Sums of arctangents and sums of products of arctangents *

Kunle Adegoke †

Department of Physics and Engineering Physics,
Obafemi Awolowo University, 220005 Ile-Ife, Nigeria

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Abstract

We present new infinite arctangent sums and infinite sums of products of arctangents. Many previously known evaluations appear as special cases of the general results derived in this paper.

1 Introduction

This paper reports the evaluation of infinite arctangent sums such as

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{3}{k^2+3k+1} = \frac{\pi}{2} \tag{1.1}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4k}{(2k^2-1)^2} = \frac{\pi}{2}\,, \tag{1.2}$$

and infinite sums of products of arctangents, such as

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} \tan^{-1} \frac{1}{k} = \frac{\pi^2}{8} \tag{1.3}$$

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†adegoke00@gmail.com

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{6k}{k^4 - 7k^2 + 2} \tan^{-1} \frac{1}{k^2 - k - 1} \tan^{-1} \frac{1}{k^2 + k - 1} = \frac{\pi^3}{64}\,, \quad (1.4)$$

and their generalizations. The well-known identity

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^2 + k + 1} = \frac{\pi}{4}$$

and the presumably new identity (1.1) are both special cases of the following identity (section 3.1.1, identity (3.8)):

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{q}{k^2 + qk + 1} = \sum_{k=1}^{q} \tan^{-1} \frac{1}{k}\,,$$

being evaluations at $q = 1$ and at $q = 3$, respectively.

Identity (1.3) is obtained at $q = 1$ from the following identity (identity (3.43) of section 3.2.2):

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2q^2}{(k+q-1)^2} \tan^{-1} \frac{q}{k+q-1}$$
$$= \frac{\pi^2}{8} + \sum_{k=2}^{q} \tan^{-1} \frac{q}{k-1} \tan^{-1} \frac{q}{k+q-1}\,.$$

Identity (1.4) comes from identity (3.48) of section 3.2.3, namely,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{6\alpha k}{k^4 - 7k^2 + \alpha^2 + 1} \tan^{-1} \frac{\alpha}{k^2 - k - 1} \tan^{-1} \frac{\alpha}{k^2 + k - 1} = \left(\tan^{-1} \alpha\right)^3\,.$$

Lemma 1. *Let α be a real number, let m and q be positive integers and let $\{f(k)\}_{k=1}^{\infty}$ be a real positive non-decreasing sequence such that $\lim_{k\to\infty} f(k) = \infty$, then*

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{\alpha(f(k+mq) - f(k))}{f(k)f(k+mq) + \alpha^2} \prod_{j=1}^{m-1} \tan^{-1} \frac{\alpha}{f(k+jq)}$$
$$= \sum_{k=1}^{q} \prod_{j=0}^{m-1} \tan^{-1} \frac{\alpha}{f(k+jq)}\,.$$

Lemma 1 follows directly from identity (2.1) of [1] and the trigonometric identity

$$\tan^{-1}\frac{\lambda}{x} - \tan^{-1}\frac{\lambda}{y} = \tan^{-1}\frac{\lambda(y-x)}{xy+\lambda^2}\,, \tag{1.5}$$

which holds for $\lambda^2/xy \geq -1$.

Throughout this paper, the principal value of the arctangent function is assumed.

We shall adopt the following conventions for empty sums and empty products:

$$\sum_{k=1}^{0} f(k) = 0, \quad \prod_{k=1}^{0} f(k) = 1\,.$$

2 Main Results

Theorem 2.1. *If α and $\beta \geq -1$ are real numbers and m and q are positive integers, then*

$$\sum_{k=1}^{\infty} \tan^{-1}\frac{\alpha m q}{k^2+(2\beta+mq)k+\beta(\beta+mq)+\alpha^2}\prod_{j=1}^{m-1}\tan^{-1}\frac{\alpha}{k+jq+\beta}$$
$$= \sum_{k=1}^{q}\prod_{j=0}^{m-1}\tan^{-1}\frac{\alpha}{k+jq+\beta}\,.$$

