A Note on Vertex Transitivity in Isomorphic Graphs

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ABSTRACT

In the graph theory, two graphs are said to be isomorphic if there is a one-one, onto mapping defined between their set of vertices so as to preserving the adjacency between vertices. An isomorphism defined on a vertex set of a graph to itself is called automorphism of the given graph. Two vertices in a graph are said to be similar if there is an automorphism defined on its vertex set mapping one vertex to the other. In this paper, it has been discussed that every such automorphism defines an equivalence relation on the set of vertices and the number of equivalence classes is same as the number of rotations that the automorphism makes on the vertex set. The set of all automorphisms of a graph is a permutation group under the composition of permutations. This group is called automorphism group of the graph. A graph is said to be vertex transitive if its automorphism group acts transitively on its vertex set. The path degree sequence of a vertex in a graph is the list of lengths of paths having this particular vertex as initial vertex. The ordered set of all such sequences is called path degree sequence of the graph. It is conjectured that two graphs are isomorphic iff they have same path degree sequence. In this paper, it has been discussed that this conjecture holds good when both the graphs are vertex transitive. The notion of functional graph has been introduced in this paper. The functional graph of any two isomorphic graphs is a graph in which the vertex set is the union of vertex sets of isomorphic graphs and two vertices are connected by an edge iff they are connected in any one of the graph when they belong to the same graph or one vertex is the image of the other under the given isomorphism when they are in different set of vertices. It has been proved that the functional graphs obtained from two isomorphic complete bipartite graphs are vertex transitive.

Keywords : graph automorphism ; functional graph ; vertex transitive graph ; path degree sequence.

1. INTRODUCTION

A simple graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$ of edges together with an incidence function ψ_G that associates with each edge of G to an unordered pair of distinct vertices of G. Two graphs G and G' are said to be isomorphic if there are bijective mappings $\theta : V(G) \to V(G')$ and $\varphi : E(G) \to E(G')$ such that $\psi_G(e) = uv$ iff $\psi_{G'}(\varphi(e)) = \theta(u)\theta(v)$. In other words, if G and G' are isomorphic then there is a bijective map $f : V(G) \to V(G')$ such that $f(u)f(v)$ is an edge whenever uv is an edge. Suppose, if f is an isomorphism defined from $V(G)$ to $V(G')$ then the functional graph of G and G' with respect to f is a graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G') \cup E(f)$ where $E(f) = \{uv/f(u) = v\}$. We denote it as $G \cdot_f G'$.

It can be easily seen that the order of the functional graph is twice the order of the graph G and its size is the sum of twice the size of G and the order of G.An isomorphism from $V(G)$ to itself is called an automorphism defined on G.

2. REVIEW OF LITERATURE

The study of structural properties of the graphs is an important area of research in graph theory. Intuitively the structural properties of two graphs remain same if their adjacency matrices with vertices ordered so that the matrices are identical. The notion of isomorphism is another way of identifying structurally equivalent graphs. The degree sequences play a vital role in studying structural properties of graphs. The distance degree sequence has been studied with a primary objective of knowing whether a graph is determined by this sequence or not. Randic [7] conjectured that a tree is determined by its distance degree sequence, and Slater [8] disproved this conjecture in the year 1982. Some recent structural properties on planar graphs have been studied in [10]. Some problems concerning distance and path degree sequence can be seen in [3]. Path degree sequence is another tool to describe the structural properties of graphs. It is interesting to know the nature of path degree sequence corresponding to two isomorphic graphs. Randic[7] conjectured that the isomorphic graphs have the same path degree sequences. Later it was proved that this conjecture holds good only for a class of graphs. Graph automorphisms help us to study mainly the enumeration properties of graphs and also to study the structural properties of graphs. There is relatively natural interaction between structure of graphs and their path degree sequences. The automorphism group which is an algebraic invariant of graph defines an important class of graphs known as vertex transitive graphs. The vertex transitive graphs are regular, but have special properties which are not shared by regular graphs[5].

Babai[2] and Thomassen[9] have obtained structure theorems for connected vertex-transitive graphs with prescribed Hadwiger number. In this attempt, the role of automorphism group of graphs has been studied to find the nature of path degree sequence of two vertex transitive graphs.

3. GRAPH AUTOMORPHISMS

3.1 Definition: Two vertices u and v in a graph G are said to be similar if there is an automorphism mapping u to v on G.

3.2 Theorem

The relation $\sim$ defined on $V(G)$ of a graph G, such that $v_1 \sim v_2$ if and only if v_1 and v_2 are similar is an equivalence relation.

