Quasi-Exact Solvability of Symmetrized Sextic Oscillators and Analyticity of the Related Quotient Polynomials

Spiros Konstantogiannis
spiroskonstantogiannis@gmail.com
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Abstract

Quasi-exactly solvable symmetrized sextic oscillators have been proposed and studied by Quesne, who categorized them based on the parity – natural or unnatural – of their known eigenfunction **[2]**. Herein, we examine the quasi-exact solvability of symmetrized sextic oscillators using a quotient-polynomial approach **[3, 4, 5]**, which, in this case, opens up the possibility to construct non-analytic sextic oscillators from analytic quotient polynomials, and thus to distinguish the oscillators resulting from analytic quotient polynomials from those resulting from non-analytic quotient polynomials. We analyze the cases n=0 and n=1, and we show that the results are in agreement with those of Quesne **[2]**. In the case n=2, we construct sextic oscillators using only analytic quotient polynomials, and focusing on the non-analytic oscillators whose known eigenfunction is of unnatural parity, we register a relation between the coefficients of the two non-analytic terms of the exponential polynomial, which then we generalize to the higher cases n=3 and n=4, to construct new non-analytic sextic oscillators whose known eigenfunction is of unnatural parity.

Keywords: polynomial oscillators, sextic oscillators, quotient polynomial, analytic sextic oscillators, non-analytic sextic oscillators, symmetrized sextic oscillators, analytic quotient polynomials, natural parity, unnatural parity, quasi-exactly solvable potentials, Bethe ansatz

Contents

Introduction

Within the framework of the Bethe ansatz **[6, 7, 8]**, the solvability of one-dimensional, attractive at (least at) long distances, analytic or symmetrized, real polynomial potentials can be examined by means of quotient polynomials **[3, 4, 5]**.
In this context, the closed-form eigenfunctions describing bound eigenstates of previous potentials are given by the ansatz

$\psi(\tilde{x};m,n) = A_n p_n(\tilde{x}) \exp(g_{2m}(\tilde{x}))$ (1)

where $p_n(\tilde{x})$ is a dimensionless, real polynomial of degree $n = 0,1,\dots$, and $g_{2m}(\tilde{x})$ is an also dimensionless, real polynomial of degree $2m = 2,3,\dots$, i.e. $m = 1,\frac{3}{2},\dots$ **[5]**.
The tilde indicates a dimensionless quantity **[3]**.

Symmetrized sextic oscillators

The general form of a symmetrized exponential polynomial $g_4(\tilde{x})$ ($m = 2$) is

$$g_4(\tilde{x}) = -\frac{{g_4}^2}{4}\tilde{x}^4 + \frac{g_3}{3}|\tilde{x}|^3 + \frac{g_2}{2}\tilde{x}^2 + g_1|\tilde{x}|$$

with ${g_4}^2 > 0$.
The factors $-\frac{1}{4},\frac{1}{3},\frac{1}{2}$ are put in for convenience, while the negative leading coefficient is necessary so that the ansatz (1) is square integrable, which, in turn, results from the fact that it describes a bound eigenstate.
The constant term g_0 of the exponential polynomial corresponds to a constant exponential factor $\exp(g_0)$, which can be incorporated into the normalization constant of the ansatz (1), and thus we omit it.
For simplicity, we set ${g_4}^2 = 1$.
Besides, using that $\tilde{x} = \frac{x}{l}$ **[3]**, we can choose the length scale l so that the coefficient of x^4 is -1.
Thus, without sacrificing generality, we write the exponential polynomial as

$$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 + \frac{g_3}{3}|\tilde{x}|^3 + \frac{g_2}{2}\tilde{x}^2 + g_1|\tilde{x}| \quad (2)$$

For $\tilde{x} > 0$

$$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 + \frac{g_3}{3}\tilde{x}^3 + \frac{g_2}{2}\tilde{x}^2 + g_1\tilde{x},$$

and thus

$$g_4'(\tilde{x}) = -\tilde{x}^3 + g_3\tilde{x}^2 + g_2\tilde{x} + g_1$$

$$g_4''(\tilde{x}) = -3\tilde{x}^2 + 2g_3\tilde{x} + g_2$$

For $\tilde{x} < 0$

$$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 - \frac{g_3}{3}\tilde{x}^3 + \frac{g_2}{2}\tilde{x}^2 - g_1\tilde{x},$$

and thus

$$g_4'(\tilde{x}) = -\tilde{x}^3 - g_3\tilde{x}^2 + g_2\tilde{x} - g_1$$

$$g_4''(\tilde{x}) = -3\tilde{x}^2 - 2g_3\tilde{x} + g_2$$

The potential is given by the general expression **[3]**

$$\tilde{V}(\tilde{x};m,n) = g_{2m}'^2(\tilde{x}) + g_{2m}''(\tilde{x}) - q_{2(m-1)}(\tilde{x};n) + \tilde{E} \quad (3)$$

with $q_{2(m-1)}(\tilde{x};n)$ being the quotient polynomial **[3]**, which is of degree $2(m-1)$.

The energy $\tilde{E}$ of the eigenstate described by the ansatz (1) is calculated by the condition $\tilde{V}(0;m,n) = 0$, provided that the potential is continuous at zero.

For $m = 2$, from (3) we obtain sextic oscillators, and since the exponential polynomial is symmetrized, the sextic oscillators are also symmetrized.

The respective quotient polynomial is an (in general) symmetrized polynomial of degree 2, which means that it is of even parity*, and thus it has the general form

$$q_2(\tilde{x};n) = q_2(n)\tilde{x}^2 + q_1(n)|\tilde{x}| + q_0(n) \quad (4)$$

* The symmetrized exponential polynomial $g_4(\tilde{x})$ is of even parity, and then $g_4'(\tilde{x})$ is of odd parity, and thus $g_4'^2(\tilde{x})$ is again of even parity, while $g_4''(\tilde{x})$ is also of even parity.

Thus, the polynomial $g_4'^2(\tilde{x}) + g_4''(\tilde{x})$ is of even parity.

Besides, the symmetrized sextic oscillator, i.e. the potential $\tilde{V}(\tilde{x};2,n)$ given by (3), is also of even parity.

Then, writing (3) as

$$q_2(\tilde{x};n) = g_4'^2(\tilde{x}) + g_4''(\tilde{x}) - \tilde{V}(\tilde{x};2,n) + \tilde{E},$$

we see that the polynomial in the right-hand side is a sum of even-parity polynomials, and thus it is itself of even parity, and then the quotient polynomial $q_2(\tilde{x};n)$ is of even parity too.

We note that, compared to the analytic sextic oscillator **[4]**, the quotient polynomial (4) has a non-analytic (symmetrized) linear term, which is also the only intermediate term in the quotient polynomial (4).

Based on the discussion in **[3]**, the presence of this non-analytic term results in finding no more than one closed-form eigenfunction for each respective symmetrized sextic oscillator.

On the contrary, in the case of symmetrized quartic oscillators, the quotient polynomial has no intermediate terms, it is a non-analytic (symmetrized) linear polynomial **[5]**.

Now, using the above expressions of the first and second derivatives of the exponential polynomial, we have

$g_4'(0^+) = g_1 \quad g_4''(0^+) = g_2$

$g_4'(0^-) = -g_1 \quad g_4''(0^-) = g_2$

Thus

$g_4'(0^-) = -g_4'(0^+)$

and then

$g_4'^2(0^-) = g_4'^2(0^+)$

Also

$g_4''(0^-) = g_4''(0^+)$

Thus, the polynomial $g_4'^2(\tilde{x}) + g_4''(\tilde{x})$ is continuous at zero.

The quotient polynomial (4) is also continuous at zero, as its non-analytic term $q_1(n)|\tilde{x}|$ is continuous at zero.

Thus, the symmetrized sextic oscillator $\tilde{V}(\tilde{x};2,n)$, given by (3) for $m=2$, is continuous at zero, and everywhere.

Since the potential is continuous at zero, both the ansatz eigenfunction (1) and its derivative must be continuous at zero **[1]**, i.e.

$\psi(0^-;2,n) = \psi(0^+;2,n)$ (5)

$\psi'(0^-;2,n) = \psi'(0^+;2,n)$ (6)

Using (1), (5) is written as

$$A_n p_n(0^-)\exp(g_4(0^-)) = A_n p_n(0^+)\exp(g_4(0^+))$$

Since $A_n \neq 0$ – otherwise the ansatz (1) is identically zero – and $g_4(0^-) = g_4(0^+) = 0$, the last equation becomes

$p_n(0^-) = p_n(0^+)$ (7)

i.e. the polynomial $p_n(\tilde{x})$ is continuous at zero, and everywhere.

Besides, the derivative of the ansatz (1), for $m=2$, is

$$\psi'(\tilde{x};2,n) = A_n\left(p_n'(\tilde{x}) + g_4'(\tilde{x})\,p_n(\tilde{x})\right)\exp(g_4(\tilde{x}))$$

Then, (6) is written as

$$A_n\left(p_n'(0^-) + g_4'(0^-)\,p_n(0^-)\right)\exp(g_4(0^-)) = A_n\left(p_n'(0^+) + g_4'(0^+)\,p_n(0^+)\right)\exp(g_4(0^+))$$

Then, using that $A_n \neq 0$, $g_4(0^-) = g_4(0^+) = 0$, and that

$g_4'\left(0^-\right)=-g_1 \quad g_4'\left(0^+\right)=g_1$,

and (7), we obtain

$$p_n'\left(0^-\right)-g_1 p_n\left(0^+\right)=p_n'\left(0^+\right)+g_1 p_n\left(0^+\right)$$

and thus

$$p_n'\left(0^-\right)=p_n'\left(0^+\right)+2g_1 p_n\left(0^+\right) \quad (8)$$

i.e. the first derivative of $p_n\left(\tilde{x}\right)$ has a finite jump at zero, which is equal to $-2g_1 p_n\left(0^+\right)$.

The polynomials $p_n\left(\tilde{x}\right)$ satisfy the differential equation **[3]**

$$p_n''\left(\tilde{x}\right)+2g_{2m}'\left(\tilde{x}\right)p_n'\left(\tilde{x}\right)=-q_{2(m-1)}\left(\tilde{x};n\right)p_n\left(\tilde{x}\right)$$

which, in our case, and in the region $\tilde{x}>0$, is written as

$$p_n''\left(\tilde{x}\right)+2\left(-\tilde{x}^3+g_3\tilde{x}^2+g_2\tilde{x}+g_1\right)p_n'\left(\tilde{x}\right)=-\left(q_2\left(n\right)\tilde{x}^2+q_1\left(n\right)\tilde{x}+q_0\left(n\right)\right)p_n\left(\tilde{x}\right) \quad (9)$$

The leading coefficient $q_2\left(n\right)$ of the quotient polynomial is calculated by equating the coefficients of the highest terms in $\tilde{x}$, in both sides of (9).

The polynomial $p_n\left(\tilde{x}\right)$ is of degree n.

As in the case of analytic polynomial potentials **[3, 4]**, we can incorporate, without loss of generality, the non-zero leading coefficient of $p_n\left(\tilde{x}\right)$ into the normalization constant A_n of the ansatz eigenfunction (1).

If $n\geq 1$, we have

In the positive region, where $\left|\tilde{x}\right|^n=\tilde{x}^n$, the leading term of $p_n\left(\tilde{x}\right)$ is $\tilde{x}^n$, and then the leading term of $p_n'\left(\tilde{x}\right)$ is $n\tilde{x}^{n-1}$.

Then, we have

$$\deg\left(p_n''\right)=n-2 \text{ if } n\geq 2,\ \deg\left(p_n''\right)=0 \text{ if } n=1$$

$$\deg\left(\tilde{x}^3 p_n'\right)=n+2 \qquad \deg\left(\tilde{x}^2 p_n'\right)=n+1 \qquad \deg\left(\tilde{x}p_n'\right)=n \quad \deg\left(p_n'\right)=n-1$$

$$\deg\left(\tilde{x}^2 p_n\right)=n+2 \qquad \deg\left(\tilde{x}p_n\right)=n+1 \qquad \deg\left(p_n\right)=n$$

Then, the highest powers in $\tilde{x}$, in both sides of (9), are of $n+2$ degree, with one term in each side of (9), with the respective coefficients being $-2n$ and $-q_2\left(n\right)$, and thus

$$q_2\left(n\right)=2n \quad (10)$$

If $n=0$, then $p_0\left(\tilde{x}\right)=1$, and thus $p_0'\left(\tilde{x}\right)=p_0''\left(\tilde{x}\right)=0$, and then (9) gives

$$-\left(q_2\left(0\right)\tilde{x}^2+q_1\left(0\right)\tilde{x}+q_0\left(0\right)\right)=0$$

and thus

$q_0(0) = q_1(0) = q_2(0) = 0$

Besides, for $n = 0$, (10) gives $q_2(0) = 0$, which is the correct result, and thus (10) also holds for $n = 0$.

Since the quotient polynomial is of even parity, its coefficients in the negative region are the same as those in the positive region.

Substituting (10) into (9) yields

$$p_n''(\tilde{x}) + 2(-\tilde{x}^3 + g_3\tilde{x}^2 + g_2\tilde{x} + g_1)p_n'(\tilde{x}) = -(2n\tilde{x}^2 + q_1(n)\tilde{x} + q_0(n))p_n(\tilde{x}) \quad (11)$$

The coefficients of the terms of degree $k \geq 0$ in $\tilde{x}$, in both sides of (11), are, respectively,

$$p_n''(\tilde{x}) \to (k+2)(k+1)p_{k+2}$$

$$2(-\tilde{x}^3 + g_3\tilde{x}^2 + g_2\tilde{x} + g_1)p_n'(\tilde{x}) \to 2(-(k-2)p_{k-2} + (k-1)g_3p_{k-1} + kg_2p_k + (k+1)g_1p_{k+1})$$

$$-(2n\tilde{x}^2 + q_1(n)\tilde{x} + q_0(n))p_n(\tilde{x}) \to -(2np_{k-2} + q_1(n)p_{k-1} + q_0(n)p_k)$$

Thus, equating the coefficients of the terms of degree k in $\tilde{x}$, in both sides of (11), we obtain

$$(k+2)(k+1)p_{k+2} + 2(-(k-2)p_{k-2} + (k-1)g_3p_{k-1} + kg_2p_k + (k+1)g_1p_{k+1}) =$$
$$= -(2np_{k-2} + q_1(n)p_{k-1} + q_0(n)p_k) \Rightarrow$$
$$\Rightarrow (k+2)(k+1)p_{k+2} - 2(k-2)p_{k-2} + 2(k-1)g_3p_{k-1} + 2kg_2p_k + 2(k+1)g_1p_{k+1} =$$
$$= -2np_{k-2} - q_1(n)p_{k-1} - q_0(n)p_k$$

Thus, we end up to the five-term recursion relation

$$(k+2)(k+1)p_{k+2} =$$
$$= -2(k+1)g_1p_{k+1} - (q_0(n) + 2kg_2)p_k - (2(k-1)g_3 + q_1(n))p_{k-1} + 2(k-n-2)p_{k-2} \quad (12)$$

We note that (12) holds in the region $\tilde{x} > 0$.
If the coefficients of the two non-analytic terms of the exponential polynomial vanish, i.e. if $g_1 = g_3 = 0$, the exponential polynomial becomes the analytic exponential polynomial of the analytic sextic oscillator, the quotient polynomial of which is also analytic **[4]**.
Thus, a smooth transition from the symmetrized to the analytic sextic oscillator requires that if $g_1 = g_3 = 0$, then $q_1(n) = 0$.
However, as we'll see below, the opposite does not hold, i.e. if $q_1(n)$ vanishes, i.e. if the quotient polynomial is analytic, g_1 and g_3 are not necessarily zero, and the resulting sextic oscillator can be non-analytic (symmetrized).

