International Journal of Algebra and Statistics
Volume 6: 1-2(2017), 56-80
DOI :10.20454/ijas.2017.1266

Published by Modern Science Publishers
Available at: http://www.m-sciences.com

Representation of Graph Structure Based on I-V Neutrosophic Sets

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(Received: 10 Febrary 2017; Accepted: 15 April 2017)

Abstract. In this research article, we apply the concept of interval-valued neutrosophic sets to graph structures. We present the concept of interval-valued neutrosophic graph structures. We describe certain operations on interval-valued neutrosophic graph structures and elaborate them with appropriate examples. Further, we investigate some relevant properties of these operators. Moreover, we propose some open problems on interval-valued neutrosophic line graph structures.

1. Introduction

Zadeh [26] proposed fuzzy set theory to deal with uncertainty in many meticulous real-life phenomenons. Fuzzy set theory is affluently applicable in real time systems having information with different levels of precision. Zadeh introduced interval-valued fuzzy sets as generalization of fuzzy sets in [27]. There are many natural phenomenons, in which with membership value it is necessary to consider non-membership value. Membership value of an event is in its favor, whereas non-membership value is in its opposition. Atanassov [10] introduced the notion of intuitionistic fuzzy sets as an extension of fuzzy sets. Intuitionistic fuzzy sets are more practical, advantageous and applicable in many real-life phenomenons. In many real-life phenomenons like information fusion, indeterminacy is doubtlessly quantified. Smarandache [19, 20] proposed the notion of neutrosophic sets, he combined non-standard analysis, tricomponent logic, and philosophy. "It is a branch of philosophy which studies the origin, nature and scope of neutralities as well as their interactions with different ideational spectra". In neutrosophic set, three independent components are: truth-membership (t), indeterminacy-membership (i) and falsity-membership (f), each membership value is a real standard or non-standard subset of the non-standard unit interval $]0^-, 1^+[$ and there is no restriction on their sum. For convenient and advantageous use of neutrosophic sets in science and engineering, Wang et al. [22] introduced the idea of single-valued neutrosophic(SVN) sets, whose three independent components t, i and f have values in standard unit interval $[0, 1]$ and their sum does not exceed three. Neutrosophic set theory is a generalization of the fuzzy set theory and intuitionistic fuzzy set theory. It is more advantageous and applicable in many fields, including medical diagnosis, control theory, topology, decision making problems and in many more real-life problems. Wang et al. [23] presented the notion of interval-valued neutrosophic sets, it is more precise and more flexible as compared to single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which values of three independent components (t, i, f) are intervals, which are subsets

2010 *Mathematics Subject Classification*. 03E72, 05C72, 05C78, 05C99.
Keywords. Graph structure, interval-valued neutrosophic graph structure.
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of standard unit interval $[0, 1]$.
On the basis of Zadeh's fuzzy relations [28], Kaufmann proposed fuzzy graph [15]. Rosenfeld [17] discussed fuzzy analogue of various graph-theoretic ideas. Later on, Bhattacharya [11] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [16]. The complement of a fuzzy graph was defined by Mordeson [16] and further studied by Sunitha and Vijayakumar [21]. Hongmei and Lianhua gave the definition of interval-valued fuzzy graph in [14]. After that, Akram et al. [1–4] considered several concepts on interval-valued fuzzy graphs. Recently, Akram and Nasir [5] dealt with interval-valued neutrosophic graphs. Akram and Shahzadi [6] introduced the notion of neutrosophic soft graphs with applications. Akram [7] introduced the notion of single-valued neutrosophic planar graphs. Dinesh and Ramakrishnan [13] defined fuzzy graph structures and discussed its properties. Akram and Akmal [9] proposed the notion of bipolar fuzzy graph structures. In this research article, we first present the concept of interval-valued neutrosophic line graphs as an extension of interval-valued fuzzy line graphs [1]. We then present concept of interval-valued neutrosophic graph structures. Further, we describe certain operations on interval-valued neutrosophic graph structures. Finally, we state some open problems on interval-valued neutrosophic line graph structures.

2. Background

Wang et al. [23] introduced the concept of interval-valued (I-V) neutrosophic sets as follows:

Definition 2.1. *[23, 24] The interval-valued neutrosophic set A in X is defined by*

$$A = \{(x, [t_A^-(x), t_A^+(x)], [i_A^-(x), i_A^+(x)], [f_A^-(x), f_A^+(x)]) : x \in X\},$$

where, $t_A^-(x)$, $t_A^+(x)$, $i_A^-(x)$, $i_A^+(x)$, $f_A^-(x)$, and $f_A^+(x)$ are neutrosophic subsets of X such that $t_A^-(x) \le t_A^+(x)$, $i_A^-(x) \le i_A^+(x)$ and $f_A^-(x) \le f_A^+(x)$ for all $x \in X$.

For any two interval-valued neutrosophic sets $A = ([t_A^-(x), t_A^+(x)], [i_A^-(x), i_A^+(x)], [f_A^-(x), f_A^+(x)])$ and $B = ([t_B^-(x), t_B^+(x)], [i_B^-(x), i_B^+(x)], [f_B^-(x), f_B^+(x)])$ in X, we define:

- $A \cup B = \{(x, \max(t_A^-(x), t_B^-(x)), \max(t_A^+(x), t_B^+(x)), \max(i_A^-(x), i_B^-(x)), \max(i_A^+(x), i_B^+(x)), \min(f_A^-(x), f_B^-(x)), \min(f_A^+(x), f_B^+(x))) : x \in X\}$.
- $A \cap B = \{(x, \min(t_A^-(x), t_B^-(x)), \min(t_A^+(x), t_B^+(x)), \min(i_A^-(x), i_B^-(x)), \min(i_A^+(x), i_B^+(x)), \max(f_A^-(x), f_B^-(x)), \max(f_A^+(x), f_B^+(x))) : x \in X\}$.

