The qq'-Calculus

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Abstract

We present here a generalisation of the q-calculus, the qq'-calculus. The calculus is however limited.

1 The δ-derivation

1.1 Definitions

The derivative of a function f at the point x is usually defined as:

$$d_h(f)(x) = f(x+h) - f(x)$$

$$\frac{df}{dx}(x) = f'(x) = \lim_{h\to 0} \frac{d_h(f)}{d_h(x)}(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

if the limit exists.

Definition 1 *Similarly, the δ-derivative of a function is defined as:*

$$\delta_{hh'}(f)(x) = f(x+h) - f(x+h')$$

$$\frac{\delta f}{\delta x}(x) = \tilde{f}(x) = \lim_{h,h'\to 0} \frac{\delta_{hh'}(f)}{\delta_{hh'}(x)} = \lim_{h,h'\to 0} \frac{f(x+h)-f(x+h')}{h-h'} =$$

$$= \lim_{x_0,x_1\to x} \frac{f(x_0)-f(x_1)}{x_0-x_1}$$

If the δ-derivative of a function exists, then the derivative of the function exists and we have $\frac{\delta f}{\delta x}(x) = \frac{df}{dx}(x)$.

1.2 A counter-example

The derivative can exist even if the δ-derivative doesn't. Indeed let be f the fonction such that $f(x) = x^2$, if $x \in \mathbf{Q}$ and $f(x) = x^3$ if $x \notin \mathbf{Q}$. This function admits a derivative in zero which is zero, but has no δ-derivative as one can verify:

$$\lim_{hh'\to 0} \frac{h^2 - h'^3}{h-h'} = \lim_{hh'\to 0} h + h' + \frac{h'^2 - h'^3}{h-h'}$$

doesn't exist because $h - h'$ can be as small as we want.

1.3 The Leibniz rule

The Leibniz rule can be verified:

$$\frac{f(x+h)g(x+h)-f(x+h')g(x+h')}{h-h'}=$$

$$=\frac{f(x+h)-f(x+h')}{h-h'}g(x+h)+\frac{g(x+h)-g(x+h')}{h-h'}f(x+h')$$

so that:

Proposition 1

$$\frac{\delta(fg)}{\delta x}=(\frac{\delta f}{\delta x})g+f(\frac{\delta g}{\delta x})$$

1.4 Some formulas

The following formulas can be easily verified:

$$(\tilde{\frac{1}{f}})=\frac{-1}{f^2}\tilde{f}$$

$$(\tilde{\frac{f}{g}})=\frac{\tilde{f}g-f\tilde{g}}{g^2}$$

and also :

$$(f\tilde{\circ}g)=(\tilde{f}\circ g)\times\tilde{g}$$

1.5 δ-derivative of a function of class $\mathcal{C}^1$

Theorem 1 *If the fonction f is of class $\mathcal{C}^1$, then the δ-derivative exists.*

Demonstration 1 *By the Taylor's formula* $\frac{f(x)-f(x')}{x-x'}=f'(c)$, $c\in]x,x'[$.

So a smooth function is also infinitely δ-derivable.

1.6 The qq'-limit

We have, if the limit exists, for $x\neq 0$:

$$\lim_{qq'\to 1}\frac{f(qx)-f(q'x)}{(q-q')x}=\tilde{f}(x)$$

1.7 Integration and δ-derivation

Theorem 2 *If f is continuous over the interval $[a,b]$, it is Riemann integrable and the primitive is δ-derivable, so that we have:*

$$\frac{\delta}{\delta x}(\int_a^x f(t)dt)=f(x)$$

Demonstration 2

$$\frac{\delta}{\delta x}(\int_a^x f(t)dt)=\lim_{hh'\to 0}\frac{\int_{h'}^h f(t+x)dt}{h-h'}=f(x)$$

by the Taylor formula.

So, a function which is $\mathcal{C}^1$, is δ-derivable.

2 qq'-quantum derivation

2.1 Definitions

Definition 2 *Let be two numbers q, q' and let be an arbitrary function f, its qq'-differential is:*

$$d_{qq'}(f)(x) = f(qx) - f(q'x)$$

In particular $d_{qq'}x = (q - q')x$.
We have the following Leibniz rule:

$$d_{qq'}(fg)(x) = f(qx)g(qx) - f(q'x)g(q'x) =$$
$$= (f(qx) - f(q'x))g(qx) + f(q'x)(g(qx) - g(q'x)$$

Proposition 2

$$d_{qq'}(fg)(x) = d_{qq'}(f)(x).g(qx) + f(q'x).d_{qq'}(g)(x)$$

Definition 3 *The following formula:*

$$D_{qq'}f(x) = \frac{d_{qq'}(f)(x)}{d_{qq'}(x)} = \frac{f(qx) - f(q'x)}{(q - q')x}$$

is called the qq'-derivative of the function f

2.2 The Leibniz rule

The Leibiz rule is:

Proposition 3

$$D_{qq'}(fg)(x) = D_{qq'}(f)(x).g(qx) + f(q'x).D_{qq'}(g)(x)$$

2.3 Some formulas

The qq'-derivative is a linear operator as we can verify:

$$D_{qq'}(af + bg) = aD_{qq'}(f) + bD_{qq'}(g)$$

for any scalars a, b and functions f, g.