In particular,

$$\sum_{k=1}^{\infty} \tan^{-1}\frac{\alpha q}{k^2+(2\beta+q)k+q\beta+\alpha^2+\beta^2} = \sum_{k=1}^{q}\tan^{-1}\frac{\alpha}{k+\beta}\,, \tag{2.1}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \tan^{-1}\frac{2\alpha q}{k^2+2(\beta+q)k+2q\beta+\alpha^2+\beta^2}\tan^{-1}\frac{\alpha}{k+q+\beta}\\ = \sum_{k=1}^{q}\tan^{-1}\frac{\alpha}{k+\beta}\tan^{-1}\frac{\alpha}{k+q+\beta}\end{aligned} \tag{2.2}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{3\alpha q}{k^2+(2\beta+3q)k+3q\beta+\alpha^2+\beta^2} \tan^{-1} \frac{\alpha}{k+q+\beta} \tan^{-1} \frac{\alpha}{k+2q+\beta}$$

$$= \sum_{k=1}^{q} \tan^{-1} \frac{\alpha}{k+\beta} \tan^{-1} \frac{\alpha}{k+q+\beta} \tan^{-1} \frac{\alpha}{k+2q+\beta} \,. \tag{2.3}$$

Proof. Use $f(k)=k+\beta$ in Lemma 1. □

Corollary 2.2. *If q and m are positive integers and $\beta \geq -1$ is a real number, then*

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2m^2q^2}{(2k+2\beta+mq)^2} \prod_{j=1}^{m-1} \tan^{-1} \frac{mq}{2(k+\beta+jq)} = \sum_{k=1}^{q} \prod_{j=0}^{m-1} \tan^{-1} \frac{mq}{2(k+\beta+jq)} \,.$$

In particular,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2q^2}{(2k+2\beta+q)^2} = \sum_{k=1}^{q} \tan^{-1} \frac{q}{2(k+\beta)} \,. \tag{2.4}$$

Proof. Set $\alpha = mq/2$ in the identity of Theorem 2.1. □

Theorem 2.3. *If α and β are real numbers and m and q are positive integers, then*

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha mqk}{k^4+(2\beta-m^2q^2)k^2+\beta^2+\alpha^2} \prod_{j=1}^{m-1} \tan^{-1} \frac{\alpha}{k^2+(2jq-mq)k+j^2q^2+\beta-mjq^2}$$

$$= \sum_{k=1}^{q} \prod_{j=0}^{m-1} \tan^{-1} \frac{\alpha}{k^2+(2jq-mq)k+j^2q^2+\beta-mjq^2} \,.$$

In particular,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha qk}{k^4+(2\beta-q^2)k^2+\beta^2+\alpha^2} = \sum_{k=1}^{q} \tan^{-1} \frac{\alpha}{k^2-qk+\beta} \,. \tag{2.5}$$

Proof. Use $f(k)=k^2-mqk+\beta$ in Lemma 1. □

3 Examples

3.1 Sums of arctangents

3.1.1 Examples from identity (2.1)

Evaluating (2.1) at $\beta = -1$ gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{\alpha q}{k^2 + (q-2)(k-1) + \alpha^2 - 1} = \frac{\pi}{2} + \sum_{k=2}^{q} \tan^{-1} \frac{\alpha}{k-1}, \quad \alpha \in \mathbb{R}^+ , \tag{3.1}$$

from which we obtain

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{q}{k^2 + (q-2)(k-1)} = \frac{\pi}{2} + \sum_{k=2}^{q} \tan^{-1} \frac{1}{k-1} , \tag{3.2}$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha}{k^2 + \alpha^2 - 1} = \frac{\pi}{2} + \tan^{-1} \alpha \tag{3.3}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{\alpha}{k^2 - k + \alpha^2} = \frac{\pi}{2} . \tag{3.4}$$

The identity (3.4) is valid for any positive real α, and in particular evaluation at $\alpha = 1$ gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^2 - k + 1} = \frac{\pi}{2} . \tag{3.5}$$

Evaluating (3.3) at $\alpha = 1$ gives the well known result,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} = \frac{3\pi}{4} . \tag{3.6}$$

Putting $\beta = 0$ in identity (2.1) provides the evaluation

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{\alpha q}{k^2 + qk + \alpha^2} = \sum_{k=1}^{q} \tan^{-1} \frac{\alpha}{k} , \quad \alpha \in \mathbb{R} , \tag{3.7}$$

from which we get

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{q}{k^2 + qk + 1} = \sum_{k=1}^{q} \tan^{-1} \frac{1}{k} \tag{3.8}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{\alpha}{k^2 + k + \alpha^2} = \tan^{-1} \alpha\,. \tag{3.9}$$

Setting $\alpha = \tan\theta$ in identity (3.9) gives

$$\theta = \sum_{k=1}^{\infty} \tan^{-1} \frac{\tan\theta}{k^2 + k + \tan^2\theta}\,, \tag{3.10}$$

providing an infinite arctangent sum expansion for any angle θ, such that $-\pi/2 < \theta < \pi/2$.