Thus, non-analytic (symmetrized) sextic oscillators can result from analytic quotient polynomials.

Also, based on the discussion in **[3]**, the absence of intermediate terms in the quotient polynomial, which in the case of symmetrized sextic oscillators implies the analyticity of the quotient polynomial, is a quasi-exact solvability condition.

Thus, non-analytic (symmetrized) sextic oscillators resulting from analytic quotient polynomials can be quasi-exactly solvable.

For $k=0$, dropping p_{-1} and p_{-2}, whose indices are negative and thus unacceptable, as they do not correspond to polynomial coefficients, (12) gives

$$2p_2=-2g_1p_1-q_0(n)p_0 \quad (13)$$

For $k=1$, dropping p_{-1}, whose index is negative, (12) gives

$$6p_3=-4g_1p_2-(q_0(n)+2g_2)p_1-q_1(n)p_0 \quad (14)$$

For $k=2,3,...,n-2$, all five terms are present in (12).
For $k=n-1$, dropping p_{n+1}, whose index exceeds the degree of $p_n(\tilde{x})$, and thus it is unacceptable, and using that $p_n=1$ in the region $\tilde{x}>0$, (12) gives

$$0=-2ng_1-(q_0(n)+2(n-1)g_2)p_{n-1}-(2(n-2)g_3+q_1(n))p_{n-2}-6p_{n-3} \quad (15)$$

For $k=n$, dropping p_{n+1} and p_{n+2}, whose indices exceed the degree of $p_n(\tilde{x})$, and using that $p_n=1$ in the region $\tilde{x}>0$, (12) gives

$$0=-(q_0(n)+2ng_2)-(2(n-1)g_3+q_1(n))p_{n-1}-4p_{n-2} \quad (16)$$

For $k=n+1$, dropping p_{n+1},p_{n+2},p_{n+3}, whose indices exceed the degree of $p_n(\tilde{x})$, and using that $p_n=1$ in the region $\tilde{x}>0$, (12) gives

$$0=-(2ng_3+q_1(n))-2p_{n-1} \quad (17)$$

For $k=n+2$, dropping $p_{n+1},p_{n+2},p_{n+3},p_{n+4}$, whose indices exceed the degree of $p_n(\tilde{x})$, and using that $p_n=1$ in the region $\tilde{x}>0$, (12) gives

$$0=2(n+2-n-2)\Rightarrow 0=0,$$

i.e. it holds identically, which is expected, since for $k=n+2$ we obtain the leading coefficient of the quotient polynomial, which we've incorporated into (12).
Thus, (12) gives non-trivial equations for

$$k=0,1,...,n+1$$

Then, we have $n+2$ equations with $n-1$ unknown coefficients of the polynomial $p_n(\tilde{x})$ – remember that $p_n=1$ – and 2 unknown coefficients of the quotient polynomial, i.e. we have $n+2$ equations with $n+1$ unknowns.

The polynomial $p_n(\tilde{x})$ must also satisfy the two continuity conditions (7) and (8).

To write the form of $p_n(\tilde{x})$ in the region $\tilde{x}<0$, we use that the symmetrized sextic oscillators are symmetric, i.e. they are potentials of even parity, and thus the ansatz eigenfunction (1) has definite parity **[9]**.
Since the symmetrized exponential polynomial (2) is of even parity, the exponential factor $\exp(g_4(\tilde{x}))$ in (1) is also of even parity, and thus the polynomial $p_n(\tilde{x})$ has the same parity as the eigenfunction (1), i.e. it is of either even or odd parity.

Examples

n=0

In the region $\tilde{x}>0$, $p_0(\tilde{x})=1$, and thus $p_0'(\tilde{x})=p_0''(\tilde{x})=0$.
Also, for $n=0$, (10) gives $q_2(0)=0$.
Then, (11) becomes

$$0=-(q_1(0)\tilde{x}+q_0(0))$$

and thus

$$q_0(0)=q_1(0)=0$$

Then

$$q_2(\tilde{x};0)=0 \quad (18)$$

as expected, since $q_{2(m-1)}(\tilde{x};0)=0$ **[3]**.
As a constant, the polynomial $p_0(\tilde{x})=1$ can be only of even parity, and this happens if (and only if) it is also 1 in the region $\tilde{x}<0$.
Then, the first continuity condition, i.e. the condition (7), is satisfied, while the second continuity condition, i.e. the condition (8), is written as

$$0=0+2g_1$$

and thus

$$g_1=0$$

The exponential polynomial (2) then becomes

$$g_4(\tilde{x})=-\frac{1}{4}\tilde{x}^4+\frac{g_3}{3}|\tilde{x}|^3+\frac{g_2}{2}\tilde{x}^2 \quad (19)$$

Then, the ansatz eigenfunction (1) takes the form

$$\psi(\tilde{x};2,0)=A_0\exp\left(-\frac{1}{4}\tilde{x}^4+\frac{g_3}{3}|\tilde{x}|^3+\frac{g_2}{2}\tilde{x}^2\right) \quad (20)$$

and since it has no (real) zeros, it describes the ground state of the symmetrized sextic oscillator we'll calculate now.
Using (19), we have, in the region $\tilde{x}>0$,

$$g_4'(\tilde{x})=-\tilde{x}^3+g_3\tilde{x}^2+g_2\tilde{x}$$

$g_4''(\tilde{x}) = -3\tilde{x}^2 + 2g_3\tilde{x} + g_2$

Plugging the previous derivatives into (3) – for $(m,n)=(2,0)$ – and using (18), we obtain that, in the region $\tilde{x}>0$,

$$\tilde{V}_+(\tilde{x};2,0) = \left(-\tilde{x}^3 + g_3\tilde{x}^2 + g_2\tilde{x}\right)^2 - 3\tilde{x}^2 + 2g_3\tilde{x} + g_2 + \tilde{E} =$$
$$= \tilde{x}^6 + g_3{}^2\tilde{x}^4 + g_2{}^2\tilde{x}^2 - 2g_3\tilde{x}^5 - 2g_2\tilde{x}^4 + 2g_2g_3\tilde{x}^3 - 3\tilde{x}^2 + 2g_3\tilde{x} + g_2 + \tilde{E} =$$
$$= \tilde{x}^6 - 2g_3\tilde{x}^5 + \left(g_3{}^2 - 2g_2\right)\tilde{x}^4 + 2g_2g_3\tilde{x}^3 + \left(g_2{}^2 - 3\right)\tilde{x}^2 + 2g_3\tilde{x} + g_2 + \tilde{E}$$

That is, in the region $\tilde{x}>0$,

$$\tilde{V}_+(\tilde{x};2,0) = \tilde{x}^6 - 2g_3\tilde{x}^5 + \left(g_3{}^2 - 2g_2\right)\tilde{x}^4 + 2g_2g_3\tilde{x}^3 + \left(g_2{}^2 - 3\right)\tilde{x}^2 + 2g_3\tilde{x} + g_2 + \tilde{E}$$

Since the potential is symmetric,

$$\tilde{V}(\tilde{x};2,0) = \tilde{x}^6 - 2g_3|\tilde{x}|^5 + \left(g_3{}^2 - 2g_2\right)\tilde{x}^4 + 2g_2g_3|\tilde{x}|^3 + \left(g_2{}^2 - 3\right)\tilde{x}^2 + 2g_3|\tilde{x}| + g_2 + \tilde{E} \quad (21)$$

The potential is continuous at zero, and thus (21) is defined for every $\tilde{x}$.
To calculate the ground-state energy $\tilde{E}$, we apply the condition $\tilde{V}(0;2,0)=0$, which gives

$$\tilde{E} = -g_2 \quad (22)$$

Then (21) is written as

$$\tilde{V}(\tilde{x};2,0) = \tilde{x}^6 - 2g_3|\tilde{x}|^5 + \left(g_3{}^2 - 2g_2\right)\tilde{x}^4 + 2g_2g_3|\tilde{x}|^3 + \left(g_2{}^2 - 3\right)\tilde{x}^2 + 2g_3|\tilde{x}| \quad (23)$$

Thus, the ground state of the symmetrized sextic oscillator (23) is described by the wave function (20) and it has energy given by (22).

Setting

$$\frac{g_3}{3} = -a \Rightarrow g_3 = -3a$$

$$\frac{g_2}{2} = b \Rightarrow g_2 = 2b$$

$$g_1 = -c,$$

we have $c=0$, and the potential (23), the ground-state wave function (20), and the ground-state energy (22) are respectively written as

$$\tilde{V}(\tilde{x};2,0) = \tilde{x}^6 + 6a|\tilde{x}|^5 + \left(9a^2 - 4b\right)\tilde{x}^4 - 12ab|\tilde{x}|^3 + \left(4b^2 - 3\right)\tilde{x}^2 - 6a|\tilde{x}|$$

$$\psi(\tilde{x};2,0) = A_0 \exp\left(-\frac{1}{4}\tilde{x}^4 - a|\tilde{x}|^3 + b\tilde{x}^2\right)$$

$$\tilde{E} = -2b,$$

in agreement with Quesne **[2]**.

n=1

For $n=1$, (12) becomes

$$(k+2)(k+1)p_{k+2} = -2(k+1)g_1p_{k+1} - (q_0(1)+2kg_2)p_k - (2(k-1)g_3+q_1(1))p_{k-1} + 2(k-3)p_{k-2} \quad (24)$$

with $k = 0,1,2$.

Also, in the region $\tilde{x} > 0$, $p_1(\tilde{x}) = \tilde{x} + p_0$.

For $k = 0$, using that $p_1 = 1$ and dropping the coefficients with negative or greater than 1 indices, (24) gives

$$0 = -2g_1 - q_0(1)p_0$$

in agreement with (13) for $p_1 = 1$ and $p_2 = 0$ ($n = 1$).

For $k = 1$, (24) gives

$$0 = -(q_0(1)+2g_2) - q_1(1)p_0$$

in agreement with (14) for $p_1 = 1$, $p_2 = p_3 = 0$ ($n = 1$).

For $k = 2$, (24) gives

$$0 = -(2g_3 + q_1(1)) - 2p_0$$

in agreement with (17) for $n = 1$.

Thus, we have the equations

$$2g_1 + q_0(1)p_0 = 0 \quad (25)$$

$$q_0(1) + 2g_2 + q_1(1)p_0 = 0 \quad (26)$$

$$2g_3 + q_1(1) + 2p_0 = 0 \quad (27)$$

This is a non-linear, non-homogeneous system of three equations with the three unknowns $q_0(1), q_1(1)$, and p_0.

The polynomial $p_1(\tilde{x})$ has definite parity, i.e. it is of either even or odd parity.

i. If $p_1(\tilde{x})$ is of odd parity, it must vanish at zero*, and thus $p_0 = 0$.

> The first continuity condition ensures that the polynomial $p_n(\tilde{x})$ is continuous at zero, and thus we define $p_n(0) \equiv p_n(0^+)$.

Substituting $p_0 = 0$ into (25), (26), and (27), we obtain, respectively,

$$g_1 = 0 \quad (28)$$

$$q_0(1) = -2g_2 \quad (29)$$

$$q_1(1) = -2g_3 \quad (30)$$

The odd-parity polynomial $p_1(\tilde{x}) = \tilde{x}$ is continuous at zero, and thus it satisfies the first continuity condition.

Using (28), the second continuity condition becomes

$$p_1'(0^-) = p_1'(0^+),$$

which is also satisfied, since $p_1(\tilde{x})$ is analytic.
Using (28), the exponential polynomial (2) is written as

$$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 + \frac{g_3}{3}|\tilde{x}|^3 + \frac{g_2}{2}\tilde{x}^2 \quad (31)$$

and then the ansatz eigenfunction (1) is written as

$$\psi(\tilde{x};2,1) = A_1\tilde{x}\exp\left(-\frac{1}{4}\tilde{x}^4 + \frac{g_3}{3}|\tilde{x}|^3 + \frac{g_2}{2}\tilde{x}^2\right) \quad (32)$$

The wave function (32) has one (real) zero, at zero, and thus it describes the first-excited state of the symmetrized sextic oscillator we'll calculate now.
Using (31), we have, in the region $\tilde{x} > 0$,

$$g_4'(\tilde{x}) = -\tilde{x}^3 + g_3\tilde{x}^2 + g_2\tilde{x}$$

$$g_4''(\tilde{x}) = -3\tilde{x}^2 + 2g_3\tilde{x} + g_2$$

The quotient polynomial is given by (4), with $n = 1$, i.e.

$$q_2(\tilde{x};1) = q_2(1)\tilde{x}^2 + q_1(1)|\tilde{x}| + q_0(1)$$

with $q_2(1) = 2$, as (10) gives.
Then, by means of (29) and (30), we end up to the quotient polynomial

$$q_2(\tilde{x};1) = 2\tilde{x}^2 - 2g_3|\tilde{x}| - 2g_2 \quad (33)$$

Thus, plugging into (3) – for $(m,n) = (2,1)$ – the first and second derivatives of the exponential polynomial and the quotient polynomial (33) in the region $\tilde{x} > 0$, we obtain

$$\tilde{V}_+(\tilde{x};2,1) = \left(-\tilde{x}^3 + g_3\tilde{x}^2 + g_2\tilde{x}\right)^2 - 3\tilde{x}^2 + 2g_3\tilde{x} + g_2 - \left(2\tilde{x}^2 - 2g_3\tilde{x} - 2g_2\right) + \tilde{E} =$$
$$= \tilde{x}^6 + g_3{}^2\tilde{x}^4 + g_2{}^2\tilde{x}^2 - 2g_3\tilde{x}^5 - 2g_2\tilde{x}^4 + 2g_2g_3\tilde{x}^3 - 3\tilde{x}^2 + 2g_3\tilde{x} + g_2 - 2\tilde{x}^2 + 2g_3\tilde{x} + 2g_2 + \tilde{E} =$$
$$= \tilde{x}^6 - 2g_3\tilde{x}^5 + \left(g_3{}^2 - 2g_2\right)\tilde{x}^4 + 2g_2g_3\tilde{x}^3 + \left(g_2{}^2 - 5\right)\tilde{x}^2 + 4g_3\tilde{x} + 3g_2 + \tilde{E}$$

That is, in the region $\tilde{x} > 0$,

$$\tilde{V}_+(\tilde{x};2,1) = \tilde{x}^6 - 2g_3\tilde{x}^5 + \left(g_3{}^2 - 2g_2\right)\tilde{x}^4 + 2g_2g_3\tilde{x}^3 + \left(g_2{}^2 - 5\right)\tilde{x}^2 + 4g_3\tilde{x} + 3g_2 + \tilde{E}$$

Since the potential is symmetric,

$$\tilde{V}(\tilde{x};2,1) = \tilde{x}^6 - 2g_3|\tilde{x}|^5 + \left(g_3{}^2 - 2g_2\right)\tilde{x}^4 + 2g_2g_3|\tilde{x}|^3 + \left(g_2{}^2 - 5\right)\tilde{x}^2 + 4g_3|\tilde{x}| + 3g_2 + \tilde{E} \quad (34)$$

The potential is continuous at zero, and thus (34) is defined for every $\tilde{x}$.
Applying the condition $\tilde{V}(0;2,1) = 0$, we obtain the energy, which is

$$\tilde{E} = -3g_2 \quad (35)$$

and we end up to the symmetrized sextic oscillator

$$\tilde{V}(\tilde{x};2,1)=\tilde{x}^6-2g_3|\tilde{x}|^5+\left(g_3^{\ 2}-2g_2\right)\tilde{x}^4+2g_2g_3|\tilde{x}|^3+\left(g_2^{\ 2}-5\right)\tilde{x}^2+4g_3|\tilde{x}| \quad (36)$$

Thus, the first-excited state of the symmetrized sextic oscillator (36) is described by the wave function (32) and it has energy given by (35).