Akram and Nasir [5] defined interval-valued neutrosophic graph as follows:

Definition 2.2. *[5] An interval-valued neutrosophic graph on a nonempty set X is a pair $G = (A, B)$, where A is an interval-valued neutrosophic set on X and B is an interval-valued neutrosophic relation on X such that:*

1. $t_B^-(xy) \le \min(t_A^-(x), t_A^-(y))$, $\quad t_B^+(xy) \le \min(t_A^+(x), t_A^+(y))$,
2. $i_B^-(xy) \le \min(i_A^-(x), i_A^-(y))$, $\quad i_B^+(xy) \le \min(i_A^+(x), i_A^+(y))$,
3. $f_B^-(xy) \le \min(f_A^-(x), f_A^-(y))$, $\quad f_B^+(xy) \le \min(f_A^+(x), f_A^+(y))$,

for all $x, y \in X$. Note that B is called symmetric relation on A.

We now introduce the notion of an interval-valued neutrosophic line graph as a generalization of an interval-valued fuzzy line graph [1].

Definition 2.3. *Let $L(\hat{G}) = (X, Y)$ be line graph of the crisp graph $\hat{G} = (V, E)$. Let $A_1 = ([t_{A_1}^-, t_{A_1}^+], [i_{A_1}^-, i_{A_1}^+], [f_{A_1}^-, f_{A_1}^+])$ and $B_1 = ([t_{B_1}^-, t_{B_1}^+], [i_{B_1}^-, i_{B_1}^+], [f_{B_1}^-, f_{B_1}^+])$ be interval-valued neutrosophic sets on V and E, $A_2 = ([t_{A_2}^-, t_{A_2}^+], [i_{A_2}^-, i_{A_2}^+], [f_{A_2}^-, f_{A_2}^+])$ and $B_2 = ([t_{B_2}^-, t_{B_2}^+], [i_{B_2}^-, i_{B_2}^+], [f_{B_2}^-, f_{B_2}^+])$ on X and Y, respectively. Then an interval-valued neutrosophic line graph of the interval-valued neutrosophic graph $G = (A_1, B_1)$ is an interval-valued neutrosophic graph $L(G) = (A_2, B_2)$ such that*

(i) $t^-_{A_2}(S_x) = t^-_{B_1}(x) = t^-_{B_1}(u_xv_x),\ \ t^+_{A_2}(S_x) = t^+_{B_1}(x) = t^+_{B_1}(u_xv_x),$

(ii) $i^-_{A_2}(S_x) = i^-_{B_1}(x) = i^-_{B_1}(u_xv_x),\ \ i^+_{A_2}(S_x) = i^+_{B_1}(x) = i^+_{B_1}(u_xv_x),$

(iii) $f^-_{A_2}(S_x) = f^-_{B_1}(x) = f^-_{B_1}(u_xv_x),\ \ f^+_{A_2}(S_x) = f^+_{B_1}(x) = f^+_{B_1}(u_xv_x),$

(iv) $t^-_{B_2}(S_xS_y) = \min(t^-_{B_1}(x), t^-_{B_1}(y)),\ \ t^+_{B_2}(S_xS_y) = \min(t^+_{B_1}(x), t^+_{B_1}(y)),$

(v) $i^-_{B_2}(S_xS_y) = \min(i^-_{B_1}(x), i^-_{B_1}(y)),\ \ i^+_{B_2}(S_xS_y) = \min(i^+_{B_1}(x), i^+_{B_1}(y)),$

(vi) $f^-_{B_2}(S_xS_y) = \min(f^-_{B_1}(x), f^-_{B_1}(y)),\ \ f^+_{B_2}(S_xS_y) = \min(f^+_{B_1}(x), f^+_{B_1}(y)),$

for all $S_x, S_y \in X$, $S_xS_y \in Y$.

Proposition 2.4. *$L(G) = (A_2, B_2)$ is an interval-valued neutrosophic line graph of some interval-valued neutrosophic graph $G = (A_1, B_1)$ if and only if*

$$t^-_{B_2}(S_xS_y) = \min(t^-_{A_2}(S_x), t^-_{A_2}(S_y)),\ \ t^+_{B_2}(S_xS_y) = \min(t^+_{A_2}(S_x), t^+_{A_2}(S_y)),$$

$$i^-_{B_2}(S_xS_y) = \min(i^-_{A_2}(S_x), i^-_{A_2}(S_y)),\ \ i^+_{B_2}(S_xS_y) = \min(i^+_{A_2}(S_x), i^+_{A_2}(S_y)),$$

$$f^-_{B_2}(S_xS_y) = \min(f^-_{A_2}(S_x), f^-_{A_2}(S_y)),\ \ f^+_{B_2}(S_xS_y) = \min(f^+_{A_2}(S_x), f^+_{A_2}(S_y)),$$

$\forall S_x, S_y \in Y$.