Upon setting $\beta = -1/2$ in identity (2.1) we obtain

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4\alpha q}{4k^2 + 4k(q-1) - 2q + 1 + 4\alpha^2} = \sum_{k=1}^{q} \tan^{-1} \frac{2\alpha}{2k-1}\,, \quad \alpha \in \mathbb{R}\,, \tag{3.11}$$

from which we also obtain, at $\alpha = q/2$,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2q^2}{(2k + q - 1)^2} = \sum_{k=1}^{q} \tan^{-1} \frac{q}{2k-1}\,, \tag{3.12}$$

with the special value

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{2k^2} = \frac{\pi}{4}\,. \tag{3.13}$$

Choosing $\alpha = \sin\theta$, $\beta = \cos\theta$ and $q = 1$ in identity (2.1) gives

$$\theta = 2\sum_{k=1}^{\infty} \tan^{-1} \frac{\sin\theta}{k^2 + (2\cos\theta + 1)k + \cos\theta + 1}\,, \tag{3.14}$$

thereby providing infinite arctangent sums for angles $-\pi/2 \le \theta \le \pi/2$. In particular,

$$\frac{\pi}{6} = \sum_{k=1}^{\infty} \tan^{-1} \frac{\sqrt{3}}{2k^2 + 4k + 3}\,. \tag{3.15}$$

Replacing β with $\beta - 1$ in identity (2.1) and evaluating at $q = 1$ gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{\alpha}{k^2 + (2\beta - 1)k - \beta + \alpha^2 + \beta^2} = \tan^{-1} \frac{\alpha}{\beta}\,, \tag{3.16}$$

which is equivalent to identity (3) of [2].

3.1.2 Examples from identity (2.4)

Evaluating identity (2.4) at $\beta = 0$ and at $\beta = -1$, respectively, we have

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2q^2}{(2k + q)^2} = \sum_{k=1}^{q} \tan^{-1} \frac{q}{2k} \tag{3.17}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2q^2}{(2k + q - 2)^2} = \frac{\pi}{2} + \sum_{k=2}^{q} \tan^{-1} \frac{q}{2(k - 1)} \tag{3.18}$$

of which a special value is

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{(2k - 1)^2} = \frac{\pi}{2}\,. \tag{3.19}$$

3.1.3 Examples from identity (2.5)

Putting $\alpha = 1 = \beta$ in identity (2.5) we obtain

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2qk}{k^4 + (2 - q^2)k^2 + 2} = \sum_{k=1}^{q} \tan^{-1} \frac{1}{k^2 - qk + 1}\,, \tag{3.20}$$

which at $q = 1$ gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2k}{k^4 + k^2 + 2} = \frac{\pi}{4}\,, \tag{3.21}$$

and at $q = 2$ gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4k}{k^4 - 2k^2 + 2} = \frac{3\pi}{4}\,, \tag{3.22}$$

a result that was also reported in [3].

At $\beta = 2$, $q = 3$, identity (2.5) gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{6\alpha k}{k^4 - 5k^2 + 4 + \alpha^2} = \pi + \tan^{-1} \frac{\alpha}{2}\,, \quad \alpha \in \mathbb{R}^+\,, \tag{3.23}$$

which at $\alpha = 2$ produces

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{12k}{k^4 - 5k^2 + 8} = \frac{5\pi}{4}\,. \tag{3.24}$$

Evaluating identity (2.5) at $\beta = 0$, $q = 1$ yields

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha k}{k^4 - k^2 + \alpha^2} = \frac{\pi}{2}\,, \tag{3.25}$$

which at $\alpha = 1/2$ gives the special value

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4k}{(2k^2 - 1)^2} = \frac{\pi}{2}\,.$$