Setting

$$\frac{g_3}{3}=-a\Rightarrow g_3=-3a$$

$$\frac{g_2}{2}=b\Rightarrow g_2=2b$$

$$g_1=-c\,,$$

we have $c=0$, and the potential (36), the first-excited-state wave function (32), and the energy (35) are respectively written as

$$\tilde{V}(\tilde{x};2,1)=\tilde{x}^6+6a|\tilde{x}|^5+\left(9a^2-4b\right)\tilde{x}^4-12ab|\tilde{x}|^3+\left(4b^2-5\right)\tilde{x}^2-12a|\tilde{x}|$$

$$\psi(\tilde{x};2,1)=A_1\tilde{x}\exp\left(-\frac{1}{4}\tilde{x}^4-a|\tilde{x}|^3+b\tilde{x}^2\right)$$

$$\tilde{E}=-6b\,,$$

in agreement with Quesne **[2]**.

ii. If $p_1(\tilde{x})$ is of even parity, then

$$p_1(\tilde{x})=|\tilde{x}|+p_0$$

The previous polynomial is continuous at zero, and thus it satisfies the first continuity condition.
Also

$$p_1'\left(0^-\right)=-1 \text{ and } p_1'\left(0^+\right)=1,$$

and then the second continuity condition is written as

$$-1=1+2g_1p_0\Rightarrow g_1p_0=-1$$

From the last equation, we derive that $g_1\neq 0$, and then, solving for p_0,

$$p_0=-\frac{1}{g_1} \quad (37)$$

Substituting (37) into (25), (26), and (27), we obtain, respectively,

$$2g_1+q_0(1)\left(-\frac{1}{g_1}\right)=0$$

$$q_0(1)+2g_2+q_1(1)\left(-\frac{1}{g_1}\right)=0$$

$$2g_3+q_1(1)+2\left(-\frac{1}{g_1}\right)=0$$

The first of the above three equations gives

$$q_0(1) = 2g_1^{\,2} \quad (38)$$

The third of the above three equations gives

$$q_1(1) = 2\left(\frac{1}{g_1} - g_3\right) \quad (39)$$

Then, substituting (38) and (39) into the second of the above three equations, we obtain

$$2g_1^{\,2} + 2g_2 + 2\left(\frac{1}{g_1} - g_3\right)\left(-\frac{1}{g_1}\right) = 0 \Rightarrow g_1^{\,2} + g_2 - \frac{1}{g_1^{\,2}} + \frac{g_3}{g_1} = 0 \Rightarrow$$
$$\Rightarrow g_1^{\,4} + g_2 g_1^{\,2} - 1 + g_3 g_1 = 0$$

Thus, the coefficient g_1 of the non-analytic linear term of the exponential polynomial must satisfy the quartic equation

$$g_1^{\,4} + g_2 g_1^{\,2} + g_3 g_1 - 1 = 0 \quad (40)$$

Using (37), the even-parity polynomial $p_1(\tilde{x})$ is written as

$$p_1(\tilde{x}) = |\tilde{x}| - \frac{1}{g_1}$$

and the ansatz eigenfunction (1) is then written as

$$\psi(\tilde{x};2,1) = A_1\left(|\tilde{x}| - \frac{1}{g_1}\right)\exp\left(-\frac{1}{4}\tilde{x}^4 + \frac{g_3}{3}|\tilde{x}|^3 + \frac{g_2}{2}\tilde{x}^2 + g_1|\tilde{x}|\right) \quad (41)$$

with g_1 satisfying (40).

For $g_1 < 0$, the wave function (41) has no (real) zeros, and thus it describes the ground state of the symmetrized sextic oscillator we calculate below.

For $g_1 > 0$, the wave function (41) has two (real) zeros, at $|\tilde{x}| = \frac{1}{g_1}$, i.e. at $\tilde{x} = \pm\frac{1}{g_1}$, and thus it describes the second-excited state of the symmetrized sextic oscillator we calculate below.

Using (38), (39), and that $q_2(1) = 2$, the quotient polynomial in this case is

$$q_2(\tilde{x};n) = 2\tilde{x}^2 + 2\left(\frac{1}{g_1} - g_3\right)|\tilde{x}| + 2g_1^{\,2} \quad (42)$$

The exponential polynomial is given by the general relation (2) with g_1 satisfying (40).

Then, the first and second derivatives of the exponential polynomial in the region $\tilde{x} > 0$ are, respectively,

$$g_4{}'(\tilde{x}) = -\tilde{x}^3 + g_3\tilde{x}^2 + g_2\tilde{x} + g_1$$

$$g_4''(\tilde{x}) = -3\tilde{x}^2 + 2g_3\tilde{x} + g_2$$

Plugging into (3) – for $(m,n)=(2,1)$ – the first and second derivatives of the exponential polynomial and the quotient polynomial (42) in the region $\tilde{x}>0$, we obtain

$$\tilde{V}_+(\tilde{x};2,1) = \left(-\tilde{x}^3 + g_3\tilde{x}^2 + g_2\tilde{x} + g_1\right)^2 - 3\tilde{x}^2 + 2g_3\tilde{x} + g_2 - \left(2\tilde{x}^2 + 2\left(\frac{1}{g_1} - g_3\right)\tilde{x} + 2g_1^{\ 2}\right) + \tilde{E} =$$
$$= \tilde{x}^6 + g_3^{\ 2}\tilde{x}^4 + g_2^{\ 2}\tilde{x}^2 + g_1^{\ 2} - 2g_3\tilde{x}^5 - 2g_2\tilde{x}^4 - 2g_1\tilde{x}^3 + 2g_2g_3\tilde{x}^3 + 2g_1g_3\tilde{x}^2 + 2g_1g_2\tilde{x} - 3\tilde{x}^2 + 2g_3\tilde{x} + g_2 -$$
$$-2\tilde{x}^2 - 2\left(\frac{1}{g_1} - g_3\right)\tilde{x} - 2g_1^{\ 2} + \tilde{E} = \tilde{x}^6 - 2g_3\tilde{x}^5 + \left(g_3^{\ 2} - 2g_2\right)\tilde{x}^4 + 2\left(g_2g_3 - g_1\right)\tilde{x}^3 +$$
$$+\left(g_2^{\ 2} + 2g_1g_3 - 5\right)\tilde{x}^2 + 2\left(g_1g_2 + g_3 - \left(\frac{1}{g_1} - g_3\right)\right)\tilde{x} + g_2 - g_1^{\ 2} + \tilde{E}$$

That is, in the region $\tilde{x}>0$,

$$\tilde{V}_+(\tilde{x};2,1) = \tilde{x}^6 - 2g_3\tilde{x}^5 + \left(g_3^{\ 2} - 2g_2\right)\tilde{x}^4 + 2\left(g_2g_3 - g_1\right)\tilde{x}^3 + \left(g_2^{\ 2} + 2g_1g_3 - 5\right)\tilde{x}^2 +$$
$$+2\left(g_1g_2 + 2g_3 - \frac{1}{g_1}\right)\tilde{x} + g_2 - g_1^{\ 2} + \tilde{E}$$

Since the potential is symmetric,

$$\tilde{V}(\tilde{x};2,1) = \tilde{x}^6 - 2g_3|\tilde{x}|^5 + \left(g_3^{\ 2} - 2g_2\right)\tilde{x}^4 + 2\left(g_2g_3 - g_1\right)|\tilde{x}|^3 + \left(g_2^{\ 2} + 2g_1g_3 - 5\right)\tilde{x}^2 +$$
$$+2\left(g_1g_2 + 2g_3 - \frac{1}{g_1}\right)|\tilde{x}| + g_2 - g_1^{\ 2} + \tilde{E} \quad (43)$$

The potential is continuous at zero, and thus (43) is defined for every $\tilde{x}$.
Applying the condition $\tilde{V}(0;2,1)=0$, we obtain the energy, which is

$$\tilde{E} = g_1^{\ 2} - g_2 \quad (44)$$

and we end up to the symmetrized sextic oscillator

$$\tilde{V}(\tilde{x};2,1) = \tilde{x}^6 - 2g_3|\tilde{x}|^5 + \left(g_3^{\ 2} - 2g_2\right)\tilde{x}^4 + 2\left(g_2g_3 - g_1\right)|\tilde{x}|^3 + \left(g_2^{\ 2} + 2g_1g_3 - 5\right)\tilde{x}^2 +$$
$$+2\left(g_1g_2 + 2g_3 - \frac{1}{g_1}\right)|\tilde{x}| \quad (45)$$

Thus, for $g_1<0$, the ground state of the symmetrized sextic oscillator (45) is described by the wave function (41), and its energy is given by (44), while for $g_1>0$, the second-excited state of the symmetrized sextic oscillator (45) is described by the wave function (41), and its energy is again given by (44).
In both cases, $g_1<0$ and $g_1>0$, g_1 satisfies the quartic equation (40).

Setting

$$\frac{g_3}{3} = -a \Rightarrow g_3 = -3a$$

$$\frac{g_2}{2} = b \Rightarrow g_2 = 2b$$

$g_1 = -c$,

the quartic equation (40) becomes

$c^4 + 2bc^2 + 3ac - 1 = 0$,

in agreement with Quesne **[2]**,

the potential (45) and the wave function (41) are respectively written as

$$\tilde{V}(\tilde{x};2,1) = \tilde{x}^6 + 6a|\tilde{x}|^5 + (9a^2 - 4b)\tilde{x}^4 + 2(-6ab + c)|\tilde{x}|^3 + (4b^2 + 6ac - 5)\tilde{x}^2 + 2\left(-2bc - 6a + \frac{1}{c}\right)|\tilde{x}| =$$

$$= \tilde{x}^6 + 6a|\tilde{x}|^5 + (9a^2 - 4b)\tilde{x}^4 - (12ab - 2c)|\tilde{x}|^3 + (4b^2 + 6ac - 5)\tilde{x}^2 - \left(12a + 4bc - \frac{2}{c}\right)|\tilde{x}|$$

$$\psi(\tilde{x};2,1) = A_1\left(|\tilde{x}| + \frac{1}{c}\right)\exp\left(-\frac{1}{4}\tilde{x}^4 - a|\tilde{x}|^3 + b\tilde{x}^2 - c|\tilde{x}|\right),$$

in agreement with Quesne **[2]**,

and the energy (44) is written as

$\tilde{E} = c^2 - 2b$,

which is the same as that found by Quesne **[2]** if we take into account the quartic equation (40), which, as we saw, is written as $c^4 + 2bc^2 + 3ac - 1 = 0$.

Indeed, since $c \neq 0$, dividing both members of the previous equation by c^2 yields

$$c^2 + 2b + \frac{3a}{c} - \frac{1}{c^2} = 0 \Rightarrow c^2 + 2b = -\frac{3a}{c} + \frac{1}{c^2} \Rightarrow -\frac{6a}{c} + \frac{2}{c^2} = 2c^2 + 4b$$

Then, substituting into the expression of Quesne **[2]**, i.e.

$$\tilde{E} = -6b - c^2 - \frac{6a}{c} + \frac{2}{c^2},$$

we obtain

$\tilde{E} = c^2 - 2b$.

Summarizing the case $n = 1$, we observe that the parameter g_1, i.e. the coefficient of the non-analytic linear term of the exponential polynomial, determines the degree of excitation of the known eigenstate.

If the parameter g_1 is negative, the known eigenstate is the ground state, if it is zero, the known eigenstate is the first-excited state, while if it is positive, the known eigenstate is the second-excited state.

We also observe that, for $n = 1$, we find a wave function of natural, i.e. of odd, parity, which corresponds to g_1 being zero and it describes the first-excited state, and a wave function of unnatural, i.e. of even, parity, which corresponds to g_1 being non-zero and it describes the ground state, if g_1 is negative, or the second-excited state, if g_1 is positive.

Both these features also appear in the case $n = 1$ of symmetrized quartic oscillators **[5]**.

The transition to analytic sextic oscillators in the cases n=0 and n=1

In the cases $n = 0$ and $n = 1$, we'll examine the transition of the symmetrized to analytic sextic oscillators.

In the case $n=0$, the parameter g_1 is always zero, and the exponential polynomial is then given by (19), i.e.

$$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 + \frac{g_3}{3}|\tilde{x}|^3 + \frac{g_2}{2}\tilde{x}^2$$

If g_3 is also zero, the exponential polynomial becomes analytic, and the respective symmetrized sextic oscillator, given by (23), then becomes

$$\tilde{V}(\tilde{x};2,0) = \tilde{x}^6 - 2g_2\tilde{x}^4 + \left(g_2{}^2 - 3\right)\tilde{x}^2$$

i.e. it becomes an analytic sextic oscillator.
The ground-state wave function (20) then becomes

$$\psi(\tilde{x};2,0) = A_0 \exp\left(-\frac{1}{4}\tilde{x}^4 + \frac{g_2}{2}\tilde{x}^2\right),$$

while the energy (22) remains as it is, as it does not depend either on g_1 or on g_3.
The previous potential, wave function, and energy are in agreement with the corresponding results in **[4]** for $g_4{}^2 = 1$.
For $n=0$, the quotient polynomial is zero, and thus it is (trivially) analytic.
In the case $n=1$, with $g_1 = 0$, setting $g_3 = 0$, the exponential polynomial (31) becomes analytic, and the symmetrized sextic oscillator (36) becomes the analytic sextic oscillator

$$\tilde{V}(\tilde{x};2,1) = \tilde{x}^6 - 2g_2\tilde{x}^4 + \left(g_2{}^2 - 5\right)\tilde{x}^2,$$

in agreement with the corresponding result in **[4]** for $g_4{}^2 = 1$.
The wave function (32) becomes

$$\psi(\tilde{x};2,1) = A_1\tilde{x}\exp\left(-\frac{1}{4}\tilde{x}^4 + \frac{g_2}{2}\tilde{x}^2\right),$$

and it is the first-excited-state wave function of the previous analytic sextic oscillator, with energy given by (35), which remains as it is.
Finally, the quotient polynomial (33) becomes the analytic quotient polynomial

$$q_2(\tilde{x};1) = 2\tilde{x}^2 - 2g_2,$$

which the previous analytic sextic oscillator results from.
Thus, in the case $n=1$, the vanishing of both parameters g_1 and g_3 results in the vanishing of the non-analytic linear term $q_1(1)|\tilde{x}|$ of the quotient polynomial, i.e. if $g_1 = g_3 = 0$, then $q_1(1) = 0$.
In the case $n=1$, with $g_1 \neq 0$, the exponential polynomial has always a non-analytic linear term $g_1|\tilde{x}|$, and the respective sextic oscillator (45) is always non-analytic, since its non-analytic terms $-2g_3|\tilde{x}|^5$ and $2(g_2g_3 - g_1)|\tilde{x}|^3$ cannot both become zero if $g_1 \neq 0$.
Thus, in the case $n=1$, with $g_1 \neq 0$, we cannot take an analytic sextic oscillator.