Proof. By using similar arguments as used in the proof of Proposition 3.7 of [1], the proof is straightforward. □

Definition 2.5. *Consider two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$. A mapping $\varphi : V_1 \to V_2$ is called homomorphism $\varphi : G_1 \to G_2$ if*

$$(a) \begin{cases} t^-_{A_1}(x_1) \le t^-_{A_2}(\varphi(x_1)), & i^-_{A_1}(x_1) \le i^-_{A_2}(\varphi(x_1)), \quad f^-_{A_1}(x_1) \le f^-_{A_2}(\varphi(x_1)), \\ t^+_{A_1}(x_1) \le t^+_{A_2}(\varphi(x_1)), & i^+_{A_1}(x_1) \le i^+_{A_2}(\varphi(x_1)), \quad f^+_{A_1}(x_1) \le f^+_{A_2}(\varphi(x_1)), \end{cases}$$

$$(b) \begin{cases} t^-_{B_1}(x_1y_1) \le t^-_{B_2}(\varphi(x_1)\varphi(y_1)), & t^+_{B_1}(x_1y_1) \le t^+_{B_2}(\varphi(x_1)\varphi(y_1)), \\ i^-_{B_1}(x_1y_1) \le i^-_{B_2}(\varphi(x_1)\varphi(y_1)), & i^+_{B_1}(x_1y_1) \le i^+_{B_2}(\varphi(x_1)\varphi(y_1)), \\ f^-_{B_1}(x_1y_1) \le f^-_{B_2}(\varphi(x_1)\varphi(y_1)), & f^+_{B_1}(x_1y_1) \le f^+_{B_2}(\varphi(x_1)\varphi(y_1)), \end{cases}$$

for all $x_1 \in V_1$, $x_1y_1 \in E_1$. Weak vertex-isomorphism of interval-valued neutrosophic graphs is a bijective homomorphism $\varphi : G_1 \to G_2$, such that

$$(c) \begin{cases} t^-_{A_1}(x_1) = t^-_{A_2}(\varphi(x_1)), & i^-_{A_1}(x_1) = i^-_{A_2}(\varphi(x_1)), \quad f^-_{A_1}(x_1) = f^-_{A_2}(\varphi(x_1)), \\ t^+_{A_1}(x_1) = t^+_{A_2}(\varphi(x_1)), & i^+_{A_1}(x_1) = i^+_{A_2}(\varphi(x_1)), \quad f^+_{A_1}(x_1) = f^+_{A_2}(\varphi(x_1)), \end{cases}$$

for all $x_1 \in V_1$ and $\varphi : G_1 \to G_2$ is called weak line-isomorphism if

$$(d) \begin{cases} t^-_{B_1}(x_1y_1) = t^-_{B_2}(\varphi(x_1)\varphi(y_1)), & t^+_{B_1}(x_1y_1) = t^+_{B_2}(\varphi(x_1)\varphi(y_1)), \\ i^-_{B_1}(x_1y_1) = i^-_{B_2}(\varphi(x_1)\varphi(y_1)), & i^+_{B_1}(x_1y_1) = i^+_{B_2}(\varphi(x_1)\varphi(y_1)), \\ f^-_{B_1}(x_1y_1) = f^-_{B_2}(\varphi(x_1)\varphi(y_1)), & f^+_{B_1}(x_1y_1) = f^+_{B_2}(\varphi(x_1)\varphi(y_1)), \end{cases}$$

for all $x_1y_1 \in V_1$. Weak isomorphism $\varphi : G_1 \to G_2$ of two interval-valued neutrosophic graphs G_1 and G_2 is bijective homomorphism and satisfies (c) and (d). Weak isomorphism may not preserve the weights of the edges but preserves the weights of vertices .

Theorem 2.6. *Let $L(G) = (A_2, B_2)$ be an interval-valued neutrosophic line graph corresponding to an interval-valued neutrosophic graph $G = (A_1, B_1)$. Suppose that $\hat{G} = (V, E)$ is a connected graph. Then*

(i) *there is a week isomorphism between G and L(G) if and only if $\hat{G}$ is a cyclic graph and $\forall\, v \in V$, $x \in E$,*

$$t^-_{A_1}(v) = t^-_{B_1}(x),\ i^-_{A_1}(v) = i^-_{B_1}(x),\ f^-_{A_1}(v) = f^-_{B_1}(x),$$
$$t^+_{A_1}(v) = t^+_{B_1}(x),\ i^+_{A_1}(v) = i^+_{B_1}(x),\ f^+_{A_1}(v) = f^+_{B_1}(x),$$

i.e., $A_1 = ([t^-_{A_1}, t^+_{A_1}], [i^-_{A_1}, i^+_{A_1}], [f^-_{A_1}, f^+_{A_1}])$ and $B_1 = ([t^-_{B_1}, t^+_{B_1}], [i^-_{B_1}, i^+_{B_1}], [f^-_{B_1}, f^+_{B_1}])$ are constant functions on the sets V and E, respectively, taking on same value.

(ii) *If φ is a weak isomorphism between G and L(G), then φ is an isomorphism.*

Proof. By using similar arguments as used in the proof of Theorem 3.11 of [1], the proof is straightforward. □

3. Operations on I-V Neutrosophic Graph Structures

In this section, we describe certain methods of construction of new interval-valued neutrosophic graph structures from old one.