Setting $\beta = q^2/2$ in identity (2.5) gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{8\alpha q k}{4k^4 + 4\alpha^2 + q^4} = \sum_{k=1}^{q} \tan^{-1} \frac{2\alpha}{2k^2 - 2qk + q^2}\,, \tag{3.26}$$

and, in particular, with $\alpha = q^2/2$, we have

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2q^3 k}{2k^4 + q^4} = \sum_{k=1}^{q} \tan^{-1} \frac{q^2}{2k^2 - 2qk + q^2}\,, \tag{3.27}$$

giving at $q = 1$, the special value

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2k}{2k^4 + 1} = \frac{\pi}{4}\,. \tag{3.28}$$

Evaluating identity (3.26) at $q = 2$ gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4\alpha k}{k^4 + \alpha^2 + 4} = \tan^{-1} \frac{\alpha}{2} + \tan^{-1} \alpha\,, \tag{3.29}$$

which was also reported in [4].

Setting $2\alpha = \tan\theta$ and $q = 1$ in identity (3.26) gives

$$\theta = \sum_{k=1}^{\infty} \tan^{-1} \frac{4k\tan\theta}{4k^4 + \sec^2\theta}\,, \tag{3.30}$$

providing an infinite arctangent sum expansion for any angle θ, such that $-\pi/2 < \theta < \pi/2$. In particular,

$$\frac{\pi}{3} = \sum_{k=1}^{\infty} \tan^{-1} \frac{k\sqrt{3}}{k^4 + 1}\,, \quad \frac{\pi}{6} = \sum_{k=1}^{\infty} \tan^{-1} \frac{k\sqrt{3}}{3k^4 + 1}\,. \tag{3.31}$$

3.2 Sums of products of arctangents

3.2.1 Examples from Theorem 2.1

Using $\beta = -1$ in the identity of Theorem 2.1, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} \tan^{-1} \frac{\alpha m q}{k^2 + (mq-2)(k-1) + \alpha^2 - 1} \prod_{j=1}^{m-1} \tan^{-1} \frac{\alpha}{k + jq - 1} \\ &\qquad = \frac{\pi}{2} \prod_{j=1}^{m-1} \tan^{-1} \frac{\alpha}{jq} + \sum_{k=2}^{q} \prod_{j=0}^{m-1} \tan^{-1} \frac{\alpha}{k + jq - 1}, \quad \alpha \in \mathbb{R}^+ \end{aligned} \tag{3.32}$$

In particular,

$$\begin{aligned} &\sum_{k=1}^{\infty} \tan^{-1} \frac{\alpha m}{k^2 + (m-2)(k-1) + \alpha^2 - 1} \prod_{j=1}^{m-1} \tan^{-1} \frac{\alpha}{k + j - 1} \\ &\qquad = \frac{\pi}{2} \prod_{j=1}^{m-1} \tan^{-1} \frac{\alpha}{j}\,, \end{aligned} \tag{3.33}$$

which at $m = 2$ gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha}{k^2 + \alpha^2 - 1} \tan^{-1} \frac{\alpha}{k} = \frac{\pi}{2} \tan^{-1} \alpha, \quad \alpha \in \mathbb{R}^+\,, \tag{3.34}$$

yielding at $\alpha = 1$, $\alpha = \sqrt{3}$ and $\alpha = 1/\sqrt{3}$, respectively,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} \tan^{-1} \frac{1}{k} = \frac{\pi^2}{8},$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\sqrt{3}}{k^2+2} \tan^{-1} \frac{\sqrt{3}}{k} = \frac{\pi^2}{6} \tag{3.35}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\sqrt{3}}{3k^2-2} \tan^{-1} \frac{\sqrt{3}}{3k} = \frac{\pi^2}{12}. \tag{3.36}$$