Proof

It can be easily seen that the identity map defined on $V(G)$ is always an automorphism. Also if f is an automorphism then f^{-1} is also an automorphism. Thus, these two properties give us that the relation $\sim$ is reflexive and symmetric. Moreover if f and g are automorphisms, then $f \cdot g$ is also an automorphism which provides us that $v_1 \sim v_3$ whenever $v_1 \sim v_2$ and $v_2 \sim v_3$. Thus $\sim$ is an equivalence relation.

2.3 Theorem

Every automorphism f on $V(G)$ defines an equivalence relation on $V(G)$, and the number of equivalence classes is same as the number of rotations that f makes on V(G).

Proof

First, we claim that e very automorphism decomposes the vertex set into disjoint subsets and rotates the vertices of each disjoint subset within themselves cyclically. For this, let f be an automorphism on $V(G) = \{v_1, v_2, \cdots v_n\}$. Let us assume that f moves v_i to v_{i+1}, v_{i+1} to v_{i+2}, $\cdots$. Since, there are only n vertices, there must be a vertex v_{i+r} , $1 \leq i + r \leq n$ such that $v_{i+r} = v_i$. Thus some part of the effect of f is equivalent to the cycle $f' = (v_i, v_{i+1}, \cdots, v_{i+r-1})$. If $i + r = n$, then all vertices have been accounted for and we have $f = f'$.

On the other hand , if $i + r < n$, let v_j be the vertex not contained in f' and suppose that f moves v_j to v_{j+1}, v_{j+1} to v_{j+2} and so on until we return to v_j, as we must do after at most $n - r$ steps. The image of v_{j+1} and its successors are certainly different from the $v_i's$ or else two vertices, one v_j and one v_i would have the same image. We have therefore isolated another cycle $f'' = (v_j, v_{j+1}, \cdots v_{j+s-1})$, for some s. If $r + s = n$, then the order in which f' and f'' are carried out is evidently irrelevant, as neither operation has an influence on the other.Hence $f = f' \cdot f'' = f'' \cdot f'$.

Suppose, if $r + s < n$, then the process may be continued and more cycles can be extracted from f until each of the n objects has been drawn into one of the cycles. Thus we get a decomposition of f as $f = (v_i, v_{i+1}, \cdots, v_{i+r-1}) \left(v_j, v_{j+1}, \cdots v_{j+s-1}\right) \cdots (v_k, v_{k+1}, \cdots v_{k+t-1})$ such that $r + s + \cdots + t = n$. This product into mutually exclusive cycles is unique, since the order in which they occur is immaterial. Thus we have proved our claim.

Next, we prove that f defines an equivalence relation $\sim$ on $V(G)$. From the representation given for f, we can write $f(v_i) = v_{i+1}, f(v_{i+1}) = v_{i+2}, \cdots, f(v_{i+r-2}) = v_{i+r-1}$ and $f(v_{i+r-1}) = v_i$. The reflexivity of the relation $\sim$ is taken care by the identity mapping defined on $V(G)$, irrespective of f defined on $V(G)$. Similarly from the inverse of f, we can prove the property of symmetry. Now we claim that the composition of f repeating required number of times provides the transitive property. In particular, it is enough if we claim that $v_{i+x} \sim v_{i+y}$, $v_{i+y} \sim v_{i+w}$ imply $v_{i+x} \sim v_{i+w}$. Note that if $f(v_i) = v_{i+1}$ and $f(v_{i+1}) = v_{i+2}$, then we can write it as $f^2(v_i) = f\big(f(v_i)\big) = f(v_{i+1}) = v_{i+2}$ i.e., f^2 is the automorphism mapping v_i to v_{i+2} when f maps v_i to v_{i+1} and v_{i+1} to v_{i+2}. Hence if $f^p(v_{i+x}) = v_{i+y}$ and $f^q\big(v_{i+y}\big) = v_{i+w}$ then $f^{p+q}(v_{i+x}) = v_{i+w}$. Therefore, it can be concluded that f defines an equivalence relation on $V(G)$. Finally, we see that the number of equivalence classes is the number of rotations that f makes on $V(G)$. Consider the decomposition of f interms of cycles as assumed above. To prove our claim we prove the following

i. $v_{i+x} \sim v_{i+y}$ for all x, y with $0 \le x, y \le r-1$

ii. $v_{i+w} \nsim v_{j+z}$ for all w, z with $0 \le w \le r-1$, $0 \le z \le s-1$

Suppose, if there are p intermediate vertices between v_{i+x} and v_{i+y} in the rotation corresponding to the cycle $(v_i , v_{i+1} ,\cdots, v_{i+r-1})$ then $f^{p+1}(v_{i+x}) = v_{i+y}$. Hence $v_{i+x} \sim v_{i+y}$. Secondly, suppose if $v_{i+w} \sim v_{j+z}$, for some z, then $v_{i+w} \sim v_{j+z}$ for all w with $0 \le w \le r-1$ then $v_{j+z} = v_{i+w}$ for some w. This is a contradiction as the cycles are mutually disjoint. This completes the proof of the theorem.