Example 1

$$D_{qq'}(x^n) = [n]_{qq'}x^{n-1}$$

with $[n]_{qq'} = \frac{q^n - q'^n}{q - q'}$

The number $[n]_{qq'}$ is called the qq'-analog of n as $\lim_{qq' \to 1}[n]_{qq'} = n$. We obtain also:

Proposition 4

$$D_{qq'}(\frac{f}{g})(x) = \frac{g(q'x)D_{qq'}(f)(x) - f(q'x)D_{qq'}(g)(x)}{g(qx)g(q'x)} =$$
$$= \frac{g(qx)D_{qq'}(f)(x) - f(qx)D_{qq'}(g)(x)}{g(qx)g(q'x)}$$

For the composition we also have, if $u = x^a$:

Proposition 5

$$D_{qq'}(f \circ u)(x) = (D_{q^a q'^a}(f) \circ u)(x) \times D_{qq'}(u)(x)$$

3 qq'-analogue of $(x-a)^n$

3.1 Definition

Definition 4

$$[0]_{qq'}! = 1$$
$$[n]_{qq'}! = [n]_{qq'} \times [n-1]_{qq'} \times \ldots \times [1]_{qq'} \ if \ n \neq 0$$

3.2 The exponential

Definition 5

$$\exp_{qq'}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_{qq'}!}$$

The derivative is :

$$D_{qq}(\exp_{qq'})(x) = \exp_{qq'}(x)$$

3.3 The qq'-analogue of $(x-a)^n$

Definition 6 *The qq'-analogue of $(x-a)^n$ is:*

$$(x-a)^n_{qq'} = \prod_{k,l,\, k+l=n-1} (x - q^k {q'}^l a)$$

We have the following theorem:

Theorem 3

$$D_{qq'}(x-a)^n_{qq'} = [n]_{qq'}(x-a)^{n-1}_{qq'}$$

Demonstration 3

$$(x-a)^n_{qq'} = (x-qa)^{n-1}_{qq'}(x - {q'}^n a)$$

so that, by induction on n, using Leibniz rule:

$$D_{qq'}(x-a)^n_{qq'} = D_{qq'}(x-a)^{n-1}_{qq'}(q'x - {q'}^{n-1}a) + (qx-qa)^{n-1}_{qq'} =$$

$$= [n-1]_{qq'}q'(x-a)^{n-2}_{qq'}(x - {q'}^{n-2}a) + q^{n-1}(x-a)^{n-1}_{qq'} = [n]_{qq'}(x-a)^{n-1}_{qq'}$$

We have also:

$$(x-a)^{n+m}_{qq'} = (x - {q'}^m a)^n_{qq'}(x - q^n a)^m_{qq'}$$

4 qq'-Taylor's Formula for polynomials

4.1 The Taylor's expansion

Theorem 4 *For any polynomial $P(X)$ of degree n, and any number a, we have the following qq'-Taylor expansion:*

$$P(x) = \sum_{j=0}^{n} (D^j_{qq'}P)(a) \frac{(x-a)^j_{qq'}}{[j]_{qq'}!}$$

Demonstration 4 *Due to the degree, we can write:*

$$P(x) = \sum_{j=0}^{n} c_j \frac{(x-a)^j_{qq'}}{[j]_{qq'}!}$$

and now, by derivation, we have inductively on the degree of P:

$$c_k = (D^k_{qq'}P)(a)$$

4.2 A formula

The qq'-Taylor formula for x^n about $x = 1$ then gives:

$$x^n = \sum_{j=0}^{n} [n]_{qq'} \dots [n-j+1]_{qq'} \frac{(x-a)^j_{qq'}}{[j]_{qq'}!}$$

Formula 1

$$x^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{qq'} (x-a)^j_{qq'}$$

with $\begin{bmatrix} n \\ j \end{bmatrix}_{qq'} = \frac{[n]_{qq'}!}{[j]_{qq'}![n-j]_{qq'}!}$.

References

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