frac{\phi^{-1}}{(2\phi)^{13}(15n+13)}-\frac{1}{(2\phi)^{14}(15n+14)}\Bigg)\end{aligned} \quad (40)$$

Remark : $F(\{a,b\},\{c\},x)$ is the hypergeometric function.

References

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