3.4 Note

The equivalence class corresponding to the automorphism f and vertex v is denoted as $E_f(v)$. Hence $O(G) = \sum_v \big|E_f(v)\big|$.

3.5 Theorem

If $E(v)$ is an equivalence class of an equivalence relation defined on the vertex set of a graph G under the relation similarity, then the corresponding automorphism rotates the vertices in $E(v)$ cyclically. Moreover if f rotates the vertices in clockwise direction, then f^{-1} does the same in anticlockwise direction.

Proof

Let G be a graph with $E(v) = \{v_1, v_2, \cdots, v_m\}$ for a $v \in V(G)$. Also let $|E(G)| = m$; $|V(G)| = n$. Let $v = v_1$ and consider the set $A = \{v_1, f(v_1), \cdots, f^{m-1}(v_1)\}$. Now we claim that $|A| = m$, i.e., the elements $v_1, f(v_1), f^2(v_1), \cdots f^{m-1}(v_1)$ are all distinct, where $f^2(v_1) = f(f(v_1))$; $f^3(v_1) = f(f\big(f(v_1)\big))$ etc.,

Suppose, let $f^i(v_1) = f^j(v_1)$, where $i \ne j$ and $1 < i,\ j < m-1$, and $f^{i-1}(v_1) \ne f^{j-1}(v_1)$. This would lead to conclude that two different elements $f^{i-1}(v_1)$ and $f^{j-1}(v_1)$ have the same image $f^i(v_1) = f^j(v_1)$. This contradicts to the fact that f is one-one. Hence $|A| = m$. Also $f^m(v_1) = v_1$. Hence f should rotate vertices from v_1 to $f^m(v_1)$ in one direction, and hence the f^{-1} rotates in the opposite direction from $f^{-1}\big(f^m(v_1)\big)$ to $f^{-1}(v_1)$. Hence the theorem.

3.6 Theorem

Let G be a graph with $|V(G)| = n$. Let G' be the graph obtained from G, by adding exactly one pendent vertex at each vertex of G. Then $|Aut(G')| = (n!)^2$

Proof

Let $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $(G') = V(G) \cup \{v_1, v_2, \cdots, v_n\}$. Note that G' has no vertex of degree one other than the vertices of $v_1, v_2, \cdots, v_n$. Also note that $u's$ should be mapped only with $u's$ and the mappings between $v's$ fix the mappings between $v's$. The $u's$ can be mapped in $n!$ ways and $v's$ can be mapped in $n!$ ways. Hence $|Aut(G')| = (n!)^2$.

3.7 Theorem

Let G be a graph and $\sim$ be an equivalence relation on V(G) with respect to similar vertices. If there exists an automorphism f such that $\left|\{E_f(u)/u \in V(G)\}\right| = n$ then G is regular.

Proof

It is necessary that $u \sim v$ iff $\deg(u) = \deg(v)$. Hence if $\left|\{E_f(u)\}\right| = n$, $u \sim v$ for all $v \in V(G)$, then $\deg(u) = \deg(v)$ for all $v \in V(G)$. Hence G is regular.

The above theorem can also be restated as follows.

3.8 Theorem

Let G be a graph with $|V(G)| = n$. If there exists an automorphism f of cycle n on $V(G)$, then G is regular.

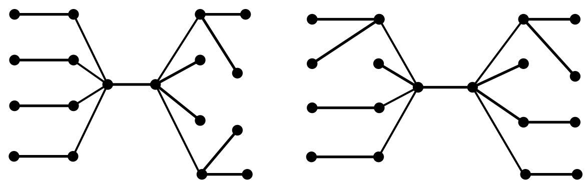
3.9 *Definitions*

In a graph G the distance degree sequence (dds) of a vertex v_i is the sequence $\left(d_{i_0}, d_{i_1}, d_{i_2}, \cdots, d_{i_j}, \cdots\right)$ where d_{i_j} denotes the number of vertices at distance *j* from v_i. The ordered set of all such sequences arranged in lexicographic order is called the distance degree sequence (DDS) of G. Similarly, the path degree sequence (pds) of v_i is the sequence $\left(p_{i_0}, p_{i_1}, p_{i_2}, \cdots, p_{i_j}, \cdots\right)$ where p_{i_j} denotes the number of paths in G of length *j* having v_i as the initial vertex. The ordered set of all such sequences arranged in lexicographic order is called the path degree sequence (PDS) of G. For any graph $d_{i_j} \leq p_{i_j}$. It is conjectured that 'Two graphs G_1 and G_2 are isomorphic if and only if $PDS(G_1) = PDS(G_2)$' [4]. The following theorem says that this conjecture holds good when G_1 and G_2 are vertex transitive graphs.