Non-analytic sextic oscillators resulting from analytic quotient polynomials

The quotient polynomial (42), which is the quotient polynomial of the case $n=1$, with $g_1 \neq 0$, becomes analytic if its non-analytic linear term $2\left(\frac{1}{g_1}-g_3\right)|\tilde{x}|$ vanishes, i.e. if $g_3=\frac{1}{g_1}$, and then we have a case of a non-analytic sextic oscillator resulting from an analytic quotient polynomial.

For $g_3=\frac{1}{g_1}$, the quartic equation (40) becomes $g_1^4+g_2g_1^2=0$, and since $g_1 \neq 0$, we obtain $g_2=-g_1^2$, and we've expressed both g_2 and g_3 in terms of g_1.

Using the previous expressions of g_2 and g_3, the sextic oscillator (45), the energy (44), and the eigenfunction (41) are respectively written as

$$\tilde{V}(\tilde{x};2,1)=\tilde{x}^6-\frac{2}{g_1}|\tilde{x}|^5+\left(\frac{1}{g_1^2}+2g_1^2\right)\tilde{x}^4-4g_1|\tilde{x}|^3+\left(g_1^4-3\right)\tilde{x}^2+2\left(-g_1^3+\frac{1}{g_1}\right)|\tilde{x}| \quad (46)$$

$$\tilde{E}=2g_1^2 \quad (47)$$

$$\psi(\tilde{x};2,1)=A_1\left(|\tilde{x}|-\frac{1}{g_1}\right)\exp\left(-\frac{1}{4}\tilde{x}^4+\frac{1}{3g_1}|\tilde{x}|^3-\frac{g_1^2}{2}\tilde{x}^2+g_1|\tilde{x}|\right) \quad (48)$$

For $g_1<0$, the wave function (48) describes the ground state of the non-analytic sextic oscillator (46), with energy given by (47), while for $g_1>0$, the wave function (48) describes the second-excited state of the non-analytic sextic oscillator (46), with energy given by (47), which is always positive, whether it is the ground-state energy or the second-excited-state energy.

The analytic quotient polynomial which the non-analytic sextic oscillator (46) results from is given by (42) for $g_3=\frac{1}{g_1}$, i.e.

$$q_2(\tilde{x};n)=2\tilde{x}^2+2g_1^2$$

We thus see that a non-analytic sextic oscillator can result from an analytic quotient polynomial.

We also observe that, in both the oscillator (46) and the exponential polynomial of (48), the coefficients of the analytic terms are of even parity in g_1, while the coefficients of the non-analytic terms are of odd parity in g_1. Additionally, the energy (47) is of even parity in g_1.

n=2 (for analytic quotient polynomials only)

In the case $n=2$, we'll calculate the sextic oscillators resulting from analytic quotient polynomials.

In the region $\tilde{x}>0$, $p_2(\tilde{x})=\tilde{x}^2+p_1\tilde{x}+p_0$.

Since we consider analytic quotient polynomials, $q_1(2)=0$, and then the recursion relation (12) becomes, for $n=2$,

$$(k+2)(k+1)p_{k+2}=-2(k+1)g_1p_{k+1}-(q_0(2)+2kg_2)p_k-2(k-1)g_3p_{k-1}+2(k-4)p_{k-2} \quad (49)$$

with $k=0,1,2,3$

For $k=0$, dropping p_{-1} and p_{-2}, and using that $p_2=1$, we obtain from (49)

$$2=-2g_1p_1-q_0(2)p_0$$

For $k=1$, dropping p_{-1} and p_3, and using that $p_2=1$, we obtain from (49)

$$0=-4g_1-(q_0(2)+2g_2)p_1$$

For $k=2$, dropping p_3 and p_4, and using that $p_2=1$, we obtain from (49)

$$0=-(q_0(2)+4g_2)-2g_3p_1-4p_0$$

For $k=3$, dropping p_3,p_4, and p_5, and using that $p_2=1$, we obtain from (49)

$$0=-4g_3-2p_1$$

Solving the last equation for p_1, we obtain

$$p_1=-2g_3 \quad (50)$$

Substituting (50) into the other three equations, we obtain

$$2=4g_1g_3-q_0(2)p_0 \quad (51)$$

$$0=-4g_1+2g_3(q_0(2)+2g_2) \quad (52)$$

$$0=-(q_0(2)+4g_2)+4g_3^{\,2}-4p_0 \quad (53)$$

The equation (52) is written as

$$g_3(q_0(2)+2g_2)=2g_1 \quad (54)$$

i. If $g_3=0$, then from (54), $g_1=0$, and the exponential polynomial (2) becomes analytic. Then, since the quotient polynomial is also analytic, the sextic oscillator

$$\tilde{V}(\tilde{x};2,2)=g_4'^2(\tilde{x})+g_4''(\tilde{x})-q_2(\tilde{x};2)+\tilde{E}$$

is analytic too, as sum of analytic polynomials.
Thus, we have the case of an analytic sextic oscillator resulting from an analytic quotient polynomial.
For $g_1=g_3=0$, (51) and (53) are respectively written as

$$2=-q_0(2)p_0 \quad (55)$$

$$0=-(q_0(2)+4g_2)-4p_0 \quad (56)$$

From (55), we derive that both $q_0(2)$ and p_0 are non-zero, and solving for p_0, we obtain

$$p_0 = -\frac{2}{q_0(2)} \quad (57)$$

Substituting (57) into (56) we obtain

$$0 = -\left(q_0(2) + 4g_2\right) - 4\left(-\frac{2}{q_0(2)}\right) \Rightarrow 0 = -q_0(2) - 4g_2 + \frac{8}{q_0(2)} \Rightarrow$$

$$\Rightarrow -q_0^2(2) - 4g_2 q_0(2) + 8 = 0$$

Thus

$$q_0^2(2) + 4g_2 q_0(2) - 8 = 0 \quad (58)$$

This is the quadratic equation that $q_0(2)$ satisfies in the case $n=2$ of the analytic sextic oscillator **[4]**, for $g_4^2 = 1$.

Since $g_1 = 0$, from the second continuity condition we obtain that the derivative of $p_2(\tilde{x})$ is analytic at zero (and thus everywhere), i.e. $p_2'(0^-) = p_2'(0^+)$.

Then, using (57) and that for $g_3 = 0$, $p_1 = 0$, as derived from (50), the polynomial $p_2(\tilde{x})$, in the region $\tilde{x} > 0$, is

$$p_2(\tilde{x}) = \tilde{x}^2 - \frac{2}{q_0(2)}$$

Since $p_2(\tilde{x})$ has definite parity and it is analytic at zero, it can only be even, and then

$$p_2(\tilde{x}) = \tilde{x}^2 - \frac{2}{q_0(2)} \quad (59)$$

for every $\tilde{x}$, with $q_0(2)$ satisfying (58).

The even-parity polynomial (59) is the polynomial we obtain in the case $n=2$ of the analytic sextic oscillator **[4]**, for $g_4^2 = 1$.

Therefore, to summarize, if $g_3 = 0$, then $g_1 = 0$ and we end up to the analytic sextic oscillator **[4]**, for $g_4^2 = 1$.

ii. If $g_3 \neq 0$, solving (54) for $q_0(2)$, we obtain

$$q_0(2) = \frac{2g_1}{g_3} - 2g_2 \quad (60)$$

Substituting (60) into (51) and (53), we obtain, respectively,

$$2 = 4g_1 g_3 - \left(\frac{2g_1}{g_3} - 2g_2\right) p_0 \quad (61)$$

$$0 = -\left(\frac{2g_1}{g_3} + 2g_2\right) + 4g_3^2 - 4p_0 \quad (62)$$

The equation (50) also holds for $g_3 \neq 0$.

The polynomial $p_2(\tilde{x})$ has definite parity, and thus we have the subcases

iia. Natural (even) parity

If $p_2(\tilde{x})$ is of even parity, then $p_2(\tilde{x}) = \tilde{x}^2 + p_1|\tilde{x}| + p_0$.

The previous polynomial is continuous at zero, and thus the first continuity condition is satisfied.

Also, $p_2'(0^-) = -p_1$ and $p_2'(0^+) = p_1$, and thus the second continuity condition is written as

$$-p_1 = p_1 + 2g_1 p_0 \Rightarrow g_1 p_0 = -p_1$$

Substituting (50) into the last equation, we obtain

$$g_1 p_0 = 2g_3$$

Since $g_3 \neq 0$, both g_1 and p_0 are also non-zero, and thus

$$p_0 = \frac{2g_3}{g_1} \quad (63)$$

Substituting (63) into (61), we obtain

$$2 = 4g_1g_3 - \left(\frac{2g_1}{g_3} - 2g_2\right)\frac{2g_3}{g_1} \Rightarrow 1 = 2g_1g_3 - \frac{2g_3}{g_1}\left(\frac{g_1}{g_3} - g_2\right) \Rightarrow$$

$$\Rightarrow 1 = 2g_1g_3 - 2 + \frac{2g_2g_3}{g_1} \Rightarrow \frac{2g_2g_3}{g_1} + 2g_1g_3 - 3 = 0 \Rightarrow$$

$$\Rightarrow 2g_2g_3 + 2g_1^2g_3 - 3g_1 = 0 \Rightarrow 2g_2g_3 = -2g_1^2g_3 + 3g_1$$

Since $g_3 \neq 0$, solving the last equation for g_2 gives

$$g_2 = \frac{-2g_1^2g_3 + 3g_1}{2g_3} \quad (64)$$

Besides, substituting (63) into (62), we obtain

$$0 = -\left(\frac{2g_1}{g_3} + 2g_2\right) + 4g_3^2 - 8\frac{g_3}{g_1} \Rightarrow -\left(\frac{g_1}{g_3} + g_2\right) + 2g_3^2 - 4\frac{g_3}{g_1} = 0 \Rightarrow$$

$$\Rightarrow \frac{g_1}{g_3} + g_2 - 2g_3^2 + 4\frac{g_3}{g_1} = 0 \Rightarrow \frac{g_1}{g_3} + 4\frac{g_3}{g_1} + g_2 - 2g_3^2 = 0$$

Substituting (64) into the last equation, we obtain

$$\frac{g_1}{g_3} + 4\frac{g_3}{g_1} + \frac{-2g_1^2g_3 + 3g_1}{2g_3} - 2g_3^2 = 0 \Rightarrow 2g_1^2 + 8g_3^2 + g_1\left(-2g_1^2g_3 + 3g_1\right) - 4g_1g_3^3 = 0 \Rightarrow$$

$$\Rightarrow 2g_1^2 + 8g_3^2 - 2g_1^3g_3 + 3g_1^2 - 4g_1g_3^3 = 0 \Rightarrow 5g_1^2 + 8g_3^2 - 2g_1^3g_3 - 4g_1g_3^3 = 0$$

Thus, we end up to the following equation

$$4g_1g_3^3 - 8g_3^2 + 2g_1^3g_3 - 5g_1^2 = 0 \quad (65)$$

Since $g_1 \neq 0$, the equation (65) is cubic in g_3, and thus it has at least one real root.

Also, since $-5{g_1}^2$ is non-zero, all roots of (65) are non-zero, as they should, since g_3 must be non-zero.
Solving (65) for g_3, we express it in terms of g_1, and then substituting into (64), we express g_2 in terms of g_1 too.
To summarize the subcase iia, the analytic quotient polynomial

$$q_2(\tilde{x};2) = q_2(2)\tilde{x}^2 + q_0(2),$$

with $q_0(2)$ given by (60) and $q_2(2)$ given by (10) for $n=2$, i.e. $q_2(2)=4$, corresponds to the even-parity polynomial $p_2(\tilde{x}) = \tilde{x}^2 + p_1|\tilde{x}| + p_0$, with p_0 given by (63) and p_1 given by (50), and the parameters g_1, g_2, g_3 satisfying the equations (64) and (65), which have at least one real solution, with $g_1 g_3 \neq 0$, giving g_2 and g_3 in terms of g_1.
Since $p_2(\tilde{x})$ is of even parity, the respective ansatz eigenfunction (1) is also of even parity.
Also, since $g_1 g_3 \neq 0$, the exponential polynomial (2) is non-analytic, and thus the resulting sextic oscillator is also non-analytic, as seen from (3).
Therefore, the previous analytic quotient polynomial corresponds to a non-analytic sextic oscillator with its closed-form eigenfunction being of natural, i.e. of even, parity.

iib. Unnatural (odd) parity

If the polynomial $p_2(\tilde{x})$ is of odd parity, then

$$p_0 = 0$$

and

$$p_2(\tilde{x}) = \begin{cases} -\tilde{x}^2 + p_1\tilde{x}, & \tilde{x} < 0 \\ \tilde{x}^2 + p_1\tilde{x}, & \tilde{x} > 0 \end{cases}$$

with p_1 given by (50).
Substituting $p_0 = 0$ into (61) and (62), we obtain, respectively,

$$2 = 4g_1 g_3$$

$$0 = -\left(\frac{2g_1}{g_3} + 2g_2\right) + 4{g_3}^2$$

From the first equation, we derive that g_1 and g_3 are both non-zero, and then, solving for g_3, we obtain

$$g_3 = \frac{1}{2g_1} \quad (66)$$

Substituting (66) into the second of the previous two equations and solving for g_2, we obtain

$$0=-\left(4g_1^{\ 2}+2g_2\right)+\frac{1}{g_1^{\ 2}}\Rightarrow -4g_1^{\ 2}-2g_2+\frac{1}{g_1^{\ 2}}=0\Rightarrow 2g_2=-4g_1^{\ 2}+\frac{1}{g_1^{\ 2}}$$

Thus

$$g_2=-2g_1^{\ 2}+\frac{1}{2g_1^{\ 2}} \quad (67)$$

The equations (66) and (67) give us g_2 and g_3 in terms of g_1.
We see that $p_2\left(0^-\right)=0=p_2\left(0^+\right)$, and thus the first continuity condition is satisfied.
Using the expression of $p_2\left(\tilde{x}\right)$, we have

$$p_2'\left(\tilde{x}\right)=\begin{cases}-2\tilde{x}+p_1, \ \tilde{x}<0\\ 2\tilde{x}+p_1, \ \tilde{x}>0\end{cases},$$

and thus

$$p_2'\left(0^-\right)=p_1=p_2'\left(0^+\right)$$

Then, the second continuity condition is written as

$$p_1=p_1+2g_1 0\Rightarrow 0=0,$$

i.e. it is also satisfied.
Using the expression of g_3, given by (66), (50) becomes

$$p_1=-\frac{1}{g_1}$$

and then the odd-parity polynomial $p_2\left(\tilde{x}\right)$ is

$$p_2\left(\tilde{x}\right)=\begin{cases}-\tilde{x}^2-\dfrac{1}{g_1}\tilde{x}, \ \tilde{x}<0\\ \tilde{x}^2-\dfrac{1}{g_1}\tilde{x}, \ \tilde{x}>0\end{cases},$$

or, since $p_2\left(\tilde{x}\right)$ is continuous at zero,

$$p_2\left(\tilde{x}\right)=\begin{cases}-\tilde{x}^2-\dfrac{1}{g_1}\tilde{x}, \ \tilde{x}\le 0\\ \tilde{x}^2-\dfrac{1}{g_1}\tilde{x}, \ \tilde{x}\ge 0\end{cases}$$

Using the sign function, we write $p_2\left(\tilde{x}\right)$ as

$$p_2\left(\tilde{x}\right)=\operatorname{sgn}\left(\tilde{x}\right)\tilde{x}^2-\frac{1}{g_1}\tilde{x}=\tilde{x}\left(\operatorname{sgn}\left(\tilde{x}\right)\tilde{x}-\frac{1}{g_1}\right),$$

and since $\operatorname{sgn}\left(\tilde{x}\right)\tilde{x}=\left|\tilde{x}\right|$, we end up to

$p_2(\tilde{x}) = \tilde{x}\left(|\tilde{x}| - \frac{1}{g_1}\right)$ (68)

Besides, by means of (66) and (67), the exponential polynomial (2) is written as

$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 + \frac{1}{6g_1}|\tilde{x}|^3 + \left(-g_1^{\ 2} + \frac{1}{4g_1^{\ 2}}\right)\tilde{x}^2 + g_1|\tilde{x}|$ (69)

As in the respective $n=1$ case, the coefficients of the two analytic terms of the exponential polynomial are of even parity in g_1, while the coefficients of the two non-analytic terms are of odd parity in g_1.