At $m = 3$, the identity (3.32) gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{3\alpha}{k^2+k+\alpha^2-2} \tan^{-1} \frac{\alpha}{k} \tan^{-1} \frac{\alpha}{k+1} = \frac{\pi}{2} \tan^{-1} \alpha \tan^{-1} \frac{\alpha}{2}, \tag{3.37}$$

from which we get the evaluations

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{3\sqrt{2}}{k^2+k} \tan^{-1} \frac{\sqrt{2}}{k} \tan^{-1} \frac{\sqrt{2}}{k+1} = \frac{\pi}{2} \tan^{-1} \sqrt{2} \tan^{-1} \frac{\sqrt{2}}{2} \tag{3.38}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{3\sqrt{3}}{k^2+k+1} \tan^{-1} \frac{\sqrt{3}}{k} \tan^{-1} \frac{\sqrt{3}}{k+1} = \frac{\pi^2}{6} \tan^{-1} \frac{\sqrt{3}}{2}, \tag{3.39}$$

at $\alpha = \sqrt{2}$ and $\alpha = \sqrt{3}$, respectively. The following four-member sum

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4\sqrt{3}}{k^2+2k} \tan^{-1} \frac{\sqrt{3}}{k} \tan^{-1} \frac{\sqrt{3}}{k+1} \tan^{-1} \frac{\sqrt{3}}{k+2} = \frac{\pi^3}{36} \tan^{-1} \frac{\sqrt{3}}{2}, \tag{3.40}$$

is produced at $m = 4$ and $\alpha = \sqrt{3}$ in identity (3.32).

3.2.2 Examples from Corollary 2.2

If we set $\beta = -1/2$ in the identity of Corollary 2.2 we have

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2m^2q^2}{(2k+mq-1)^2} \prod_{j=1}^{m-1} \tan^{-1} \frac{mq}{2k+2jq-1} = \sum_{k=1}^{q} \prod_{j=0}^{m-1} \tan^{-1} \frac{mq}{2k+2jq-1}, \tag{3.41}$$

while at $\beta = -1$ we have

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2m^2q^2}{(2k+mq-2)^2} \prod_{j=1}^{m-1} \tan^{-1} \frac{mq}{2(k+jq-1)} = \sum_{k=1}^{q} \prod_{j=0}^{m-1} \tan^{-1} \frac{mq}{2(k+jq-1)}\,, \tag{3.42}$$

which at $m = 2$ gives

$$\begin{aligned} &\sum_{k=1}^{\infty} \tan^{-1} \frac{2q^2}{(k+q-1)^2} \tan^{-1} \frac{q}{k+q-1} \\ &\quad = \frac{\pi^2}{8} + \sum_{k=2}^{q} \tan^{-1} \frac{q}{k-1} \tan^{-1} \frac{q}{k+q-1}\,, \end{aligned} \tag{3.43}$$

which at $q = 1$ gives identity (1.3).

3.2.3 Examples from Theorem 2.3

Upon setting $q = 1$ in Theorem 2.3 we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha mk}{k^4+(2\beta-m^2)k^2+\beta^2+\alpha^2} \prod_{j=1}^{m-1} \tan^{-1} \frac{\alpha}{k^2-(m-2j)k+j^2+\beta-mj} \\ &\quad = \prod_{j=0}^{m-1} \tan^{-1} \frac{\alpha}{j^2+2j+\beta-mj-m+1}, \quad \alpha,\beta \in \mathbb{R}\,. \end{aligned} \tag{3.44}$$

In particular, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} \tan^{-1} \frac{4\alpha k}{k^4+2(\beta-2)k^2+\alpha^2+\beta^2} \tan^{-1} \frac{\alpha}{k^2+\beta-1} \\ &\quad = \tan^{-1} \frac{\alpha}{\beta-1} \tan^{-1} \frac{\alpha}{\beta} \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} &\sum_{k=1}^{\infty} \tan^{-1} \frac{6\alpha k}{k^4+(2\beta-9)k^2+\alpha^2+\beta^2} \tan^{-1} \frac{\alpha}{k^2-k+\beta-2} \tan^{-1} \frac{\alpha}{k^2+k+\beta-2} \\ &\quad = \left(\tan^{-1} \frac{\alpha}{\beta-2}\right)^2 \tan^{-1} \frac{\alpha}{\beta}\,, \end{aligned} \tag{3.46}$$

which at $\beta = 1$ give

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4\alpha k}{k^4 - 2k^2 + \alpha^2 + 1} \tan^{-1} \frac{\alpha}{k^2} = \frac{\pi}{2} \tan^{-1} \alpha \tag{3.47}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{6\alpha k}{k^4 - 7k^2 + \alpha^2 + 1} \tan^{-1} \frac{\alpha}{k^2 - k - 1} \tan^{-1} \frac{\alpha}{k^2 + k - 1} = \left(\tan^{-1} \alpha\right)^3 , \tag{3.48}$$

producing at $\alpha = 1$,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4k}{k^4 - 2k^2 + 2} \tan^{-1} \frac{1}{k^2} = \frac{\pi^2}{8} \tag{3.49}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{6k}{k^4 - 7k^2 + 2} \tan^{-1} \frac{1}{k^2 - k - 1} \tan^{-1} \frac{1}{k^2 + k - 1} = \frac{\pi^3}{64} .$$