Definition 3.1. *$\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is called an interval-valued neutrosophic graph structure(IVNGS) of graph structure $G_s = (U, U_1, U_2, \ldots, U_t)$ if $I =< r, [t^-(r), t^+(r)], [i^-(r), i^+(r)], [f^-(r), f^+(r)] >$ and $I_j =< rs, [t^-(rs), t^+(rs)], [i^-(rs), i^+(rs)], [f^-(rs), f^+(rs)] >$ are interval-valued neutrosophic(IVN) sets on U and U_j, respectively, such that:*

1. $t^-_j(rs) \le \min\{t^-(r), t^-(s)\}$, $\quad t^+_j(rs) \le \min\{t^+(r), t^+(s)\}$,
2. $i^-_j(rs) \le \min\{i^-(r), i^-(s)\}$, $\quad i^+_j(rs) \le \min\{i^+(r), i^+(s)\}$,
3. $f^-_j(rs) \le \min\{f^-(r), f^-(s)\}$, $\quad f^+_j(rs) \le \min\{f^+(r), f^+(s)\}$,

for all $r, s \in U$. Note that $0 \le t_j(rs) + i_j(rs) + f_j(rs) \le 3$, for all $(rs) \in U_j$, $j = 1, 2, \ldots, t$.

Example 3.2. *Consider graph structure $G_s = (U, U_1, U_2)$ such that $U = \{r_1, r_2, r_3, r_4, r_5, r_6\}$, $U_1 = \{r_1r_4, r_2r_4, r_2r_6\}$, $U_2 = \{r_1r_5, r_3r_5, r_3r_6\}$. Let I be an IVN set on U given in Table. 1 and I_1, I_2 be IVN sets on U_1, U_2, respectively given in Table. 2. and Table. 3.*

Table 1: IVN set I on vertex set U

I	r_1	r_2	r_3	r_4	r_5	r_6
t^-	0.2	0.3	0.3	0.1	0.3	0.3
t^+	0.3	0.4	0.4	0.2	0.4	0.4
i^-	0.3	0.4	0.4	0.4	0.4	0.1
i^+	0.4	0.5	0.5	0.5	0.5	0.2
f^-	0.4	0.3	0.3	0.2	0.3	0.3
f^+	0.5	0.4	0.4	0.3	0.4	0.4

Table 2: IVN set I_1 on set U_1

I_1	r_1r_4	r_2r_4	r_2r_6
t^-	0.1	0.1	0.3
t^+	0.2	0.2	0.4
i^-	0.3	0.4	0.1
i^+	0.4	0.5	0.2
f^-	0.2	0.2	0.3
f^+	0.3	0.3	0.4

Table 3: IVN set I_2 on set U_2

I_2	r_1r_5	r_3r_5	r_3r_6
t^-	0.2	0.3	0.3
t^+	0.3	0.4	0.4
i^-	0.3	0.4	0.1
i^+	0.4	0.5	0.2
f^-	0.3	0.3	0.3
f^+	0.4	0.4	0.4

Routine calculations show that $\check{G}_{iv} = (I, I_1, I_2)$ is an interval-valued neutrosophic graph structure, as shown in Fig. 1.

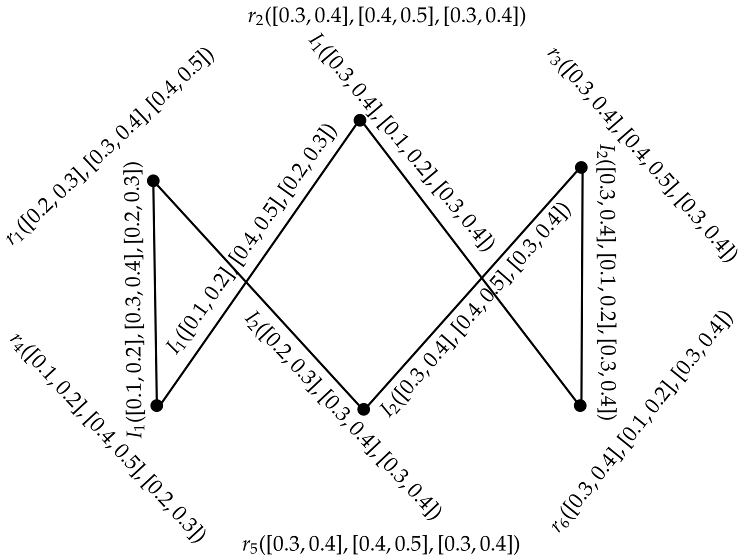

Figure 1: An IVNGS $\check{G}_{iv} = (I, I_1, I_2)$

We now present some operations on IVNGSs.

Definition 3.3. *Let $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ and $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ be two IVNGSs of graph structures $G_{s1} = (U_1, U_{11}, U_{12}, \ldots, U_{1t})$ and $G_{s2} = (U_2, U_{21}, U_{22}, \ldots, U_{2t})$, respectively.*
Cartesian product of $\check{G}_{iv1}$ and $\check{G}_{iv2}$, denoted by

$$\check{G}_{iv1} \times \check{G}_{iv2} = (I_1 \times I_2, I_{11} \times I_{21}, I_{12} \times I_{22}, \ldots, I_{1t} \times I_{2t}),$$

is defined as:

$$\text{(i)} \begin{cases} t^-_{(I_1\times I_2)}(rs) = (t^-_{I_1} \times t^-_{I_2})(rs) = t^-_{I_1}(r) \wedge t^-_{I_2}(s) \\ i^-_{(I_1\times I_2)}(rs) = (i^-_{I_1} \times i^-_{I_2})(rs) = i^-_{I_1}(r) \wedge i^-_{I_2}(s) \\ f^-_{(I_1\times I_2)}(rs) = (f^-_{I_1} \times f^-_{I_2})(rs) = f^-_{I_1}(r) \wedge f^-_{I_2}(s) \end{cases}$$