Using (68) and (69), the ansatz eigenfunction (1) is written as

$\psi(\tilde{x};2,2) = A_2\tilde{x}\left(|\tilde{x}| - \frac{1}{g_1}\right)\exp\left(-\frac{1}{4}\tilde{x}^4 + \frac{1}{6g_1}|\tilde{x}|^3 + \left(-g_1^{\ 2} + \frac{1}{4g_1^{\ 2}}\right)\tilde{x}^2 + g_1|\tilde{x}|\right)$ (70)

If $g_1 < 0$, $|\tilde{x}| - \frac{1}{g_1} > 0$, and the wave function (70) has only one (real) zero, at zero, and thus it describes the first-excited state of the symmetrized sextic oscillator we calculate below.

If $g_1 > 0$, the equation $|\tilde{x}| - \frac{1}{g_1} = 0$ has two real roots, at $\pm\frac{1}{g_1}$, and then the wave function (70) has three (real) zeros, and thus it describes the third-excited state of the symmetrized sextic oscillator we calculate below.

Using (66) and (67), the constant term $q_0(2)$ of the quotient polynomial, given by (60), is written as

$$q_0(2) = \frac{2g_1}{\frac{1}{2g_1}} - 2\left(-2g_1^{\ 2} + \frac{1}{2g_1^{\ 2}}\right) = 4g_1^{\ 2} + 4g_1^{\ 2} - \frac{1}{g_1^{\ 2}} = 8g_1^{\ 2} - \frac{1}{g_1^{\ 2}}$$

That is

$q_0(2) = 8g_1^{\ 2} - \frac{1}{g_1^{\ 2}}$ (71)

The leading coefficient $q_2(2)$ of the quotient polynomial is the same for all $n=2$ cases and it is given by (10) for $n=2$, i.e. $q_2(2)=4$.

Thus, since $q_1(2)=0$, as we consider analytic quotient polynomials, the quotient polynomial in this case is

$q_2(\tilde{x};2) = 4\tilde{x}^2 + 8g_1^{\ 2} - \frac{1}{g_1^{\ 2}}$ (72)

Using (69), the first and second derivatives of the exponential polynomial in the region $\tilde{x} > 0$ is

$$g_4'(\tilde{x}) = -\tilde{x}^3 + \frac{1}{2g_1}\tilde{x}^2 + 2\left(-g_1^{\ 2} + \frac{1}{4g_1^{\ 2}}\right)\tilde{x} + g_1$$

$$g_4''(\tilde{x}) = -3\tilde{x}^2 + \frac{1}{g_1}\tilde{x} + 2\left(-g_1^2 + \frac{1}{4g_1^2}\right)$$

Plugging into (3) – for $(m,n)=(2,2)$ – the first and second derivatives of the exponential polynomial and the quotient polynomial (72), we obtain that, in the region $\tilde{x} > 0$,

$$\tilde{V}_+(\tilde{x};2,2) = \left(-\tilde{x}^3 + \frac{1}{2g_1}\tilde{x}^2 + 2\left(-g_1^2 + \frac{1}{4g_1^2}\right)\tilde{x} + g_1\right)^2 - 3\tilde{x}^2 + \frac{1}{g_1}\tilde{x} + 2\left(-g_1^2 + \frac{1}{4g_1^2}\right) -$$

$$-\left(4\tilde{x}^2 + 8g_1^2 - \frac{1}{g_1^2}\right) + \tilde{E} = \tilde{x}^6 + \frac{1}{4g_1^2}\tilde{x}^4 + 4\left(-g_1^2 + \frac{1}{4g_1^2}\right)^2 \tilde{x}^2 + g_1^2 - \frac{1}{g_1}\tilde{x}^5 - 4\left(-g_1^2 + \frac{1}{4g_1^2}\right)\tilde{x}^4 -$$

$$-2g_1\tilde{x}^3 + 2\left(-g_1 + \frac{1}{4g_1^3}\right)\tilde{x}^3 + \tilde{x}^2 + 4\left(-g_1^3 + \frac{1}{4g_1}\right)\tilde{x} - 3\tilde{x}^2 + \frac{1}{g_1}\tilde{x} + 2\left(-g_1^2 + \frac{1}{4g_1^2}\right) - 4\tilde{x}^2 - 8g_1^2 +$$

$$+\frac{1}{g_1^2} + \tilde{E} = \tilde{x}^6 - \frac{1}{g_1}\tilde{x}^5 + \left(\frac{1}{4g_1^2} - 4\left(-g_1^2 + \frac{1}{4g_1^2}\right)\right)\tilde{x}^4 + 2\left(-g_1 + \frac{1}{4g_1^3} - g_1\right)\tilde{x}^3 +$$

$$+\left(4\left(-g_1^2 + \frac{1}{4g_1^2}\right)^2 - 6\right)\tilde{x}^2 + \left(4\left(-g_1^3 + \frac{1}{4g_1}\right) + \frac{1}{g_1}\right)\tilde{x} + g_1^2 + 2\left(-g_1^2 + \frac{1}{4g_1^2}\right) - 8g_1^2 + \frac{1}{g_1^2} + \tilde{E} =$$

$$= \tilde{x}^6 - \frac{1}{g_1}\tilde{x}^5 + \left(\frac{1}{4g_1^2} + 4g_1^2 - \frac{1}{g_1^2}\right)\tilde{x}^4 + 2\left(-2g_1 + \frac{1}{4g_1^3}\right)\tilde{x}^3 + 2\left(2\left(g_1^4 + \frac{1}{16g_1^4} - \frac{1}{2}\right) - 3\right)\tilde{x}^2 +$$

$$+\left(-4g_1^3 + \frac{1}{g_1} + \frac{1}{g_1}\right)\tilde{x} + g_1^2 - 2g_1^2 + \frac{1}{2g_1^2} - 8g_1^2 + \frac{1}{g_1^2} + \tilde{E} =$$

$$= \tilde{x}^6 - \frac{1}{g_1}\tilde{x}^5 + \left(4g_1^2 - \frac{3}{4g_1^2}\right)\tilde{x}^4 + 2\left(-2g_1 + \frac{1}{4g_1^3}\right)\tilde{x}^3 + 2\left(2g_1^4 + \frac{1}{8g_1^4} - 4\right)\tilde{x}^2 + \left(-4g_1^3 + \frac{2}{g_1}\right)\tilde{x} -$$

$$-9g_1^2 + \frac{3}{2g_1^2} + \tilde{E}$$

That is, in the region $\tilde{x} > 0$,

$$\tilde{V}_+(\tilde{x};2,2) = \tilde{x}^6 - \frac{1}{g_1}\tilde{x}^5 + \left(4g_1^2 - \frac{3}{4g_1^2}\right)\tilde{x}^4 + 2\left(-2g_1 + \frac{1}{4g_1^3}\right)\tilde{x}^3 + 2\left(2g_1^4 + \frac{1}{8g_1^4} - 4\right)\tilde{x}^2 +$$

$$+\left(-4g_1^3 + \frac{2}{g_1}\right)\tilde{x} - 9g_1^2 + \frac{3}{2g_1^2} + \tilde{E}$$

Since the potential is symmetric,

$$\tilde{V}(\tilde{x};2,2) = \tilde{x}^6 - \frac{1}{g_1}|\tilde{x}|^5 + \left(4g_1^2 - \frac{3}{4g_1^2}\right)\tilde{x}^4 + 2\left(-2g_1 + \frac{1}{4g_1^3}\right)|\tilde{x}|^3 + 2\left(2g_1^4 + \frac{1}{8g_1^4} - 4\right)\tilde{x}^2 +$$

$$+\left(-4g_1^3 + \frac{2}{g_1}\right)|\tilde{x}| - 9g_1^2 + \frac{3}{2g_1^2} + \tilde{E} \quad (73)$$

The potential is continuous at zero, and thus (73) is defined for every $\tilde{x}$.
Applying the condition $\tilde{V}(0;2,2) = 0$, we obtain the energy, which is

$\tilde{E} = 3\left(3g_1^{\ 2} - \frac{1}{2g_1^{\ 2}}\right)$ (74)

and we end up to the symmetrized sextic oscillator

$$\tilde{V}(\tilde{x};2,2) = \tilde{x}^6 - \frac{1}{g_1}|\tilde{x}|^5 + \left(4g_1^{\ 2} - \frac{3}{4g_1^{\ 2}}\right)\tilde{x}^4 + 2\left(-2g_1 + \frac{1}{4g_1^{\ 3}}\right)|\tilde{x}|^3 + 2\left(2g_1^{\ 4} + \frac{1}{8g_1^{\ 4}} - 4\right)\tilde{x}^2 + \left(-4g_1^{\ 3} + \frac{2}{g_1}\right)|\tilde{x}| \quad (75)$$

The coefficient $-\frac{1}{g_1}$ of the non-analytic term $|\tilde{x}|^5$ is non-zero, and thus the sextic oscillator (75) is non-analytic.

The non-analytic sextic oscillator (75) has an eigenstate described by the closed-form wave function (70), with energy given by (74).

If $g_1 < 0$, the eigenstate is the first-excited state, while if $g_1 > 0$, the eigenstate is the third-excited state.

The energy (74) is of even parity in g_1, and thus it is the same in both cases, i.e. the energy of the first-excited state of the oscillator (75) with $g_1 < 0$ is the same as the energy of the third-excited state of the oscillator (75) with $g_1 > 0$.

The wave function (70) is of odd parity and since it corresponds to even n ($n = 2$), it is of unnatural (or opposite) parity.

As in the respective $n = 1$ case, the coefficients of the analytic terms of the oscillator (75) are of even parity in g_1, while the coefficients of the non-analytic terms are of odd parity in g_1.

Summary of the case n=2 with analytic quotient polynomial and comparison with the cases n=1 and n=0

Summarizing the case $n = 2$ with analytic quotient polynomial, we have

i. If $g_3 = 0$, then g_1 is also zero, and we obtain the known analytic sextic oscillator **[4]**, with $g_4^{\ 2} = 1$.

ii. If $g_3 \neq 0$, we have the subcase where the polynomial $p_2(\tilde{x})$, and thus the eigenfunction too, are of natural, i.e. of even, parity, and the subcase where the polynomial $p_2(\tilde{x})$ and the eigenfunction are of unnatural, i.e. of odd, parity. In both subcases, g_1 is non-zero and, also, the resulting sextic oscillators are non-analytic.

In the natural-parity subcase, the parameter g_3 satisfies the cubic equation (65), which, as an odd-degree equation, has at least one real root, while in the unnatural-parity subcase, the parameter g_3 is given by the relation $g_3 = \frac{1}{2g_1}$.

Paying our attention to non-analytic sextic oscillators resulting from analytic quotient polynomials in the case $n = 2$, we see that the closed-form eigenfunction can be of either natural, i.e. even, or unnatural, i.e. odd, parity, depending on whether the parameter g_3 satisfies the cubic equation (65) or the relation $g_3 = \frac{1}{2g_1}$.

In the case $n=1$, when a non-analytic sextic oscillator results from an analytic quotient polynomial, the closed-form eigenfunction is always of unnatural, i.e. of even, parity, and a similar relation holds, i.e. $g_3 = \frac{1}{g_1}$.

In the case $n=0$, the quotient polynomial is zero, and thus it is (trivially) analytic, the resulting sextic oscillator is non-analytic when (and only when) $g_3 \neq 0$, as seen from (23), but the closed-form eigenfunction is always of natural, i.e. of even, parity.

Unnatural-parity states in non-analytic sextic oscillators resulting from analytic quotient polynomials, for n=3 and n=4

The previous relations are generalized to the cases $n=3$ and $n=4$, in the sense that for $g_3 = \frac{1}{3g_1}$ ($n=3$) and $g_3 = \frac{1}{4g_1}$ ($n=4$), we can find non-analytic sextic oscillators resulting from analytic quotient polynomials, with the closed-form eigenfunction being of unnatural parity, i.e. of even ($n=3$) or odd ($n=4$) parity.

n=3

We'll search for unnatural (even) parity eigenfunctions of non-analytic sextic oscillators resulting from analytic quotient polynomials, when $g_3 = \frac{1}{3g_1}$, with $g_1 \neq 0$.

Since the polynomial $p_n(\tilde{x})$ and the ansatz eigenfunction (1) have the same parity, the polynomial $p_3(\tilde{x})$ is of even parity, and thus it has the form

$$p_3(\tilde{x}) = |\tilde{x}|^3 + p_2\tilde{x}^2 + p_1|\tilde{x}| + p_0 \quad (76)$$

The polynomial (76) is continuous at zero, and thus the first continuity condition (7) is satisfied.