Special values from (3.47) include

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4k\sqrt{3}}{k^4 - 2k^2 + 4} \tan^{-1} \frac{\sqrt{3}}{k^2} = \frac{\pi^2}{6} \tag{3.50}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4k\sqrt{3}}{3k^4 - 6k^2 + 4} \tan^{-1} \frac{\sqrt{3}}{3k^2} = \frac{\pi^2}{12} . \tag{3.51}$$

By first replacing β with $\beta - 1$ and then setting $\alpha = \sin\theta$ and $\beta = \cos\theta$, the identity (3.45) can be put in the form,

$$\frac{\theta^2}{2} = \sum_{k=1}^{\infty} \tan^{-1} \frac{4k\sin\theta}{k^4 - 4k^2\sin^2(\theta/2) + 4\cos^2(\theta/2)} \tan^{-1} \frac{\sin\theta}{k^2 + \cos\theta} , \tag{3.52}$$

suitable for expressing the square of any angle θ with magnitude less that $\pi/2$ as an infinite sum of products of two arctangents. In particular, at $\theta = \pi/3$, we have

$$\frac{\pi^2}{18} = \sum_{k=1}^{\infty} \tan^{-1} \frac{2k\sqrt{3}}{k^4 - k^2 + 3} \tan^{-1} \frac{\sqrt{3}}{2k^2 + 1} . \tag{3.53}$$

At $m=5$, $\alpha=1=\beta=q$ in identity (3.44) we have

$$\begin{aligned}\sum_{k=1}^{\infty}\tan^{-1}\frac{10k}{k^4-23k^2+2}\tan^{-1}\frac{1}{k^2-3k-3}\tan^{-1}\frac{1}{k^2-k-5}\times\\ \times\tan^{-1}\frac{1}{k^2-k+5}\tan^{-1}\frac{1}{k^2+3k-3}\\ =\frac{\pi}{4}\left(\tan^{-1}\frac{1}{3}\right)^2\left(\tan^{-1}\frac{1}{5}\right)^2 .\end{aligned} \tag{3.54}$$

3.3 More examples

Using $f(k)=(k-1)^3$ in Lemma 1 at $m=1$ gives

$$\sum_{k=1}^{\infty}\tan^{-1}\frac{\alpha(3k^2-3k+1)}{k^3(k-1)^3+\alpha^2}=\frac{\pi}{2} \tag{3.55}$$

at $q=1$ and

$$\sum_{k=1}^{\infty}\tan^{-1}\frac{2\alpha(3k^2+1)}{(k^2-1)^3+\alpha^2}=\frac{\pi}{2}+\tan^{-1}\alpha\,, \tag{3.56}$$

at $q=2$, for α a positive real number. At $\alpha=1$, identity (3.55) gives

$$\sum_{k=1}^{\infty}\tan^{-1}\frac{(3k^2-3k+1)}{(k^2-k+1)(k^4-2k^3+k+1)}=\frac{\pi}{2} \tag{3.57}$$

while identity (3.56) gives

$$\sum_{k=1}^{\infty}\tan^{-1}\frac{2(3k^2+1)}{k^2(k^4-3k^2+3)}=\frac{3\pi}{4}\,. \tag{3.58}$$

At $m=2$ and $q=1$, we have

$$\sum_{k=1}^{\infty}\tan^{-1}\frac{2\alpha(3k^2+1)}{(k^2-1)^3+\alpha^2}\tan^{-1}\frac{\alpha}{k^3}=\frac{\pi}{2}\tan^{-1}\alpha,\quad \alpha\in\mathbb{R}^+\,. \tag{3.59}$$