3.10 Theorem

Two vertex transitive graphs G' and G'' are isomorphic iff $PDS(G') = PDS(G'')$.

In general this theorem is not true for arbitrary graphs. The following is a pair of non-isomorphic trees with same PDS



Slater[8] conjectured that the above pair of non-isomorphic graphs is the smallest pair of such graphs. But Gargano and Quintas [6] proved that Slater's conjecture hold good only for a pair of graphs which have no independent cycles. They constructed a pair of such graphs on 14 vertices, each graph having exactly one independent cycle.

Proof of the above theorem

Let G' and G'' be vertex transitive graphs and $f: V(G') \to V(G'')$ be an isomorphism. Let x and y are any two vertices in G'. Let P be the smallest path connecting x and y and n be the distance between x and y. i.e., let P be the path containing the vertices $x, v_1, v_2, \cdots v_{n-1}, y$. Now consider the vertices $f(x), f(v_1), \cdots, f(v_{n-1}), f(y)$. By the definition of isomorphism, the adjacency between the vertices is preserved and therefore the sequence $f(x)$, $f(v_1)$, $\cdots, f(v_{n-1}), f(y)$ forms a path $f(P)$ from $f(x)$ to $f(y)$. We claim that the length of $f(P)$ is also n. Suppose, let the length of $f(P)$ be d, and $d < n$. Also let $f(P)$ is formed by the vertices $f(x), u_1, u_2, \cdots, u_{d-1}, f(y)$ in G''. Since f is onto, there are vertices $w_1, w_2, \cdots, w_{d-1}$ in G' such that $f(w_i) = u_i$, $i = 1,2, \cdots d-1$. Thus $x, w_1, w_2, \cdots, w_{n-1}, y$ define a path of length n in G' which contradicts our assumption that P is the smallest path connecting x and y. Thus $f(x), f(v_1), \cdots, f(v_{d-1}), f(y)$ is also the smallest path connecting $f(x)$ and $f(y)$. Hence $PDS(G') = PDS(G'')$.

Conversely assume that G' and G'' are vertex transitive graphs and $PDS(G') = PDS(G'') = \{A, B, C, D, \cdots\}$ where $A, B, C, D, \cdots$ are the path sequences of vertices.

Let $f : G' \to G''$ be defined such that

i. $f(u) = v$, if u and v have same path degree sequence
ii. For any two vertices u and v in G', $f(u) = x$; $f(v) = y$ iff
(a) $d(u, v) = d(x, y)$

(b) u and v are in same equivalence class in G' iff x and y are in same equivalence class in G''

From the above conditions, it can be seen that f is one- to –one and onto and the vertices adjacent to u are mapped to the vertices adjacent to x. Hence f is an isomorphism.

4. FUNCTIONAL GRAPHS AND AUTOMORPHISMS

4.1 Lemma

The functional graph obtained from the complete graph k_n with respect to an automorphism defined on it is an $(n-1)$ -semiregular graph. In general, it is an $(n, 2, n-1)$ - graph.

Proof

Let f be an automorphism defined from k_n to k_n with different vertex labelings $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$. Let $f(u_i) = v_i$. Then it can be seen that $N(u_i) = \{u_1, u_2, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n, v_i\}$. Hence $k_n \ \cdot_f \ k_n$ is n-regular and the vertices which are at distance 2 from u_i are $v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n$.

Hence $k_n \ \cdot_f \ k_n$ is an $(n, 2, n-1)$ – graph

4.2 Theorem

Let G' and G'' are complete bipartite graphs and $f : G' \to G''$ is an isomorphism. Then $G' \cdot_f G''$ is vertex transitive.