Also, $p_3'(0^-) = -p_1$, $p_3'(0^+) = p_1$, and thus the second continuity condition (8) is written as

$$-p_1 = p_1 + 2g_1p_0$$

and thus

$$p_1 = -g_1p_0 \quad (77)$$

Using that $q_1(3)=0$, as we consider analytic quotient polynomials, and that $g_3 = \frac{1}{3g_1}$, the recursion relation (12) becomes, for $n=3$,

$$(k+2)(k+1)p_{k+2} = -2(k+1)g_1p_{k+1} - (q_0(3) + 2kg_2)p_k - \frac{2(k-1)}{3g_1}p_{k-1} + 2(k-5)p_{k-2} \quad (78)$$

with $k = 0,1,2,3,4$

For $k=0$, dropping p_{-1} and p_{-2}, and using (77), (78) gives

$$2p_2 = -2g_1(-g_1p_0) - q_0(3)p_0 = (2g_1^2 - q_0(3))p_0$$

That is

$$2p_2 = \left(2g_1^{\ 2} - q_0\left(3\right)\right)p_0$$

For $k=1$, dropping p_{-1}, using that $p_3=1$ and (77), (78) gives

$$6 = -4g_1p_2 - \left(q_0\left(3\right)+2g_2\right)\left(-g_1p_0\right)$$

and thus

$$-4g_1p_2 + g_1\left(q_0\left(3\right)+2g_2\right)p_0 = 6$$

For $k=2$, dropping p_4, using that $p_3=1$ and (77), (78) gives

$$0 = -6g_1 - \left(q_0\left(3\right)+4g_2\right)p_2 - \frac{2}{3g_1}\left(-g_1p_0\right) - 6p_0 = -6g_1 - \left(q_0\left(3\right)+4g_2\right)p_2 + \frac{2}{3}p_0 - 6p_0 =$$

$$= -6g_1 - \left(q_0\left(3\right)+4g_2\right)p_2 - \frac{16}{3}p_0$$

That is

$$6g_1 + \left(q_0\left(3\right)+4g_2\right)p_2 + \frac{16}{3}p_0 = 0$$

For $k=3$, dropping p_4 and p_5, using that $p_3=1$ and (77), (78) gives

$$0 = -\left(q_0\left(3\right)+6g_2\right) - \frac{4}{3g_1}p_2 - 4\left(-g_1p_0\right)$$

and thus

$$-\left(q_0\left(3\right)+6g_2\right) - \frac{4}{3g_1}p_2 + 4g_1p_0 = 0$$

For $k=4$, dropping p_4, p_5, and p_6, and using that $p_3=1$, (78) gives

$$0 = -\frac{6}{3g_1} - 2p_2$$

and thus

$$p_2 = -\frac{1}{g_1} \quad (79)$$

We see that p_2 is of odd parity in g_1.
Substituting (79) into the equations for $k=0,1,2,3$, we obtain

$$-\frac{2}{g_1} = \left(2g_1^{\ 2} - q_0\left(3\right)\right)p_0 \quad (80)$$

$$g_1\left(q_0\left(3\right)+2g_2\right)p_0 = 2 \quad (81)$$

$$6g_1 - \frac{q_0\left(3\right)+4g_2}{g_1} + \frac{16}{3}p_0 = 0 \quad (82)$$

$$-\left(q_0\left(3\right)+6g_2\right)+\frac{4}{3g_1^{\ 2}}+4g_1p_0=0 \quad (83)$$

We ended up to a non-linear, non-homogeneous system of 4 equations with 4 unknowns, which are g_1, g_2, p_0, and $q_0\left(3\right)$.

From (80) and (81), we see that $p_0 \neq 0$, otherwise both equations are impossible.

Thus, from (80) we obtain

$$q_0\left(3\right)-2g_1^{\ 2}=\frac{2}{g_1p_0} \quad (84)$$

Also, from (81) we obtain, g_1 is also non-zero,

$$q_0\left(3\right)+2g_2=\frac{2}{g_1p_0} \quad (85)$$

Comparing (84) and (85), we obtain

$$q_0\left(3\right)-2g_1^{\ 2}=q_0\left(3\right)+2g_2$$

and thus

$$g_2=-g_1^{\ 2} \quad (86)$$

We see that g_2 is of even parity in g_1, while g_3 is of odd parity in g_1.

Substituting (86) into (82) and (83), we obtain

$$6g_1-\frac{q_0\left(3\right)-4g_1^{\ 2}}{g_1}+\frac{16}{3}p_0=0 \quad (87)$$

$$-\left(q_0\left(3\right)-6g_1^{\ 2}\right)+\frac{4}{3g_1^{\ 2}}+4g_1p_0=0 \quad (88)$$

Solving (87) for p_0 yields

$$\frac{16}{3}p_0=\frac{q_0\left(3\right)-4g_1^{\ 2}}{g_1}-6g_1 \Rightarrow p_0=\frac{3\left(q_0\left(3\right)-4g_1^{\ 2}\right)}{16g_1}-\frac{18g_1}{16}$$

Thus

$$p_0=\frac{3\left(q_0\left(3\right)-4g_1^{\ 2}\right)}{16g_1}-\frac{9g_1}{8} \quad (89)$$

Substituting (89) into (88) yields

$$-\left(q_0\left(3\right)-6g_1^{\ 2}\right)+\frac{4}{3g_1^{\ 2}}+4g_1\left(\frac{3\left(q_0\left(3\right)-4g_1^{\ 2}\right)}{16g_1}-\frac{9g_1}{8}\right)=0 \Rightarrow$$

$$\Rightarrow -q_0\left(3\right)+6g_1^{\ 2}+\frac{4}{3g_1^{\ 2}}+\frac{3\left(q_0\left(3\right)-4g_1^{\ 2}\right)}{4}-\frac{9g_1^{\ 2}}{2}=0 \Rightarrow$$

$$\Rightarrow -q_0(3)+6g_1^2+\frac{4}{3g_1^2}+\frac{3q_0(3)}{4}-3g_1^2-\frac{9g_1^2}{2}=0\Rightarrow$$

$$\Rightarrow -\frac{q_0(3)}{4}-\frac{3g_1^2}{2}+\frac{4}{3g_1^2}=0\Rightarrow q_0(3)+6g_1^2-\frac{16}{3g_1^2}=0$$

Thus

$$q_0(3)=-6g_1^2+\frac{16}{3g_1^2} \quad (90)$$

$q_0(3)$ is of even parity in g_1.
Substituting (90) into (89) yields

$$p_0=\frac{3\left(-6g_1^2+\frac{16}{3g_1^2}-4g_1^2\right)}{16g_1}-\frac{9g_1}{8}=\frac{-18g_1^2+\frac{16}{g_1^2}-12g_1^2}{16g_1}-\frac{9g_1}{8}=$$

$$=-\frac{30g_1}{16}+\frac{1}{g_1^3}-\frac{9g_1}{8}=\frac{1}{g_1^3}-\frac{15g_1}{8}-\frac{9g_1}{8}=\frac{1}{g_1^3}-3g_1$$

That is

$$p_0=\frac{1}{g_1^3}-3g_1 \quad (91)$$

p_0 is of odd parity in g_1.
Substituting (91) into (77) yields

$$p_1=3g_1^2-\frac{1}{g_1^2} \quad (92)$$

p_1 is of even parity in g_1.
Through the equations (86),(90),(91),(92), and (79), we've expressed $g_2, q_0(3), p_0, p_1$, and p_2 in terms of g_1, and we have yet to use one of (80) or (81), to find g_1.
Thus, substituting (86),(90), and (91) into (81), we obtain

$$g_1\left(-6g_1^2+\frac{16}{3g_1^2}-2g_1^2\right)\left(\frac{1}{g_1^3}-3g_1\right)=2\Rightarrow\left(-6g_1^2+\frac{16}{3g_1^2}-2g_1^2\right)\left(\frac{1}{g_1^2}-3g_1^2\right)=2\Rightarrow$$

$$\Rightarrow -6+18g_1^4+\frac{16}{3g_1^4}-16-2+6g_1^4=2\Rightarrow 24g_1^4-26+\frac{16}{3g_1^4}=0\Rightarrow 12g_1^4-13+\frac{8}{3g_1^4}=0$$

Thus

$$36g_1^8-39g_1^4+8=0$$

Setting $g_1^4=y\geq 0$, the previous equation becomes

$$36y^2-39y+8=0$$

The discriminant of the trinomial in the left-hand side of the last equation is

$$(-39)^2 - 36*32 = 1521 - 1152 = 369 > 0$$

Thus

$$y_{1,2} = \frac{39 \pm \sqrt{369}}{72} \simeq \frac{39 \pm 19.21}{72} \simeq 0.81,\ 0.27$$

Both roots are positive, and thus accepted, and then we have

$$g_1^{\ 4} = y_{1,2} \Rightarrow g_1^{\ 2} = \sqrt{y_{1,2}} \Rightarrow g_1 = \pm\sqrt[4]{y_{1,2}}$$

Therefore, we obtain four accepted values of g_1, two positive values, which are $\sqrt[4]{y_{1,2}}$, and the opposite values $-\sqrt[4]{y_{1,2}}$, with $y_{1,2} \simeq 0.81,\ 0.27$

Besides, using (79),(91), and (92), the polynomial $p_3(\tilde{x})$ is written as

$$p_3(\tilde{x}) = |\tilde{x}|^3 - \frac{1}{g_1}\tilde{x}^2 + \left(3g_1^{\ 2} - \frac{1}{g_1^{\ 2}}\right)|\tilde{x}| + \frac{1}{g_1^{\ 3}} - 3g_1 = \tilde{x}^2\left(|\tilde{x}| - \frac{1}{g_1}\right) + \left(3g_1^{\ 2} - \frac{1}{g_1^{\ 2}}\right)|\tilde{x}| - \left(3g_1 - \frac{1}{g_1^{\ 3}}\right) =$$

$$= \tilde{x}^2\left(|\tilde{x}| - \frac{1}{g_1}\right) + \left(3g_1^{\ 2} - \frac{1}{g_1^{\ 2}}\right)|\tilde{x}| - \frac{1}{g_1}\left(3g_1^{\ 2} - \frac{1}{g_1^{\ 2}}\right) = \tilde{x}^2\left(|\tilde{x}| - \frac{1}{g_1}\right) + \left(3g_1^{\ 2} - \frac{1}{g_1^{\ 2}}\right)\left(|\tilde{x}| - \frac{1}{g_1}\right) =$$

$$= \left(\tilde{x}^2 + 3g_1^{\ 2} - \frac{1}{g_1^{\ 2}}\right)\left(|\tilde{x}| - \frac{1}{g_1}\right)$$

That is

$$p_3(\tilde{x}) = \left(\tilde{x}^2 + \frac{3g_1^{\ 4} - 1}{g_1^{\ 2}}\right)\left(|\tilde{x}| - \frac{1}{g_1}\right) \quad (93)$$

Since $g_1^{\ 4} = y_{1,2} \simeq 0.81,\ 0.27$, the expression $3g_1^{\ 4} - 1$ is positive for y_1 and negative for y_2.

Besides, using that $g_3 = \frac{1}{3g_1}$ and (86), the exponential polynomial (2) is written as

$$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 + \frac{1}{9g_1}|\tilde{x}|^3 - \frac{g_1^{\ 2}}{2}\tilde{x}^2 + g_1|\tilde{x}| \quad (94)$$

As in the respective $n=1$ and $n=2$ cases, the coefficients of the analytic terms of the exponential polynomial are of even parity in g_1, while the coefficients of its non-analytic terms are of odd parity in g_1.

By means of (93) and (94), the ansatz eigenfunction (1) is written as

$$\psi(\tilde{x};2,3) = A_3\left(\tilde{x}^2 + \frac{3g_1^{\ 4} - 1}{g_1^{\ 2}}\right)\left(|\tilde{x}| - \frac{1}{g_1}\right)\exp\left(-\frac{1}{4}\tilde{x}^4 + \frac{1}{9g_1}|\tilde{x}|^3 - \frac{g_1^{\ 2}}{2}\tilde{x}^2 + g_1|\tilde{x}|\right) \quad (95)$$

The (real) zeros of the wave function (95) are the (real) zeros of the polynomial $p_3(\tilde{x})$.

Then, we have

i. $g_1 = \sqrt[4]{y_1}$

Since $g_1^{\ 4} = y_1$, $3g_1^{\ 4} - 1 > 0$, and thus $\tilde{x}^2 + \frac{3g_1^{\ 4} - 1}{g_1^{\ 2}} > 0$.

Also, since $g_1 > 0$, the equation $|\tilde{x}| - \frac{1}{g_1} = 0$ has two roots, at $\tilde{x} = \pm\frac{1}{g_1}$.

In this case, the polynomial $p_3(\tilde{x})$ has two zeros, at $\pm\frac{1}{g_1}$, and then the wave function (95) is the second-excited-state wave function of the oscillator we calculate below.

ii. $g_1 = -\sqrt[4]{y_1}$

Since $g_1^{\ 4} = y_1$, $3g_1^{\ 4} - 1 > 0$, and thus $\tilde{x}^2 + \frac{3g_1^{\ 4} - 1}{g_1^{\ 2}} > 0$.

Also, since $g_1 < 0$, $|\tilde{x}| - \frac{1}{g_1} > 0$.

In this case, the polynomial $p_3(\tilde{x})$ has no zeros, and then the wave function (95) is the ground-state wave function of the oscillator we calculate below.

iii. $g_1 = \sqrt[4]{y_2}$

Since $g_1^{\ 4} = y_2$, $3g_1^{\ 4} - 1 < 0$, and thus the equation $\tilde{x}^2 + \frac{3g_1^{\ 4} - 1}{g_1^{\ 2}} = 0$ has two roots, at

$$\pm\frac{\sqrt{1-3g_1^{\ 4}}}{|g_1|} \underbrace{=}_{g_1>0} \pm\frac{\sqrt{1-3g_1^{\ 4}}}{g_1}.$$

Also, since $g_1 > 0$, the equation $|\tilde{x}| - \frac{1}{g_1} = 0$ has also two roots, at $\tilde{x} = \pm\frac{1}{g_1}$, which are different from $\pm\frac{\sqrt{1-3g_1^{\ 4}}}{g_1}$.

In this case, the polynomial $p_3(\tilde{x})$ has four zeros, and then the wave function (95) is the fourth-excited-state wave function of the oscillator we calculate below.

iv. $g_1 = -\sqrt[4]{y_2}$

Since $g_1^{\ 4} = y_2$, $3g_1^{\ 4} - 1 < 0$, and thus the equation $\tilde{x}^2 + \frac{3g_1^{\ 4} - 1}{g_1^{\ 2}} = 0$ has two roots, at

$$\pm\frac{\sqrt{1-3g_1^{\ 4}}}{|g_1|} \underbrace{=}_{g_1<0} \pm\frac{\sqrt{1-3g_1^{\ 4}}}{-g_1} = \mp\frac{\sqrt{1-3g_1^{\ 4}}}{g_1}.$$

Also, since $g_1 < 0$, $|\tilde{x}| - \frac{1}{g_1} > 0$.

In this case, the polynomial $p_3(\tilde{x})$ has two zeros, and then the wave function (95) is the second-excited-state wave function of the oscillator we calculate below.

Let us now calculate the symmetrized sextic oscillator.