$$\text{(ii)} \begin{cases} t^+_{(I_1\times I_2)}(rs) = (t^+_{I_1} \times t^+_{I_2})(rs) = t^+_{I_1}(r) \wedge t^+_{I_2}(s) \\ i^+_{(I_1\times I_2)}(rs) = (i^+_{I_1} \times i^+_{I_2})(rs) = i^+_{I_1}(r) \wedge i^+_{I_2}(s) \\ f^+_{(I_1\times I_2)}(rs) = (f^+_{I_1} \times f^+_{I_2})(rs) = f^+_{I_1}(r) \wedge f^+_{I_2}(s) \end{cases}$$

for all $rs \in U_1 \times U_2$,

$$\text{(iii)} \begin{cases} t^-_{(I_{1j}\times I_{2j})}(rs_1)(rs_2) = (t^-_{I_{1j}} \times t^-_{I_{2j}})(rs_1)(rs_2) = t^-_{I_1}(r) \wedge t^-_{I_{2j}}(s_1s_2) \\ i^-_{(I_{1j}\times I_{2j})}(rs_1)(rs_2) = (i^-_{I_{1j}} \times i^-_{I_{2j}})(rs_1)(rs_2) = i^-_{I_1}(r) \wedge i^-_{I_{2j}}(s_1s_2) \\ f^-_{(I_{1j}\times I_{2j})}(rs_1)(rs_2) = (f^-_{I_{1j}} \times f^-_{I_{2j}})(rs_1)(rs_2) = f^-_{I_1}(r) \wedge f^-_{I_{2j}}(s_1s_2) \end{cases}$$

$$\text{(iv)} \begin{cases} t^+_{(I_{1j}\times I_{2j})}(rs_1)(rs_2) = (t^+_{I_{1j}} \times t^+_{I_{2j}})(rs_1)(rs_2) = t^+_{I_1}(r) \wedge t^+_{I_{2j}}(s_1s_2) \\ i^+_{(I_{1j}\times I_{2j})}(rs_1)(rs_2) = (i^+_{I_{1j}} \times i^+_{I_{2j}})(rs_1)(rs_2) = i^+_{I_1}(r) \wedge i^+_{I_{2j}}(s_1s_2) \\ f^+_{(I_{1j}\times I_{2j})}(rs_1)(rs_2) = (f^+_{I_{1j}} \times f^+_{I_{2j}})(rs_1)(rs_2) = f^+_{I_1}(r) \wedge f^+_{I_{2j}}(s_1s_2) \end{cases}$$

for all $r \in U_1$, $s_1s_2 \in U_{2j}$,

$$\text{(v)} \begin{cases} t^-_{(I_{1j}\times I_{2j})}(r_1s)(r_2s) = (t^-_{I_{1j}} \times t^-_{I_{2j}})(r_1s)(r_2s) = t^-_{I_2}(s) \wedge t^-_{I_{1j}}(r_1r_2) \\ i^-_{(I_{1j}\times I_{2j})}(r_1s)(r_2s) = (i^-_{I_{1j}} \times i^-_{I_{2j}})(r_1s)(r_2s) = i^-_{I_2}(s) \wedge i^-_{I_{2j}}(r_1r_2) \\ f^-_{(I_{1j}\times I_{2j})}(r_1s)(r_2s) = (f^-_{I_{1j}} \times f^-_{I_{2j}})(r_1s)(r_2s) = f^-_{I_2}(s) \wedge f^-_{I_{2j}}(r_1r_2) \end{cases}$$

(vi) $\begin{cases} t^{+}_{(I_{1j}\times I_{2j})}(r_1s)(r_2s) = (t^{+}_{I_{1j}} \times t^{+}_{I_{2j}})(r_1s)(r_2s) = t^{+}_{I_2}(s) \wedge t^{+}_{I_{1j}}(r_1r_2) \\ i^{+}_{(I_{1j}\times I_{2j})}(r_1s)(r_2s) = (i^{+}_{I_{1j}} \times i^{+}_{I_{2j}})(r_1s)(r_2s) = i^{+}_{I_2}(s) \wedge i^{+}_{I_{2j}}(r_1r_2) \\ f^{+}_{(I_{1j}\times I_{2j})}(r_1s)(r_2s) = (f^{+}_{I_{1j}} \times f^{+}_{I_{2j}})(r_1s)(r_2s) = f^{+}_{I_2}(s) \wedge f^{+}_{I_{2j}}(r_1r_2) \end{cases}$
for all $s \in U_2$, $r_1r_2 \in U_{1j}$, $j = 1, 2, \ldots, t$.

Example 3.4. *Consider* $\check{G}_{iv1} = (I_1, I_{11}, I_{12})$ *and* $\check{G}_{iv2} = (I_2, I_{21}, I_{22})$ *are two IVNGSs of graph structures* $\check{G}_{s1} = (U_1, U_{11}, U_{12})$ *and* $\check{G}_{s2} = (U_2, U_{21}, U_{22})$*, respectively, as shown in Fig. 2, where* $U_{11} = \{r_1r_2\}$, $U_{12} = \{r_3r_4\}$, $U_{21} = \{s_1s_2\}$, $U_{22} = \{s_2s_3\}$.