Using $f(k)=k^2-3k+2$ and $m=1$ in Lemma 1, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\tan^{-1}\frac{\alpha q(2k+q-3)}{(k-1)(k-2)(k+q-1)(k+q-2)+\alpha^2}\\ =\pi+\sum_{k=1}^{q-2}\tan^{-1}\frac{\alpha}{k(k+1)}\,,\end{aligned} \tag{3.60}$$

from which we get

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha(k-1)}{(k-1)^2(k-2)k+\alpha^2} = \frac{\pi}{2}\,, \tag{3.61}$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha(2k-1)}{(k-1)(k-2)(k+1)k+\alpha^2} = \pi \tag{3.62}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{6\alpha k}{(k-1)(k-2)(k+2)(k+1)+\alpha^2} = \pi + \tan^{-1}\frac{\alpha}{2}\,, \tag{3.63}$$

for positive real α. Evaluating at $\alpha = 1$, identity (3.62) gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2(2k-1)}{(k^2-k-1)^2} = \pi\,, \tag{3.64}$$

while at $\alpha = 2$, identity (3.63) reproduces identity (3.24).

With $f(k) = k^2 - 3k + 2$ and $m = 2$, $q = 1$ in Lemma 1, we have

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\alpha(2k+1)}{(k-1)k(k+1)(k+2)+\alpha^2} \tan^{-1}\frac{\alpha}{k(k+1)} = \frac{\pi}{2}\tan^{-1}\frac{\alpha}{2}\,, \tag{3.65}$$

giving, in particular,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2(2k+1)}{(k^2+k-1)^2} \tan^{-1}\frac{1}{k(k+1)} = \frac{\pi}{2}\tan^{-1}\frac{1}{2}\,. \tag{3.66}$$

Using $f(k) = (k^2 - 5k + 5)^2$ in Lemma 1 gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{8\alpha(2k-1)(k^2-k+3)}{(k^2+3k+1)^2(k^2-5k+5)^2+\alpha^2} \tan^{-1}\frac{\alpha}{(k^2-k-1)^2} = 2(\tan^{-1}\alpha)^2\,, \tag{3.67}$$

at $q = 2$, $m = 2$,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{12\alpha(k-1)(k^2-2k+2)}{(k^2+k-1)^2(k^2-5k+5)^2+\alpha^2} \tan^{-1}\frac{\alpha}{(k^2-3k+1)^2}\times \times \tan^{-1}\frac{\alpha}{(k^2-k-1)^2} = (\tan^{-1}\alpha)^3 \tag{3.68}$$

at $m=3$, $q=1$ and

$$\sum_{k=1}^{\infty}\tan^{-1}\frac{8\alpha(2k-1)(k^2-k+3)}{(k^2+3k+1)^2(k^2-5k+5)^2+\alpha^2}\tan^{-1}\frac{\alpha}{(k^2-3k+1)^2}\times$$
$$\times\tan^{-1}\frac{\alpha}{(k^2-k-1)^2}\tan^{-1}\frac{\alpha}{(k^2+k-1)^2}=(\tan^{-1}\alpha)^4\,, \tag{3.69}$$

at $m=4$, $q=1$, for real numbers α.

3.4 Alternating sums

Lemma 2. *Let α be a real number, let m and q be positive integers and let $\{f(k)\}_{k=1}^{\infty}$ be a real positive non-decreasing sequence such that $\lim_{k\to\infty}f(k)=\infty$, then*

$$\sum_{k=1}^{\infty}(-1)^{k-1}\tan^{-1}\frac{\alpha(f(k+mq)\mp f(k))}{f(k)f(k+mq)\pm\alpha^2}\prod_{j=1}^{m-1}\tan^{-1}\frac{\alpha}{f(k+jq)}$$
$$=\sum_{k=1}^{q}(-1)^{k-1}\prod_{j=0}^{m-1}\tan^{-1}\frac{\alpha}{f(k+jq)}\,,$$

where the upper signs apply if q is even, and the lower if q is odd.

