

The qq' -Calculus

Antoine Balan

Abstract

We present here a generalisation of the q -calculus, the qq' -calculus. The calculus is however limited.

1 The δ -derivation

1.1 Definitions

The derivative of a function f at the point x is usually defined as:

$$d_h(f)(x) = f(x+h) - f(x)$$
$$\frac{df}{dx}(x) = f'(x) = \lim_{h \rightarrow 0} \frac{d_h(f)}{d_h(x)}(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

Definition 1 *Similarly, the δ -derivative of a function is defined as:*

$$\delta_{hh'}(f)(x) = f(x+h) - f(x+h')$$
$$\frac{\delta f}{\delta x}(x) = \tilde{f}(x) = \lim_{h, h' \rightarrow 0} \frac{\delta_{hh'}(f)}{\delta_{hh'}(x)} = \lim_{h, h' \rightarrow 0} \frac{f(x+h) - f(x+h')}{h-h'} =$$
$$= \lim_{x_0, x_1 \rightarrow x} \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

If the δ -derivative of a function exists, then the derivative of the function exists and we have $\frac{\delta f}{\delta x}(x) = \frac{df}{dx}(x)$.

1.2 A counter-example

The derivative can exist even if the δ -derivative doesn't. Indeed let be f the function such that $f(x) = x^2$, if $x \in \mathbf{Q}$ and $f(x) = x^3$ if $x \notin \mathbf{Q}$. This function admits a derivative in zero which is zero, but has no δ -derivative as one can verify:

$$\lim_{hh' \rightarrow 0} \frac{h^2 - h'^3}{h - h'} = \lim_{hh' \rightarrow 0} h + h' + \frac{h'^2 - h'^3}{h - h'}$$

doesn't exist because $h - h'$ can be as small as we want.

1.3 The Leibniz rule

The Leibniz rule can be verified:

$$\begin{aligned} & \frac{f(x+h)g(x+h) - f(x+h')g(x+h')}{h-h'} = \\ &= \frac{f(x+h) - f(x+h')}{h-h'}g(x+h) + \frac{g(x+h) - g(x+h')}{h-h'}f(x+h') \end{aligned}$$

so that:

Proposition 1

$$\frac{\delta(fg)}{\delta x} = \left(\frac{\delta f}{\delta x}\right)g + f\left(\frac{\delta g}{\delta x}\right)$$

1.4 Some formulas

The following formulas can be easily verified:

$$\begin{aligned} \left(\frac{\tilde{1}}{f}\right) &= \frac{-1}{f^2}\tilde{f} \\ \left(\frac{\tilde{f}}{g}\right) &= \frac{\tilde{f}g - f\tilde{g}}{g^2} \end{aligned}$$

and also :

$$(f \tilde{\circ} g) = (\tilde{f} \circ g) \times \tilde{g}$$

1.5 δ -derivative of a function of class \mathcal{C}^3

Theorem 1 *If the function f is of class \mathcal{C}^3 , then the δ -derivative exists.*

Demonstration 1 *As we have :*

$$(1/2) \int_0^h f^{(2)}(t)(h-t)dt = \int_0^h f'(t)dt - f'(0)h = f(h) - f(0) - f'(0)h$$

so that :

$$(1/2)h^2 \int_0^1 f^{(2)}(ht)(1-t)dt - (1/2)h'^2 \int_0^1 f^{(2)}(h't)(1-t)dt = f(h) - f(h') - f'(0)(h-h') =$$

$$= (1/2)(h^2 - h'^2) \int_0^1 f^{(2)}(ht)(1-t)dt + h'^2 \int_0^1 [f^{(2)}(ht) - f^{(2)}(h't)](1-t)dt$$

and

$$|f^{(2)}(ht) - f^{(2)}(h't)| \leq (\max|f^{(3)}|)(h-h')t$$

by the Taylor's formula.

So a smooth function is also infinitely δ -derivable.

2 The qq' -limit

We have, if the limit exists, for $x \neq 0$:

$$\lim_{qq' \rightarrow 1} \frac{f(qx) - f(q'x)}{(q - q')x} = \tilde{f}(x)$$

3 Integration and δ -derivation

Theorem 2 *If f is continuous over the interval $[a, b]$, it is Riemann integrable and the primitive is δ -derivable, so that we have:*

$$\frac{\delta}{\delta x} \left(\int_a^x f(t) dt \right) = f(x)$$

Demonstration 2

$$\frac{\delta}{\delta x} \left(\int_a^x f(t) dt \right) = \lim_{hh' \rightarrow 0} \frac{\int_{h'}^h f(t+x) dt}{h-h'} = f(x)$$

by the Taylor formula.

4 qq' -quantum derivation

4.1 Definitions

Definition 2 *Let be two numbers q, q' and let be an arbitrary function f , its qq' -differential is:*

$$d_{qq'}(f)(x) = f(qx) - f(q'x)$$

In particular $d_{qq'}x = (q - q')x$.

