Higher Integrals

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abstract

In this note we explore an elliptic integral

1. Introduction

The elliptic integral of the first kind:

$$F(\varphi,k)=\int_0^{\varphi}\frac{1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta=\int_0^{\sin\varphi}\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx \quad ,0\le k\le 1 \tag{1}$$

Complete elliptic integral of the first kind:

$$K(k)=F\left(\frac{\pi}{2},k\right)=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) \quad ,0\le k\le 1 \tag{2}$$

Remark 1: ${}_2F_1(a,b;c;x)$ is the hypergeometric function.

Example:

$$I=\int_0^{\infty}\frac{1}{\sqrt{x(2x+1)(9x+4)}}\,dx=\frac{2}{3}K\left(\frac{1}{3}\right)=\frac{\pi}{3}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{9}\right) \tag{3}$$

Proof:

If $x=t^2, dx=2tdt$, we have

$$I=\frac{\sqrt{2}}{3}\int_0^{\infty}\frac{1}{\sqrt{\left(t^2+1/2\right)\left(t^2+4/9\right)}}\,dt \tag{4}$$

If $t=\frac{2}{3}\tan\theta, dt=\frac{2}{3}\sec^2\theta\,d\theta$, we have:

$$I=\frac{2}{3}\int_0^{\pi/2}\frac{1}{\sqrt{1-(1/9)\sin^2\theta}}\,d\theta=\frac{2}{3}K\left(\frac{1}{3}\right) \tag{5}$$

A related integral is

$$I=\frac{1}{6}\int_0^1\sqrt{\frac{\sqrt{288+x^2}-17x}{x}}\,dx \tag{6}$$

another integral is

$$I=\frac{\pi}{2\sqrt{2}}-2\int_0^1\frac{x\sin^{-1}x}{\left(9-x^2\right)^{3/2}}\,dx \tag{7}$$

2. Higher Integrals

For $u=\frac{1}{4}\sqrt{1190+146\sqrt{73}}$ we have

$$I=\int_0^u\left(\frac{R(x)^{1/3}}{54x}+\frac{73x}{54R(x)^{1/3}}-\frac{17}{54}\right)dx+\int_u^\infty\left(-\frac{17}{54}+\frac{\sqrt{73}}{27}\cos\left(\frac{1}{3}\cos^{-1}\left(Q(x)\right)\right)\right)dx \tag{8}$$

$$R(x)=4374x+595x^3+54x\sqrt{6561+1785x^2-12x^4} \tag{9}$$

$$Q(x)=\frac{\sqrt{73}}{5329}\left(595+\frac{4374}{x^2}\right) \tag{10}$$

3. Some relations

For $u=\frac{1}{4}\sqrt{1190+146\sqrt{73}}$, $v=\frac{\sqrt{73}}{27}-\frac{17}{54}$ we have

$$\int_0^v\frac{1}{\sqrt{x(2x+1)(9x+4)}}\,dx=u\,v+\int_u^\infty\left(-\frac{17}{54}+\frac{\sqrt{73}}{27}\cos\left(\frac{1}{3}\cos^{-1}\left(Q(x)\right)\right)\right)dx \tag{11}$$

$$\int_v^\infty\frac{1}{\sqrt{x(2x+1)(9x+4)}}\,dx=\int_0^u\left(\frac{R(x)^{1/3}}{54x}+\frac{73x}{54R(x)^{1/3}}-\frac{17}{54}\right)dx-u\,v \tag{12}$$

$$\int_0^v\frac{27\sqrt{6\sqrt{73}}}{\sqrt{5329\cos\left(3\cos^{-1}\left(S(x)\right)\right)-595\sqrt{73}}}\,dx=$$
$$u\,v+\int_u^\infty\left(-\frac{17}{54}+\frac{\sqrt{73}}{27}\cos\left(\frac{1}{3}\cos^{-1}\left(Q(x)\right)\right)\right)dx \tag{13}$$

$$S(x)=\frac{27x}{\sqrt{73}}+\frac{17}{2\sqrt{73}} \tag{14}$$

$$\int_0^v \frac{1}{\sqrt{x(2x+1)(9x+4)}}\,dx=\int_0^v \frac{27\sqrt{6\sqrt{73}}}{\sqrt{5329\cos\left(3\cos^{-1}\left(S(x)\right)\right)-595\sqrt{73}}}\,dx \tag{15}$$

$$\int_0^v \frac{1}{\sqrt{x(2x+1)(9x+4)}}\,dx=\frac{2}{3}F\left(\sin^{-1}\left(\sqrt{\frac{2\sqrt{73}-17}{2\sqrt{73}+7}}\right),\frac{1}{3}\right) \tag{16}$$

$$\int_v^\infty \frac{1}{\sqrt{x(2x+1)(9x+4)}}\,dx=\frac{2}{3}K\left(\frac{1}{3}\right)-\frac{2}{3}F\left(\sin^{-1}\left(\sqrt{\frac{2\sqrt{73}-17}{2\sqrt{73}+7}}\right),\frac{1}{3}\right) \tag{17}$$

$$\int_0^v \frac{1}{\sqrt{x(2x+1)(9x+4)}}\,dx=\sqrt{v}\sum_{n=0}^{\infty}\left(-\frac{9}{16}v\right)^n\frac{1}{2n+1}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\left(\frac{8}{9}\right)^k \tag{18}$$

References

1. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals, Series, and Products, seventh edition. Edited by Alan Jeffrey and Daniel Zwillinger, Academic Press, 2007.
2. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., and Clark, C.W.: NIST Handbook of Matematical Functions. Cambridge University Press, 2010.