Proof

Assume that G' and G'' are complete bipartite graphs, $f : G' \to G''$ is an isomorphism. Let f'_{uv} is the automorphism on G' mapping u to v, then f'_{uv} itself defines an equivalence relation on $V(G')$ and the number of equivalence classes is the number of rotations that f'_{uv} makes on $V(G')$. Hence f'_{uv} can be written as a product of disjoint cycles uniquely. Suppose, if f'_{uv} makes m rotations on $V(G')$, then we have

$$f'_{uv} = (u_{11}, u_{12}, \cdots, u_{1m_1})(u_{21}, u_{22}, \cdots, u_{2m_2}) \cdots (u_{m1}, u_{m2}, \cdots, u_{mm_r})$$

$$= C_1 C_2 \cdots C_m$$

where $C_i = (u_{i1}, u_{i2}, \cdots, u_{ir_i})$

Since f is an isomorphism from G' to G'', $f''_{f(u)f(v)}$ mapping $f(u)$ to $f(v)$ in G'' is an isomorphism in G'' and it can be written as

$$f''_{f(u)f(v)} = C'_1 C'_2 \cdots C'_m$$

where $C'_i = f(c_i) = (f(u_{i1}), f(u_{i2}), \cdots, f(u_{ir_i}))$

Also let $V(G') = A \cup B$ and $|V(G')| = n$ and $v(G'') = C \cup D$. Also let A$= \{u_1, u_2, \cdots, u_n\}$; B$= \{v_1, v_2, \cdots, v_n\}$; C$= \{x_1, x_2, \cdots, x_n\}$; D$= \{y_1, y_2, \cdots, y_n\}$

Note that any automorphism mapping the vertex $u \in A$ to the vertex v $\in B$ can be written as the product of $\frac{n}{2}$ cycles of length 2 and the automorphism mapping two vertices from A to A (or B to

B) can be written as the product cycles that vary from one to n. Among these automorphisms, consider the automorphism that rotates all the vertices of A cyclically to themselves i.e., consider the automorphism $f'_{u_i u_j} = C_1 C_2$, where $C_1 = (u_1, u_2, \cdots, u_n)$, $C_2 = (v_1, v_2, \cdots, v_n)$ Then $f''_{f(u_i)f(u_j)} = C'_1 C'_2$ where $C'_1 = (f(u_1),\ f(u_2),\ \cdots,f(u_n))$ and $C'_2 = (f(v_1), f(v_2), \cdots, f(v_n))$.

Now we define the automorphism g on $G' \cdot_f G''$ as follows

i. The automorphism that maps within cycles is given by $h_{uv}(w) = \begin{cases} f'_{uv}(w)\ ,\ if\ w \in V(G') \\ f''_{f(u)f(v)}\ ,if\ \ w \in V(G'') \end{cases}$

ii. The automorphism that maps $u \in A$ to $v \in B$ is given by $h_{uv}(w) = w'$ if $w \in V(G')$ and $d(u,w)$ in

C_1 is the same as $d(v,w')$ in C_2. Here, $d(u,w)$ in C_1 is the number of times f_{uv} is to be operated so that to get w in the image of u . Similarly for every $w \in C$, $h_{f(u)f(v)}(w) = f''_{f(u)f(v)}(w)$ is the required automorphism.

iii. The automorphism mapping a vertex $u \in A$ to $v \in C$ is given by $h_{uv}(w) = f''_{f(u)f(v)}(w)$, $if\ \ w \in V(G')$.

iv. The automorphism mapping a vertex $u \in A$ to $v \in C$ is given by $h_{uv}(w) = w'$ if $w \in C_1$ and $d(u,w)$ in C_1 is the same as $d(v,w')$ in C'_2.

Suppose, if $w \in C_2$ then $h_{uv}(w) = w'$ if $f'^{n}_{uv}(w) = f''^{n}_{f(u)f(v)}(w'')$

Now the mappings h and h^{-1} give the required automorphisms mapping any two pair vertices in $G' \cdot_f G''$. Hence $G' \cdot_f G''$ is vertex transitive.

5. CONCLUSION

The study of degree sequences and structured properties of certain classes of graphs is not new. In general, the sequence derived from any means in graph posses many structural properties. There are lot of works carried out to characterize graphs based on distance related sequences. Path degree sequence of a graph has its application in describing atomic environments and in various classification schemes for molecules. In this attempt the graph automorphisms and their role in obtaining the necessary and sufficient conditions for a certain class of graphs containing the same path degree sequence has been discussed. The construction of desired structures is an interesting problem in graph theory. The construction of a graph from a permutation group has been widely studied. Many of the sporadic simple graphs have been constructed as groups of automorphisms of particular graphs. In some cases the group and the graph are more closely related. In this attempt, the more structured class of graphs has been constructed from isomorphic graphs. In particular it has been discussed that the equivalence class obtained from the equivalence relation on similar vertices have played a major role in some of the graph-theoretic concepts. This study can also be extended to construct more class of distance degree regular graphs.

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