We have the following Leibniz rule:

$$\begin{aligned} d_{qq'}(fg)(x) &= f(qx)g(qx) - f(q'x)g(q'x) = \\ &= (f(qx) - f(q'x))g(qx) + f(q'x)(g(qx) - g(q'x)) \end{aligned}$$

Proposition 2

$$d_{qq'}(fg)(x) = d_{qq'}(f)(x).g(qx) + f(q'x).d_{qq'}(g)(x)$$

Definition 3 *The following formula:*

$$D_{qq'}f(x) = \frac{d_{qq'}(f)(x)}{d_{qq'}(x)} = \frac{f(qx) - f(q'x)}{(q - q')x}$$

is called the qq' -derivative of the function f

4.2 The Leibniz rule

The Leibniz rule is:

Proposition 3

$$D_{qq'}(fg)(x) = D_{qq'}(f)(x).g(qx) + f(q'x).D_{qq'}(g)(x)$$

4.3 Some formulas

The qq' -derivative is a linear operator as we can verify:

$$D_{qq'}(af + bg) = aD_{qq'}(f) + bD_{qq'}(g)$$

for any scalars a, b and functions f, g .

Example 1

$$D_{qq'}(x^n) = [n]_{qq'} x^{n-1}$$

with $[n]_{qq'} = \frac{q^n - q'^n}{q - q'}$

The number $[n]_{qq'}$ is called the qq' -analog of n as $\lim_{qq' \rightarrow 1} [n]_{qq'} = n$. We obtain also:

Proposition 4

$$\begin{aligned} D_{qq'}\left(\frac{f}{g}\right)(x) &= \frac{g(q'x)D_{qq'}(f)(x) - f(q'x)D_{qq'}(g)(x)}{g(qx)g(q'x)} = \\ &= \frac{g(qx)D_{qq'}(f)(x) - f(qx)D_{qq'}(g)(x)}{g(qx)g(q'x)} \end{aligned}$$

For the composition we also have, if $u = x^a$:

Proposition 5

$$D_{qq'}(f \circ u)(x) = (D_{q^a q'^a}(f) \circ u)(x) \times D_{qq'}(u)(x)$$

5 qq' -analogue of $(x - a)^n$

5.1 Definition

Definition 4

$$\begin{aligned} [0]_{qq'}! &= 1 \\ [n]_{qq'}! &= [n]_{qq'} \times [n-1]_{qq'} \times \dots \times [1]_{qq'} \text{ if } n \neq 0 \end{aligned}$$

5.2 The exponential

Definition 5

$$\exp_{qq'}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_{qq'}!}$$

The derivative is :

$$D_{qq'}(\exp_{qq'})(x) = \exp_{qq'}(x)$$

5.3 The qq' -analogue of $(x - a)^n$

Definition 6 The qq' -analogue of $(x - a)^n$ is:

$$(x - a)_{qq'}^n = \prod_{k,l, k+l=n-1} (x - q^k q'^l a)$$

We have the following theorem:

Theorem 3

$$D_{qq'}(x-a)_{qq'}^n = [n]_{qq'}(x-a)_{qq'}^{n-1}$$

Demonstration 3

$$(x-a)_{qq'}^n = (x-qa)_{qq'}^{n-1}(x-q^na)$$

so that, by induction on n , using Leibniz rule:

$$\begin{aligned} D_{qq'}(x-a)_{qq'}^n &= D_{qq'}(x-a)_{qq'}^{n-1}(q'x-q'^{n-1}a) + (qx-qa)_{qq'}^{n-1} = \\ &= [n-1]_{qq'}q'(x-a)_{qq'}^{n-2}(x-q'^{n-2}a) + q^{n-1}(x-a)_{qq'}^{n-1} = [n]_{qq'}(x-a)_{qq'}^{n-1} \end{aligned}$$

We have also:

$$(x-a)_{qq'}^{n+m} = (x-q^ma)_{qq'}^n(x-q^na)_{qq'}^m$$

6 qq' -Taylor's Formula for polynomials

6.1 The Taylor's expansion

Theorem 4 For any polynomial $P(X)$ of degree n , and any number a , we have the following qq' -Taylor expansion:

$$P(x) = \sum_{j=0}^n (D_{qq'}^j P)(a) \frac{(x-a)_{qq'}^j}{[j]_{qq'}!}$$

Demonstration 4 Due to the degree, we can write:

$$P(x) = \sum_{j=0}^n c_j \frac{(x-a)_{qq'}^j}{[j]_{qq'}!}$$

and now, by derivation, we have inductively on the degree of P :

$$c_k = (D_{qq'}^k P)(a)$$

6.2 A formula

The qq' -Taylor formula for x^n about $x = 1$ then gives:

$$x^n = \sum_{j=0}^n [n]_{qq'} \dots [n-j+1]_{qq'} \frac{(x-a)_{qq'}^j}{[j]_{qq'}!}$$

Formula 1

$$x^n = \sum_{j=0}^n \left[\begin{matrix} n \\ j \end{matrix} \right]_{qq'} (x-a)_{qq'}^j$$

with $\left[\begin{matrix} n \\ j \end{matrix} \right]_{qq'} = \frac{[n]_{qq'}!}{[j]_{qq'}! [n-j]_{qq'}!}$.

References

- [KC] V.Kac, P.Cheung, "Quantum Calculus", Springer-Verlag, Berlin, 2002.
 [K] C.Kassel, "Quantum Groups", Springer-Verlag 155, New-York, 1995.