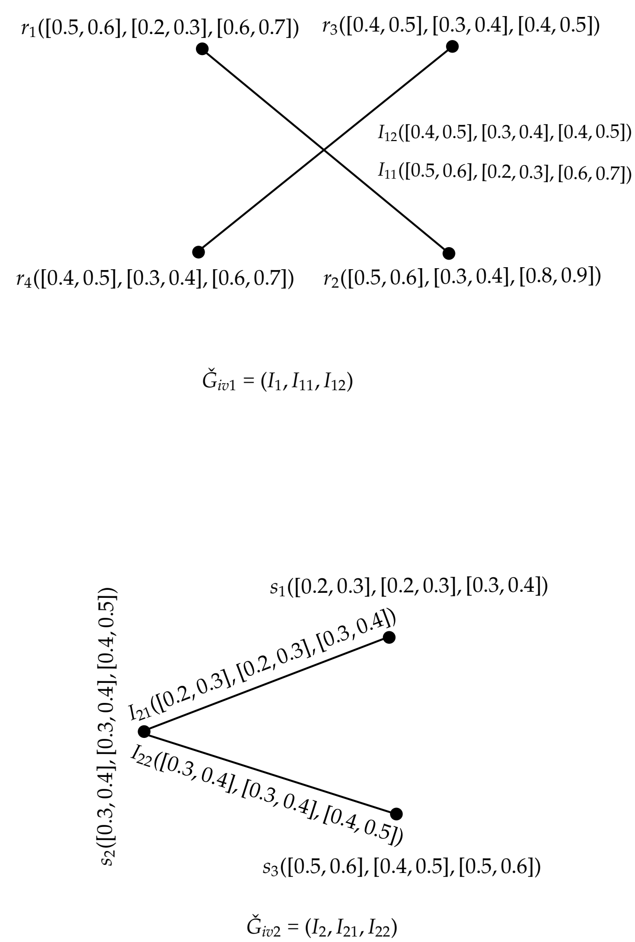

Figure 2: Two IVNGSs $\check{G}_{iv1}$ and $\check{G}_{iv2}$

Cartesian product of $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$ *defined as* $\check{G}_{iv1} \times \check{G}_{iv2} = \{I_1 \times I_2, I_{11} \times I_{21}, I_{12} \times I_{22}\}$ *is shown in Fig. 3 and Fig. 4.*

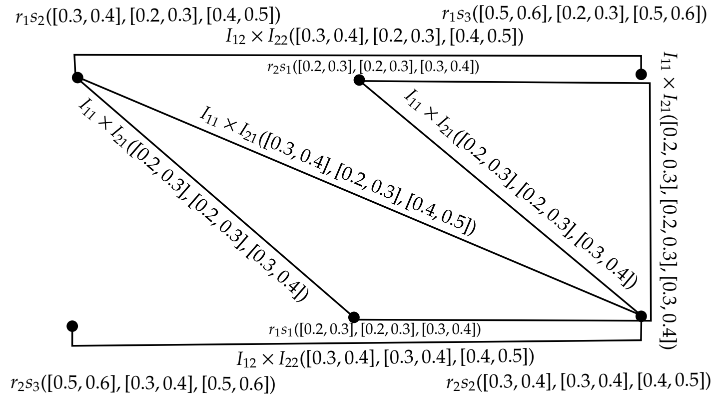

Figure 3: $\check{G}_{iv1} \times \check{G}_{iv2}$

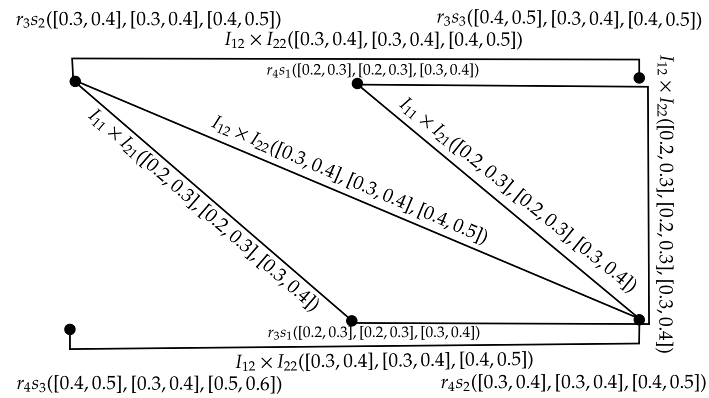

Figure 4: $\check{G}_{iv1} \times \check{G}_{iv2}$

Theorem 3.5. *Cartesian product* $\check{G}_{iv1} \times \check{G}_{iv2} = (I_1 \times I_2, I_{11} \times I_{21}, I_{12} \times I_{22}, \ldots, I_{1t} \times I_{2t})$ *of two IVNGSs* $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$ *of graph structures* G_{s1} *and* G_{s2}*, respectively is an IVNGS of* $G_{s1} \times G_{s2}$.