For $n = 3$, $q_2(3) = 6$, as derived from (10), and then, using also (90) and that $q_1(3) = 0$, the quotient polynomial in this case is

$q_2(\tilde{x};3) = 6\tilde{x}^2 - 6g_1^{\,2} + \frac{16}{3g_1^{\,2}}$ (96)

Besides, using (94), we have, in the region $\tilde{x} > 0$,

$$g_4'(\tilde{x}) = -\tilde{x}^3 + \frac{1}{3g_1}\tilde{x}^2 - g_1^{\,2}\tilde{x} + g_1$$

$$g_4''(\tilde{x}) = -3\tilde{x}^2 + \frac{2}{3g_1}\tilde{x} - g_1^{\,2}$$

Plugging into (3) – for $(m,n)=(2,3)$ – the previous derivatives and the quotient polynomial (96), we obtain that, in the region $\tilde{x} > 0$,

$$\tilde{V}_+(\tilde{x};2,3) = \left(-\tilde{x}^3 + \frac{1}{3g_1}\tilde{x}^2 - g_1^{\,2}\tilde{x} + g_1\right)^2 - 3\tilde{x}^2 + \frac{2}{3g_1}\tilde{x} - g_1^{\,2} - \left(6\tilde{x}^2 - 6g_1^{\,2} + \frac{16}{3g_1^{\,2}}\right) + \tilde{E} =$$

$$= \tilde{x}^6 + \frac{1}{9g_1^{\,2}}\tilde{x}^4 + g_1^{\,4}\tilde{x}^2 + g_1^{\,2} - \frac{2}{3g_1}\tilde{x}^5 + 2g_1^{\,2}\tilde{x}^4 - 2g_1\tilde{x}^3 - \frac{2g_1}{3}\tilde{x}^3 + \frac{2}{3}\tilde{x}^2 - 2g_1^{\,3}\tilde{x} -$$

$$-3\tilde{x}^2 + \frac{2}{3g_1}\tilde{x} - g_1^{\,2} - 6\tilde{x}^2 + 6g_1^{\,2} - \frac{16}{3g_1^{\,2}} + \tilde{E} =$$

$$= \tilde{x}^6 - \frac{2}{3g_1}\tilde{x}^5 + \left(2g_1^{\,2} + \frac{1}{9g_1^{\,2}}\right)\tilde{x}^4 - \frac{8g_1}{3}\tilde{x}^3 + \left(g_1^{\,4} + \frac{2}{3} - 9\right)\tilde{x}^2 + 2\left(\frac{1}{3g_1} - g_1^{\,3}\right)\tilde{x} + 6g_1^{\,2} - \frac{16}{3g_1^{\,2}} + \tilde{E}$$

That is, in the region $\tilde{x} > 0$,

$$\tilde{V}_+(\tilde{x};2,3) = \tilde{x}^6 - \frac{2}{3g_1}\tilde{x}^5 + \left(2g_1^{\,2} + \frac{1}{9g_1^{\,2}}\right)\tilde{x}^4 - \frac{8g_1}{3}\tilde{x}^3 + \left(g_1^{\,4} - \frac{25}{3}\right)\tilde{x}^2 + 2\left(\frac{1}{3g_1} - g_1^{\,3}\right)\tilde{x} + 6g_1^{\,2} - \frac{16}{3g_1^{\,2}} + \tilde{E}$$

Since the potential is symmetric,

$$\tilde{V}(\tilde{x};2,3) = \tilde{x}^6 - \frac{2}{3g_1}|\tilde{x}|^5 + \left(2g_1^{\,2} + \frac{1}{9g_1^{\,2}}\right)\tilde{x}^4 - \frac{8g_1}{3}|\tilde{x}|^3 + \left(g_1^{\,4} - \frac{25}{3}\right)\tilde{x}^2 + 2\left(\frac{1}{3g_1} - g_1^{\,3}\right)|\tilde{x}| +$$

$$+6g_1^{\,2} - \frac{16}{3g_1^{\,2}} + \tilde{E} \quad (97)$$

The potential is continuous at zero, and thus (97) is defined for every $\tilde{x}$.
Applying the condition $\tilde{V}(0;2,3) = 0$, we obtain the energy, which is

$$\tilde{E} = 2\left(\frac{8}{3g_1^{\,2}} - 3g_1^{\,2}\right) \quad (98)$$

and we end up to the non-analytic (symmetrized) sextic oscillator

$$\tilde{V}(\tilde{x};2,3) = \tilde{x}^6 - \frac{2}{3g_1}|\tilde{x}|^5 + \left(2g_1^{\,2} + \frac{1}{9g_1^{\,2}}\right)\tilde{x}^4 - \frac{8g_1}{3}|\tilde{x}|^3 + \left(g_1^{\,4} - \frac{25}{3}\right)\tilde{x}^2 + 2\left(\frac{1}{3g_1} - g_1^{\,3}\right)|\tilde{x}| \quad (99)$$

with $g_1 = \pm\sqrt[4]{y_{1,2}}$, $y_{1,2} \simeq 0.81,\ 0.27$

As in the respective $n=1$ and $n=2$ cases, the coefficients of the analytic terms of the oscillator are of even parity in g_1, while the coefficients of its non-analytic terms are of odd parity in g_1.

n=4

We'll search for unnatural (odd) parity eigenfunctions of non-analytic sextic oscillators resulting from analytic quotient polynomials, when $g_3=\dfrac{1}{4g_1}$, with $g_1\neq 0$.

Since the eigenfunction is of odd parity, the polynomial $p_4(\tilde{x})$ is also of odd parity, and then it has the form

$$p_4(\tilde{x})=\begin{cases}\tilde{x}^4+p_3\tilde{x}^3+p_2\tilde{x}^2+p_1\tilde{x},\ \tilde{x}>0\\ -\tilde{x}^4+p_3\tilde{x}^3-p_2\tilde{x}^2+p_1\tilde{x},\ \tilde{x}<0\end{cases},$$

which satisfies the first continuity condition, as $p_4(0^-)=0=p_4(0^+)$.

Since the polynomial $p_4(\tilde{x})$ is continuous at zero, we could include zero in its domain.

The first derivative of $p_4(\tilde{x})$ is

$$p_4'(\tilde{x})=\begin{cases}4\tilde{x}^3+3p_3\tilde{x}^2+2p_2\tilde{x}+p_1,\ \tilde{x}>0\\ -4\tilde{x}^3+3p_3\tilde{x}^2-2p_2\tilde{x}+p_1,\ \tilde{x}<0\end{cases},$$

and thus

$$p_4'(0^-)=p_1=p_4'(0^+),$$

i.e. the first derivative of $p_4(\tilde{x})$ is also continuous at zero, and thus we could also include zero in the domain of $p_4'(\tilde{x})$.

The second continuity condition is then written as

$$p_1=p_1+2g_1 0\Rightarrow 0=0,$$

i.e. it holds.

Using that $q_1(4)=0$, as we consider analytic quotient polynomials, and that $g_3=\dfrac{1}{4g_1}$, the recursion relation (12) becomes, for $n=4$,

$$(k+2)(k+1)p_{k+2}=-2(k+1)g_1p_{k+1}-\left(q_0(4)+2kg_2\right)p_k-\frac{(k-1)}{2g_1}p_{k-1}+2(k-6)p_{k-2} \quad (100)$$

with $k=0,1,2,3,4,5$

We note that the recursion relation holds in the region $\tilde{x}>0$.

For $k=0$, dropping p_{-1} and p_{-2}, and using that $p_0=0$, (100) gives

$$2p_2=-2g_1p_1$$

and thus

$p_2 = -g_1 p_1$ (101)

For $k=1$, dropping p_{-1} and using that $p_0 = 0$, (100) gives

$$6p_3 = -4g_1 p_2 - \left(q_0(4) + 2g_2\right) p_1$$

For $k=2$, using that $p_4 = 1$ and $p_0 = 0$, (100) gives

$$12 = -6g_1 p_3 - \left(q_0(4) + 4g_2\right) p_2 - \frac{1}{2g_1} p_1$$

For $k=3$, dropping p_5 and using that $p_4 = 1$, (100) gives

$$0 = -8g_1 - \left(q_0(4) + 6g_2\right) p_3 - \frac{1}{g_1} p_2 - 6p_1$$

For $k=4$, dropping p_5 and p_6, and using that $p_4 = 1$, (100) gives

$$0 = -\left(q_0(4) + 8g_2\right) - \frac{3}{2g_1} p_3 - 4p_2$$

For $k=5$, dropping p_5,p_6, and p_7, and using that $p_4 = 1$, (100) gives

$$0 = -\frac{2}{g_1} - 2p_3$$

and thus

$p_3 = -\frac{1}{g_1}$ (102)

We see that p_3 is of odd parity in g_1.
Substituting (101) and (102) into the equations for $k=1,2,3,4$, we obtain

$$-\frac{6}{g_1} = -4g_1\left(-g_1 p_1\right) - \left(q_0(4) + 2g_2\right) p_1 = \left(4g_1^2 - 2g_2 - q_0(4)\right) p_1$$

and thus

$\left(4g_1^2 - 2g_2 - q_0(4)\right) p_1 = -\frac{6}{g_1}$ (103)

And

$$12 = -6g_1\left(-\frac{1}{g_1}\right) - \left(q_0(4) + 4g_2\right)\left(-g_1 p_1\right) - \frac{1}{2g_1} p_1 = 6 + \left(g_1\left(q_0(4) + 4g_2\right) - \frac{1}{2g_1}\right) p_1 =$$

$$= 6 + g_1\left(q_0(4) + 4g_2 - \frac{1}{2g_1^2}\right) p_1$$

and thus

$g_1\left(q_0(4) + 4g_2 - \frac{1}{2g_1^2}\right) p_1 = 6$ (104)

And

$$0=-8g_1-\left(q_0\left(4\right)+6g_2\right)\left(-\frac{1}{g_1}\right)-\frac{1}{g_1}\left(-g_1p_1\right)-6p_1=-8g_1+\frac{q_0\left(4\right)+6g_2}{g_1}+p_1-6p_1$$

and thus

$$-8g_1+\frac{q_0\left(4\right)+6g_2}{g_1}-5p_1=0 \;\; (105)$$

And

$$0=-\left(q_0\left(4\right)+8g_2\right)-\frac{3}{2g_1}\left(-\frac{1}{g_1}\right)-4\left(-g_1p_1\right)=-\left(q_0\left(4\right)+8g_2\right)+\frac{3}{2{g_1}^2}+4g_1p_1$$

and thus

$$-\left(q_0\left(4\right)+8g_2\right)+\frac{3}{2{g_1}^2}+4g_1p_1=0 \;\; (106)$$

As in the case $n=3$, we end up to a non-linear, non-homogeneous system of 4 equations with 4 unknowns, which now are g_1, g_2, p_1, and $q_0\left(4\right)$.

From (103) and (104), we see that $p_1\neq 0$, otherwise both equations are impossible. Thus, from (103) we obtain

$$q_0\left(4\right)-4{g_1}^2+2g_2=\frac{6}{g_1p_1} \;\; (107)$$

Also, from (104) we obtain, g_1 is also non-zero,

$$q_0\left(4\right)+4g_2-\frac{1}{2{g_1}^2}=\frac{6}{g_1p_1} \;\; (108)$$

Comparing (107) and (108), we obtain

$$q_0\left(4\right)-4{g_1}^2+2g_2=q_0\left(4\right)+4g_2-\frac{1}{2{g_1}^2}\Rightarrow -4{g_1}^2+\frac{1}{2{g_1}^2}=2g_2$$

Thus

$$g_2=-2{g_1}^2+\frac{1}{4{g_1}^2} \;\; (109)$$

As in the case $n=3$, g_2 is of even parity in g_1, while g_3 is of odd parity in g_1. Substituting (109) into (105), we obtain

$$-8g_1+\frac{q_0\left(4\right)+6\left(-2{g_1}^2+\frac{1}{4{g_1}^2}\right)}{g_1}-5p_1=0\Rightarrow -8g_1+\frac{q_0\left(4\right)}{g_1}-12g_1+\frac{6}{4{g_1}^3}-5p_1=0\Rightarrow$$

$$\Rightarrow\frac{q_0\left(4\right)}{g_1}-20g_1+\frac{3}{2{g_1}^3}-5p_1=0$$

Solving the last equation for p_1, we obtain

$$p_1 = \frac{q_0(4)}{5g_1} - 4g_1 + \frac{3}{10g_1^3} \quad (110)$$

Substituting (109) and (110) into (106), we obtain

$$-\left(q_0(4) + 8\left(-2g_1^2 + \frac{1}{4g_1^2}\right)\right) + \frac{3}{2g_1^2} + 4g_1\left(\frac{q_0(4)}{5g_1} - 4g_1 + \frac{3}{10g_1^3}\right) = 0 \Rightarrow$$

$$\Rightarrow -q_0(4) + 16g_1^2 - \frac{2}{g_1^2} + \frac{3}{2g_1^2} + \frac{4q_0(4)}{5} - 16g_1^2 + \frac{6}{5g_1^2} = 0 \Rightarrow$$

$$\Rightarrow -\frac{q_0(4)}{5} + \frac{-20+15+12}{10g_1^2} = 0 \Rightarrow -\frac{q_0(4)}{5} + \frac{7}{10g_1^2} = 0 \Rightarrow -q_0(4) + \frac{7}{2g_1^2} = 0$$

Thus

$$q_0(4) = \frac{7}{2g_1^2} \quad (111)$$

$q_0(4)$ is of even parity in g_1.

Substituting (111) into (110), we obtain

$$p_1 = \frac{\frac{7}{2g_1^2}}{5g_1} - 4g_1 + \frac{3}{10g_1^3} = \frac{7}{10g_1^3} - 4g_1 + \frac{3}{10g_1^3} = -4g_1 + \frac{1}{g_1^3}$$

Thus

$$p_1 = -4g_1 + \frac{1}{g_1^3} \quad (112)$$

p_1 is of odd parity in g_1.

Substituting (112) into (101), we obtain

$$p_2 = 4g_1^2 - \frac{1}{g_1^2} \quad (113)$$

p_2 is of even parity in g_1.

Through the equations (109),(111),(112),(113), and (102), we've expressed $g_2, q_0(4), p_1, p_2$, and p_3 in terms of g_1.

As in the case $n=3$, all parameters have definite parity in g_1, i.e. they are of either even or odd parity in g_1.

Similarly to the case $n=3$, to calculate g_1, we substitute the previous expressions into one of the equations (103) or (104).