Lemma 2 follows from identity (2.4) of [1] and the trigonometric identities (1.5) and

$$\tan^{-1}\frac{\lambda}{x}+\tan^{-1}\frac{\lambda}{y}=\tan^{-1}\frac{\lambda(y+x)}{xy-\lambda^2}\,,\quad \lambda^2/xy<1\,.$$

On account of Lemma 2 with the upper signs, alternating versions of the results obtained in the previous sections are readily obtained by replacing q with $2q$. For example, corresponding to identity (3.7) is the following alternating sum

$$\sum_{k=1}^{\infty}(-1)^{k-1}\tan^{-1}\frac{2\alpha q}{(k+q)^2+\alpha^2-q^2}=\sum_{k=1}^{2q}(-1)^{k-1}\tan^{-1}\frac{\alpha}{k}\,,\quad \alpha\in\mathbb{R}\,, \tag{3.70}$$

which gives, in particular,

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tan^{-1} \frac{2q^2}{(k+q)^2} = \sum_{k=1}^{2q} (-1)^{k-1} \tan^{-1} \frac{q}{k}\,, \tag{3.71}$$

from which we get the special value

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tan^{-1} \frac{2}{k^2} = \frac{\pi}{4}\,. \tag{3.72}$$

Similarly, the alternating version of identity (3.43) is

$$\begin{aligned}\sum_{k=1}^{\infty} (-1)^{k-1} \tan^{-1} \frac{8q^2}{(k+2q-1)^2} \tan^{-1} \frac{2q}{(k+2q-1)} \\ = \frac{\pi^2}{8} + \sum_{k=2}^{2q} (-1)^{k-1} \tan^{-1} \frac{2q}{k-1} \tan^{-1} \frac{2q}{(k+2q-1)}\,.\end{aligned} \tag{3.73}$$

Additional alternating sums can be obtained from Lemma 2 with the lower signs. For example, using $f(k) = k + \beta$ in Lemma 2 with the lower signs we have the following result.

Theorem 3.1. *If α and β are real numbers and m and q are positive integers such that q is odd, then*

$$\begin{aligned}\sum_{k=1}^{\infty} (-1)^{k-1} \tan^{-1} \frac{\alpha(2k+2\beta+mq)}{k^2+(2\beta+mq)k+\beta(\beta+mq)-\alpha^2} \prod_{j=1}^{m-1} \tan^{-1} \frac{\alpha}{k+jq+\beta} \\ = \sum_{k=1}^{q} (-1)^{k-1} \prod_{j=0}^{m-1} \tan^{-1} \frac{\alpha}{k+jq+\beta}\,.\end{aligned}$$

Corollary 3.2. *If m and q are positive integers such that q is odd, then*

$$\begin{aligned}\sum_{k=1}^{\infty} (-1)^{k-1} \tan^{-1} \frac{m(2k+m(q-1))}{2k^2+2mk(q-1)-m^2q} \prod_{j=1}^{m-1} \tan^{-1} \frac{m}{2k+2jq-m} \\ = \sum_{k=1}^{q} (-1)^{k-1} \prod_{j=0}^{m-1} \tan^{-1} \frac{m}{2k+2jq-m}\,.\end{aligned}$$

Proof. Set $\alpha = m/2 = -\beta$ in Theorem 3.1. ☐

In particular, we have

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^{k-1}\tan^{-1}\frac{2mk}{2k^2-m^2}\prod_{j=1}^{m-1}\tan^{-1}\frac{m}{2k+2j-m}\\ =\prod_{j=0}^{m-1}\tan^{-1}\frac{m}{2j-m+2}\,,\end{aligned}\tag{3.74}$$

giving the special value

$$\sum_{k=1}^{\infty}(-1)^{k-1}\tan^{-1}\frac{2k}{2k^2-1}=\frac{\pi}{4}\,,\tag{3.75}$$

and

$$\begin{aligned}&\sum_{k=1}^{\infty}(-1)^{k}\tan^{-1}\frac{2(k+q-1)}{k^2+2k(q-1)-2q}\tan^{-1}\frac{1}{k+q-1}\\ &=\frac{\pi}{2}\tan^{-1}\frac{1}{q}+\sum_{k=2}^{q}(-1)^{k}\tan^{-1}\frac{1}{k-1}\tan^{-1}\frac{1}{k+q-1}\,,\quad q\text{ odd}\,,\end{aligned}\tag{3.76}$$

giving the special value,

$$\sum_{k=1}^{\infty}(-1)^{k}\tan^{-1}\frac{2k}{k^2-2}\tan^{-1}\frac{1}{k}=\frac{\pi^2}{8}\,.\tag{3.77}$$

References

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