Definition 3.6. *Let* $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ *and* $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ *be two IVNGSs.*
Cross product of $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$*, denoted by*

$$\check{G}_{iv1} * \check{G}_{iv2} = (I_1 * I_2, I_{11} * I_{21}, I_{12} * I_{22}, \ldots, I_{1t} * I_{2t}),$$

is defined as:

$$\text{(i)} \begin{cases} t^-_{(I_1 * I_2)}(rs) = (t^-_{I_1} * t^-_{I_2})(rs) = t^-_{I_1}(r) \wedge t^-_{I_2}(s) \\ i^-_{(I_1 * I_2)}(rs) = (i^-_{I_1} * i^-_{I_2})(rs) = i^-_{I_1}(r) \wedge i^-_{I_2}(s) \\ f^-_{(I_1 * I_2)}(rs) = (f^-_{I_1} * f^-_{I_2})(rs) = f^-_{I_1}(r) \wedge f^-_{I_2}(s) \end{cases}$$

$$\text{(ii)} \begin{cases} t^+_{(I_1 * I_2)}(rs) = (t^+_{I_1} * t^+_{I_2})(rs) = t^+_{I_1}(r) \wedge t^+_{I_2}(s) \\ i^+_{(I_1 * I_2)}(rs) = (i^+_{I_1} * i^+_{I_2})(rs) = i^+_{I_1}(r) \wedge i^+_{I_2}(s) \\ f^+_{(I_1 * I_2)}(rs) = (f^+_{I_1} * f^+_{I_2})(rs) = f^+_{I_1}(r) \wedge f^-_{I_2}(s) \end{cases}$$

for all $(rs) \in U_1 \times U_2$,

(iii) $\begin{cases} t^-_{(I_{1j}*I_{2j})}(r_1s_1)(r_2s_2) = (t^-_{I_{1j}} * t^-_{I_{2j}})(r_1s_1)(r_2s_2) = t^-_{I_{1j}}(r_1r_2) \wedge t^-_{I_{2j}}(s_1s_2) \\ i^-_{(I_{1j}*I_{2j})}(r_1s_1)(r_2s_2) = (i^-_{I_{1j}} * i^-_{I_{2j}})(r_1s_1)(r_2s_2) = i^-_{I_{1j}}(r_1r_2) \wedge i^-_{I_{2j}}(s_1s_2) \\ f^-_{(I_{1j}*I_{2j})}(r_1s_1)(r_2s_2) = (f^-_{I_{1j}} * f^-_{I_{2j}})(r_1s_1)(r_2s_2) = f^-_{I_{1j}}(r_1r_2) \wedge f^-_{I_{2j}}(s_1s_2) \end{cases}$

(iv) $\begin{cases} t^+_{(I_{1j}*I_{2j})}(r_1s_1)(r_2s_2) = (t^+_{I_{1j}} * t^+_{I_{2j}})(r_1s_1)(r_2s_2) = t^+_{I_{1j}}(r_1r_2) \wedge t^+_{I_{2j}}(s_1s_2) \\ i^+_{(I_{1j}*I_{2j})}(r_1s_1)(r_2s_2) = (i^+_{I_{1j}} * i^+_{I_{2j}})(r_1s_1)(r_2s_2) = i^+_{I_{1j}}(r_1r_2) \wedge i^+_{I_{2j}}(s_1s_2) \\ f^+_{(I_{1j}*I_{2j})}(r_1s_1)(r_2s_2) = (f^+_{I_{1j}} * f^+_{I_{2j}})(r_1s_1)(r_2s_2) = f^+_{I_{1j}}(r_1r_2) \wedge f^+_{I_{2j}}(s_1s_2) \end{cases}$

for all $r_1r_2 \in U_{1j}$, $s_1s_2 \in U_{2j}$.

Example 3.7. *Cross product of IVNGSs* $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$ *shown in Fig. 2 is defined as* $\check{G}_{iv1} * \check{G}_{iv2} = \{I_1 * I_2, I_{11} * I_{21}, I_{12} * I_{22}\}$ *and is shown in Fig. 5 and Fig. 6.*

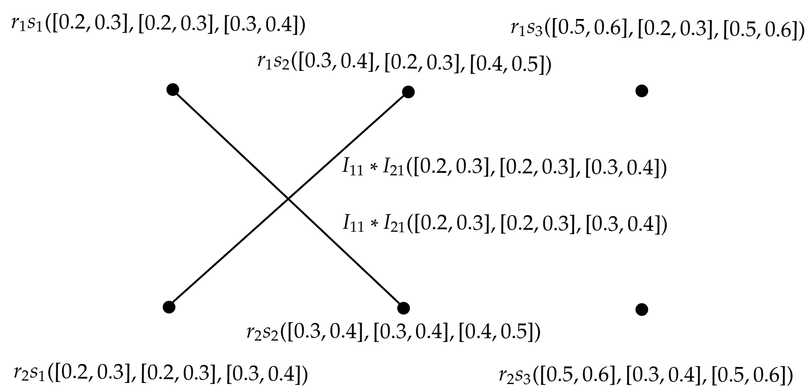

Figure 5: $\check{G}_{iv1} * \check{G}_{iv2}$

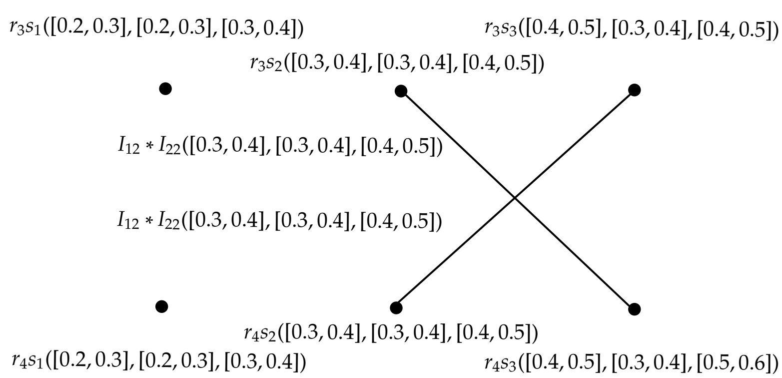

Figure 6: $\check{G}_{iv1} * \check{G}_{iv2}$

Theorem 3.8. *Cross product* $\check{G}_{iv1} * \check{G}_{iv2} = (I_1 * I_2, I_{11} * I_{21}, I_{12} * I_{22}, \ldots, I_{1t} * I_{2t})$ *of two IVNGSs of graph structures* G_{s1} *and* G_{s2} *is an IVNGS of* $G_{s1} * G_{s2}$.