Thus, substituting (109),(111), and (112) into (103) yields

$$\left(4g_1^2 - 2\left(-2g_1^2 + \frac{1}{4g_1^2}\right) - \frac{7}{2g_1^2}\right)\left(-4g_1 + \frac{1}{g_1^3}\right) = -\frac{6}{g_1} \Rightarrow$$

$$\Rightarrow \left(4g_1^2 + 4g_1^2 - \frac{1}{2g_1^2} - \frac{7}{2g_1^2}\right)\left(-4g_1 + \frac{1}{g_1^3}\right) = -\frac{6}{g_1} \Rightarrow$$

$$\Rightarrow\left(8g_1^2-\frac{4}{g_1^2}\right)\left(-4g_1+\frac{1}{g_1^3}\right)=-\frac{6}{g_1}\Rightarrow\left(4g_1^2-\frac{2}{g_1^2}\right)\left(-4g_1+\frac{1}{g_1^3}\right)=-\frac{3}{g_1}\Rightarrow$$

$$\Rightarrow -16g_1^3+\frac{4}{g_1}+\frac{8}{g_1}-\frac{2}{g_1^5}=-\frac{3}{g_1}\Rightarrow -16g_1^3+\frac{15}{g_1}-\frac{2}{g_1^5}=0\Rightarrow -16g_1^8+15g_1^4-2=0$$

Thus, g_1 satisfies the equation

$$16g_1^8-15g_1^4+2=0$$

Setting $g_1^4=y\geq 0$, the previous equation becomes

$$16y^2-15y+2=0$$

The discriminant of the trinomial in the left-hand side of the last equation is

$$(-15)^2-16*8=225-128=97$$

Thus

$$y_{1,2}=\frac{15\pm\sqrt{97}}{32}\simeq\frac{15\pm 9.85}{32}\simeq 0.78,\ 0.16$$

Both roots are positive, and thus accepted, and then we have

$$g_1^4=y_{1,2}\Rightarrow g_1^2=\sqrt{y_{1,2}}\Rightarrow g_1=\pm\sqrt[4]{y_{1,2}}$$

Therefore, we obtain four accepted values of g_1, two positive values, which are $\sqrt[4]{y_{1,2}}$, and their opposite values, which are $-\sqrt[4]{y_{1,2}}$, with $y_{1,2}\simeq 0.78,\ 0.16$

Besides, using the sign function, we write the polynomial $p_4(\tilde{x})$ as

$$p_4(\tilde{x})=\operatorname{sgn}(\tilde{x})\tilde{x}^4+p_3\tilde{x}^3+\operatorname{sgn}(\tilde{x})p_2\tilde{x}^2+p_1\tilde{x}$$

Substituting the expressions of p_1, p_2, and p_3, from (112),(113), and (102), $p_4(\tilde{x})$ is written as

$$p_4(\tilde{x})=\operatorname{sgn}(\tilde{x})\tilde{x}^4-\frac{1}{g_1}\tilde{x}^3+\operatorname{sgn}(\tilde{x})\left(4g_1^2-\frac{1}{g_1^2}\right)\tilde{x}^2+\left(-4g_1+\frac{1}{g_1^3}\right)\tilde{x}=$$

$$=\tilde{x}^3\left(\operatorname{sgn}(\tilde{x})\tilde{x}-\frac{1}{g_1}\right)+\operatorname{sgn}(\tilde{x})\left(4g_1^2-\frac{1}{g_1^2}\right)\tilde{x}^2-\left(4g_1-\frac{1}{g_1^3}\right)\tilde{x}=$$

$$=\tilde{x}^3\left(\operatorname{sgn}(\tilde{x})\tilde{x}-\frac{1}{g_1}\right)+\operatorname{sgn}(\tilde{x})\left(4g_1^2-\frac{1}{g_1^2}\right)\tilde{x}^2-\frac{1}{g_1}\left(4g_1^2-\frac{1}{g_1^2}\right)\tilde{x}=$$

$$=\tilde{x}^3\left(\operatorname{sgn}(\tilde{x})\tilde{x}-\frac{1}{g_1}\right)+\left(4g_1^2-\frac{1}{g_1^2}\right)\tilde{x}\left(\operatorname{sgn}(\tilde{x})\tilde{x}-\frac{1}{g_1}\right)=$$

$$=\left(\tilde{x}^3+\left(4g_1^2-\frac{1}{g_1^2}\right)\tilde{x}\right)\left(\operatorname{sgn}(\tilde{x})\tilde{x}-\frac{1}{g_1}\right)=\tilde{x}\left(\tilde{x}^2+4g_1^2-\frac{1}{g_1^2}\right)\left(\operatorname{sgn}(\tilde{x})\tilde{x}-\frac{1}{g_1}\right)=$$

$$=\tilde{x}\left(\tilde{x}^2+\frac{4g_1^4-1}{g_1^2}\right)\left(\operatorname{sgn}(\tilde{x})\tilde{x}-\frac{1}{g_1}\right)$$

Using that $\operatorname{sgn}(\tilde{x})\tilde{x} = |\tilde{x}|$, we end up to

$$p_4(\tilde{x}) = \tilde{x}\left(\tilde{x}^2 + \frac{4g_1^{\ 4}-1}{g_1^{\ 2}}\right)\left(|\tilde{x}| - \frac{1}{g_1}\right) \quad (114)$$

Observe the similarity of $p_4(\tilde{x})$ to $p_3(\tilde{x})$, which is given by (93), i.e.

$$p_3(\tilde{x}) = \left(\tilde{x}^2 + \frac{3g_1^{\ 4}-1}{g_1^{\ 2}}\right)\left(|\tilde{x}| - \frac{1}{g_1}\right)$$

Since $g_1^{\ 4} = y_{1,2} \simeq 0.78,\ 0.16$, the expression $4g_1^{\ 4}-1$ is positive for y_1 and negative for y_2.

Besides, using (109) and that $g_3 = \dfrac{1}{4g_1}$, the exponential polynomial (2) is written as

$$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 + \frac{1}{12g_1}|\tilde{x}|^3 + \frac{-2g_1^{\ 2} + \dfrac{1}{4g_1^{\ 2}}}{2}\tilde{x}^2 + g_1|\tilde{x}| =$$

$$= -\frac{1}{4}\tilde{x}^4 + \frac{1}{12g_1}|\tilde{x}|^3 + \left(-g_1^{\ 2} + \frac{1}{8g_1^{\ 2}}\right)\tilde{x}^2 + g_1|\tilde{x}|$$

That is

$$g_4(\tilde{x}) = -\frac{1}{4}\tilde{x}^4 + \frac{1}{12g_1}|\tilde{x}|^3 + \left(-g_1^{\ 2} + \frac{1}{8g_1^{\ 2}}\right)\tilde{x}^2 + g_1|\tilde{x}| \quad (115)$$

Again, the coefficients of the analytic terms of the exponential polynomial are of even parity in g_1, while the coefficients of its non-analytic terms are of odd parity in g_1.
Using (114) and (115), the ansatz eigenfunction (1) is written as

$$\psi(\tilde{x};2,4) = A_4\tilde{x}\left(\tilde{x}^2 + \frac{4g_1^{\ 4}-1}{g_1^{\ 2}}\right)\left(|\tilde{x}| - \frac{1}{g_1}\right)\exp\left(-\frac{1}{4}\tilde{x}^4 + \frac{1}{12g_1}|\tilde{x}|^3 + \left(-g_1^{\ 2} + \frac{1}{8g_1^{\ 2}}\right)\tilde{x}^2 + g_1|\tilde{x}|\right) \quad (116)$$

The (real) zeros of the wave function (116) are the (real) zeros of the polynomial $p_4(\tilde{x})$.
Then, we have
i. $g_1 = \sqrt[4]{y_1}$

Since $g_1^{\ 4} = y_1$, $4g_1^{\ 4}-1>0$, and thus $\tilde{x}^2 + \dfrac{4g_1^{\ 4}-1}{g_1^{\ 2}} > 0$.

Also, since $g_1 > 0$, the equation $|\tilde{x}| - \dfrac{1}{g_1} = 0$ has two roots, at $\tilde{x} = \pm\dfrac{1}{g_1}$.

In this case, the polynomial $p_4(\tilde{x})$ has three zeros, at $\pm\frac{1}{g_1}$ and at zero, and then the wave function (116) is the third-excited-state wave function of the oscillator we calculate below.

ii. $g_1 = -\sqrt[4]{y_1}$

Since $g_1^4 = y_1$, $4g_1^4 - 1 > 0$, and thus $\tilde{x}^2 + \frac{4g_1^4 - 1}{g_1^2} > 0$.

Also, since $g_1 < 0$, $|\tilde{x}| - \frac{1}{g_1} > 0$.

In this case, the polynomial $p_4(\tilde{x})$ has one zero, at zero, and then the wave function (116) is the first-excited-state wave function of the oscillator we calculate below.

iii. $g_1 = \sqrt[4]{y_2}$

Since $g_1^4 = y_2$, $4g_1^4 - 1 < 0$, and thus the equation $\tilde{x}^2 + \frac{4g_1^4 - 1}{g_1^2} = 0$ has two roots, at $\pm\frac{\sqrt{1-4g_1^4}}{|g_1|} \underbrace{=}_{g_1>0} \pm\frac{\sqrt{1-4g_1^4}}{g_1}$.

Also, since $g_1 > 0$, the equation $|\tilde{x}| - \frac{1}{g_1} = 0$ has two roots, at $\tilde{x} = \pm\frac{1}{g_1}$, which are different from $\pm\frac{\sqrt{1-4g_1^4}}{g_1}$.

In this case, the polynomial $p_4(\tilde{x})$ has five zeros, and then the wave function (116) is the fifth-excited-state wave function of the oscillator we calculate below.

iv. $g_1 = -\sqrt[4]{y_2}$

Since $g_1^4 = y_2$, $4g_1^4 - 1 < 0$, and thus the equation $\tilde{x}^2 + \frac{4g_1^4 - 1}{g_1^2} = 0$ has two roots, at $\pm\frac{\sqrt{1-4g_1^4}}{|g_1|} \underbrace{=}_{g_1<0} \pm\frac{\sqrt{1-4g_1^4}}{-g_1} = \mp\frac{\sqrt{1-4g_1^4}}{g_1}$.

Also, since $g_1 < 0$, $|\tilde{x}| - \frac{1}{g_1} > 0$.

In this case, the polynomial $p_4(\tilde{x})$ has three zeros, and then the wave function (116) is the third-excited-state wave function of the oscillator we calculate below.

Let us now calculate the symmetrized sextic oscillator.

For $n = 4$, $q_2(4) = 8$, as derived from (10), and then, using also (111) and that $q_1(4) = 0$, the quotient polynomial in this case is

$$q_2(\tilde{x};4) = 8\tilde{x}^2 + \frac{7}{2g_1^2} \quad (117)$$

Using (115), we have, in the region $\tilde{x} > 0$,

$$g_4'(\tilde{x}) = -\tilde{x}^3 + \frac{1}{4g_1}\tilde{x}^2 + 2\left(-g_1^2 + \frac{1}{8g_1^2}\right)\tilde{x} + g_1$$

$$g_4''(\tilde{x}) = -3\tilde{x}^2 + \frac{1}{2g_1}\tilde{x} + 2\left(-g_1^2 + \frac{1}{8g_1^2}\right)$$

Plugging into (3) – for $(m,n)=(2,4)$ – the previous derivatives and the quotient polynomial (117), we obtain that, in the region $\tilde{x} > 0$,

$$\tilde{V}_+(\tilde{x};2,4) = \left(-\tilde{x}^3 + \frac{1}{4g_1}\tilde{x}^2 + 2\left(-g_1^2 + \frac{1}{8g_1^2}\right)\tilde{x} + g_1\right)^2 - 3\tilde{x}^2 + \frac{1}{2g_1}\tilde{x} + 2\left(-g_1^2 + \frac{1}{8g_1^2}\right) -$$

$$-\left(8\tilde{x}^2 + \frac{7}{2g_1^2}\right) + \tilde{E} = \tilde{x}^6 + \frac{1}{16g_1^2}\tilde{x}^4 + 4\left(-g_1^2 + \frac{1}{8g_1^2}\right)^2\tilde{x}^2 + g_1^2 - \frac{1}{2g_1}\tilde{x}^5 - 4\left(-g_1^2 + \frac{1}{8g_1^2}\right)\tilde{x}^4 - 2g_1\tilde{x}^3 +$$

$$+\left(-g_1 + \frac{1}{8g_1^3}\right)\tilde{x}^3 + \frac{1}{2}\tilde{x}^2 + 4\left(-g_1^3 + \frac{1}{8g_1}\right)\tilde{x} - 3\tilde{x}^2 + \frac{1}{2g_1}\tilde{x} + 2\left(-g_1^2 + \frac{1}{8g_1^2}\right) - 8\tilde{x}^2 - \frac{7}{2g_1^2} + \tilde{E} =$$

$$= \tilde{x}^6 - \frac{1}{2g_1}\tilde{x}^5 + \left(\frac{1}{16g_1^2} - 4\left(-g_1^2 + \frac{1}{8g_1^2}\right)\right)\tilde{x}^4 + \left(-2g_1 - g_1 + \frac{1}{8g_1^3}\right)\tilde{x}^3 +$$

$$+\left(4\left(-g_1^2 + \frac{1}{8g_1^2}\right)^2 + \frac{1}{2} - 3 - 8\right)\tilde{x}^2 + \left(4\left(-g_1^3 + \frac{1}{8g_1}\right) + \frac{1}{2g_1}\right)\tilde{x} + g_1^2 + 2\left(-g_1^2 + \frac{1}{8g_1^2}\right) - \frac{7}{2g_1^2} + \tilde{E} =$$

$$= \tilde{x}^6 - \frac{1}{2g_1}\tilde{x}^5 + \left(4g_1^2 + \frac{1}{16g_1^2} - \frac{1}{2g_1^2}\right)\tilde{x}^4 + \left(-3g_1 + \frac{1}{8g_1^3}\right)\tilde{x}^3 + \left(4g_1^4 + \frac{1}{16g_1^4} - 1 + \frac{1}{2} - 11\right)\tilde{x}^2 +$$

$$+\left(-4g_1^3 + \frac{1}{2g_1} + \frac{1}{2g_1}\right)\tilde{x} - g_1^2 + \frac{1}{4g_1^2} - \frac{7}{2g_1^2} + \tilde{E} =$$

$$= \tilde{x}^6 - \frac{1}{2g_1}\tilde{x}^5 + \left(4g_1^2 - \frac{7}{16g_1^2}\right)\tilde{x}^4 + \left(-3g_1 + \frac{1}{8g_1^3}\right)\tilde{x}^3 + \left(4g_1^4 + \frac{1}{16g_1^4} - \frac{23}{2}\right)\tilde{x}^2 + \left(-4g_1^3 + \frac{1}{g_1}\right)\tilde{x} -$$

$$-g_1^2 - \frac{13}{4g_1^2} + \tilde{E}$$

That is, in the region $\tilde{x} > 0$,

$$\tilde{V}_+(\tilde{x};2,4) = \tilde{x}^6 - \frac{1}{2g_1}\tilde{x}^5 + \left(4g_1^2 - \frac{7}{16g_1^2}\right)\tilde{x}^4 + \left(-3g_1 + \frac{1}{8g_1^3}\right)\tilde{x}^3 + \left(4g_1^4 + \frac{1}{16g_1^4} - \frac{23}{2}\right)\tilde{x}^2 +$$

$$+\left(-4g_1^3 + \frac{1}{g_1}\right)\tilde{x} - g_1^2 - \frac{13}{4g_1^2} + \tilde{E}$$

Since the potential is symmetric,

$$\tilde{V}(\tilde{x};2,4) = \tilde{x}^6 - \frac{1}{2g_1}|\tilde{x}|^5 + \left(4g_1^2 - \frac{7}{16g_1^2}\right)\tilde{x}^4 + \left(-3g_1 + \frac{1}{8g_1^3}\right)|\tilde{x}|^3 + \left(4g_1^4 + \frac{1}{16g_1^4} - \frac{23}{2}\right)\tilde{x}^2 +$$

$$+\left(-4g_1^3 + \frac{1}{g_1}\right)|\tilde{x}| - g_1^2 - \frac{13}{4g_1^2} + \tilde{E} \quad (118)$$

The potential is continuous at zero, and thus (118) is defined for every $\tilde{x}$.
Applying the condition $\tilde{V}(0;2,4)=0$, we obtain the energy, which is

$$\tilde{E}=g_1^{\ 2}+\frac{13}{4g_1^{\ 2}} \quad (119)$$

and we end up to the non-analytic (symmetrized) sextic oscillator

$$\tilde{V}(\tilde{x};2,4)=\tilde{x}^6-\frac{1}{2g_1}|\tilde{x}|^5+\left(4g_1^{\ 2}-\frac{7}{16g_1^{\ 2}}\right)\tilde{x}^4+\left(-3g_1+\frac{1}{8g_1^{\ 3}}\right)|\tilde{x}|^3+\left(4g_1^{\ 4}+\frac{1}{16g_1^{\ 4}}-\frac{23}{2}\right)\tilde{x}^2+$$
$$+\left(-4g_1^{\ 3}+\frac{1}{g_1}\right)|\tilde{x}| \quad (120)$$

with $g_1=\pm\sqrt[4]{y_{1,2}}$, $y_{1,2}\simeq 0.78,\ 0.16$
Again, the coefficients of the analytic terms of the oscillator are of even parity in g_1, while the coefficients of its non-analytic terms are of odd parity in g_1.

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