An "Approximate" Weierstrass Form Connecting Alexander Polynomials for Knots 8_6 and 8_7

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"For now we see through a glass, darkly; but then face to face: now I know in part; but then shall I know even as also I am known." - 1 Corinthians 13:12.

Abstract. In this paper, we construct an equation involving Alexander polynomials for knots 8_6 and 8_7.

1. Introduction

The Alexander polynomial is of an oriented link is a Laurent polynomial associated with the link in an invariant way.

We will use the Corollary [1, p. 58]:

Corollary 1. *For any knot K,*

$$\Delta_K(t) = a_0 + a_1(t + t^{-1}) + a_2(t^2 + t^{-2})...$$

where the a_i are integers and a_0 is odd.

The Weierstrass form is a general form into which an elliptic curve over any field K can be transformed, given by

$$y^2 + ay = x^3 + bx^2 + cxy + dx + e,$$

where a, b, c, d and e are elements of K. See [2].

In this paper we built an "approximate" Weierstrass form for the equation

$$x^3 + x^2 + (12y - 17)x + 8y^2 - 28y + 23 = 0,$$

where $x = \Delta_{8_6}(t)$ and $y = \Delta_{8_7}(t)$, which is given by

$$y_0^2 = -x_0^3 + \frac{16}{3}x_0 + \frac{128}{27}.$$

This equation is more similiar with the Weierstrass form above. But it is not a elliptic curve. The genus is 0.

There are other algebraic curves that will are explored in the next papers.

2. The Equation

2.1. Equation Between $\Delta_{8_6}(t)$ and $\Delta_{8_7}(t)$.

Theorem 2. *If $x = \Delta_{8_6}(t)$ and $y = \Delta_{8_7}(t)$, then x and y satisfies the equation*

$$x^3 + x^2 + (12y - 17)x + 8y^2 - 28y + 23 = 0.$$

Proof. I know the Alexander Polynomials for knots 8_6 and 8_7 [1, pp. 58 and 59], given by

$$\Delta_{8_6}(t) = -7 + 6(t^{-1} + t) - 2(t^{-2} + t^2) \quad (1)$$

and

$$\Delta_{8_7}(t) = -5 + 5(t^{-1} + t) - 3(t^{-2} + t^2) + (t^{-3} + t^3) \quad (2)$$

Let $t \to (1 - z)/(1 + z)$ in the right hand side of (1) and (2)

$$\Delta_{8_6}(z) = \frac{1 - 10z^2 - 23z^4}{(1 - z^2)^2} \quad (3)$$

and

$$\Delta_{8_7}(z) = \frac{1 + 5z^2 + 35z^4 + 23z^6}{(1 - z^2)^3}. \tag{4}$$

Let $\Delta_{8_6}(z) = x$ and $\Delta_{8_7}(z) = y$ in (3) and (4); eliminate z from (3) and (4) and encounter

$$x^3 + x^2 + (12y - 17)x + 8y^2 - 28y + 23 = 0,$$

which is the desired result. □

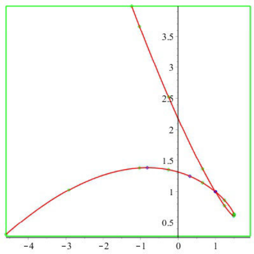

Figure 1.

Remark 3. The algebraic curve $f(x, y) = x^3 + x^2 + (12y - 17)x + 8y^2 - 28y + 23$ have genus 0. See Fig. 1. So, in thesis, not exist a Weierstrass form. But we managed to dribbe this. See below.

2.2. The "Approximate" Weierstrass Form.

Theorem 4. *The equation*

$$x^3 + x^2 + (12y - 17)x + 8y^2 - 28y + 23 = 0,$$

have the "approximate" Weierstrass form

$$y_0^2 = -x_0^3 + \frac{16}{3}x_0 + \frac{128}{27}.$$

Proof. Substitute

$$x = \frac{x_0}{8} + \frac{7}{6}. \tag{5}$$

and

$$y = -\frac{3}{32}x_0 + \frac{1}{64}y_0 + \frac{7}{8} \tag{6}$$

in equation from Theorem 1 and multiply by 512. This completes the proof. □

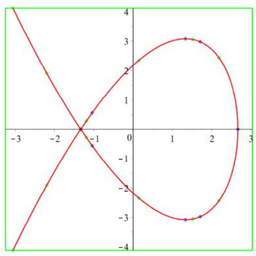

Figure 2.

Remark 5. The algebraic curve $g(x, y) = y^2 + x^3 - \frac{16}{3}x - \frac{128}{27}$ have genus 0. See Fig. 2.

3. AN ELLIPTIC CURVE

If we consider the equation

$$x^3 + x^2 + (12y - 7)x + 8y^2 - 28y + 23 = 0,$$

obviously, we have the elliptic curve with genus 1. See Fig. 3, below:

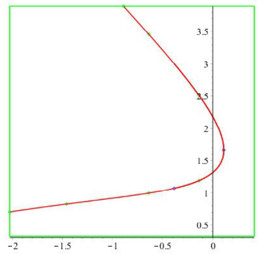

Figure 3.

Substituting

$$x = \frac{x_0}{8} + \frac{7}{6}.$$

and

$$y = -\frac{3}{32}x_0 + \frac{1}{64}y_0 + \frac{7}{8}$$

we encounter the Weierstrass form of $f(x,y)$ is

$$y_0^2 = -\left(x_0^3 + \frac{1904}{3}x_0 + \frac{161152}{27}\right).$$

Now, we have the elliptic curve, see Fig. 4, below:

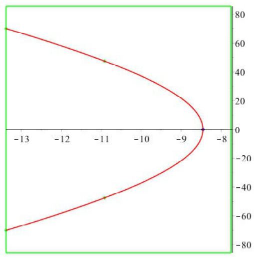

Figure 4.

Notice that the generator for the field K, i. e., a function in $\mathbb{C}(x,y)$, is $8x - 28/3$. Under the birational map this corresponds to x_0 is $64y + 48x - 112$.

References

[1] Lickorish, W. B. Raymond, *An Introduction for Knot Theory*, Graduate Texts in Mathematics, 175, Springer-Verlag, New York, 1997.

[2] Weisstein, Eric W., "Weierstrass Form." From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/WeierstrassForm.html.