Definition 3.9. *Let* $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ *and* $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ *be two IVNGSs. Lexicographic product of* $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$*, denoted by*

$$\check{G}_{iv1} \bullet \check{G}_{iv2} = (I_1 \bullet I_2, I_{11} \bullet I_{21}, I_{12} \bullet I_{22}, \ldots, I_{1t} \bullet I_{2t}),$$

is defined as:

$$\text{(i)}\begin{cases} t^-_{(I_1\bullet I_2)}(rs)=(t^-_{I_1}\bullet t^-_{I_2})(rs)=t^-_{I_1}(r)\wedge t^-_{I_2}(s)\\ i^-_{(I_1\bullet I_2)}(rs)=(i^-_{I_1}\bullet i^-_{I_2})(rs)=i^-_{I_1}(r)\wedge i^-_{I_2}(s)\\ f^-_{(I_1\bullet I_2)}(rs)=(f^-_{I_1}\bullet f^-_{I_2})(rs)=f^-_{I_1}(r)\wedge f^-_{I_2}(s)\end{cases}$$

$$\text{(ii)}\begin{cases} t^+_{(I_1\bullet I_2)}(rs)=(t^+_{I_1}\bullet t^+_{I_2})(rs)=t^+_{I_1}(r)\wedge t^+_{I_2}(s)\\ i^+_{(I_1\bullet I_2)}(rs)=(i^+_{I_1}\bullet i^+_{I_2})(rs)=i^+_{I_1}(r)\wedge i^+_{I_2}(s)\\ f^+_{(I_1\bullet I_2)}(rs)=(f^+_{I_1}\bullet f^+_{I_2})(rs)=f^+_{I_1}(r)\wedge f^+_{I_2}(s)\end{cases}$$

for all $rs\in U_1\times U_2$,

$$\text{(iii)}\begin{cases} t^-_{(I_{1j}\bullet I_{2j})}(rs_1)(rs_2)=(t^-_{I_{1j}}\bullet t^-_{I_{2j}})(rs_1)(rs_2)=t^-_{I_1}(r)\wedge t^-_{I_{2j}}(s_1s_2)\\ i^-_{(I_{1j}\bullet I_{2j})}(rs_1)(rs_2)=(i^-_{I_{1j}}\bullet i^-_{I_{2j}})(rs_1)(rs_2)=i^-_{I_1}(r)\wedge i^-_{I_{2j}}(s_1s_2)\\ f^-_{(I_{1j}\bullet I_{2j})}(rs_1)(rs_2)=(f^-_{I_{1j}}\bullet f^-_{I_{2j}})(rs_1)(rs_2)=f^-_{I_1}(r)\wedge f^-_{I_{2j}}(s_1s_2)\end{cases}$$

$$\text{(iv)}\begin{cases} t^+_{(I_{1j}\bullet I_{2j})}(rs_1)(rs_2)=(t^+_{I_{1j}}\bullet t^+_{I_{2j}})(rs_1)(rs_2)=t^+_{I_1}(r)\wedge t^+_{I_{2j}}(s_1s_2)\\ i^+_{(I_{1j}\bullet I_{2j})}(rs_1)(rs_2)=(i^+_{I_{1j}}\bullet i^+_{I_{2j}})(rs_1)(rs_2)=i^+_{I_1}(r)\wedge i^+_{I_{2j}}(s_1s_2)\\ f^+_{(I_{1j}\bullet I_{2j})}(rs_1)(rs_2)=(f^+_{I_{1j}}\bullet f^+_{I_{2j}})(rs_1)(rs_2)=f^+_{I_1}(r)\wedge f^+_{I_{2j}}(s_1s_2)\end{cases}$$

for all $r\in U_1$, $s_1s_2\in U_{2j}$,

$$\text{(v)}\begin{cases} t^-_{(I_{1j}\bullet I_{2j})}(r_1s_1)(r_2s_2)=(t^-_{I_{1j}}\bullet t^-_{I_{2j}})(r_1s_1)(r_2s_2)=t^-_{I_{1j}}(r_1r_2)\wedge t^-_{I_{2j}}(s_1s_2)\\ i^-_{(I_{1j}\bullet I_{2j})}(r_1s_1)(r_2s_2)=(i^-_{I_{1j}}\bullet i^-_{I_{2j}})(r_1s_1)(r_2s_2)=i^-_{I_{1j}}(r_1r_2)\wedge i^-_{I_{2j}}(s_1s_2)\\ f^-_{(I_{1j}\bullet I_{2j})}(r_1s_1)(r_2s_2)=(f^-_{I_{1j}}\bullet f^-_{I_{2j}})(r_1s_1)(r_2s_2)=f^-_{I_{1j}}(r_1r_2)\wedge f^-_{I_{2j}}(s_1s_2)\end{cases}$$

$$\text{(vi)}\begin{cases} t^+_{(I_{1j}\bullet I_{2j})}(r_1s_1)(r_2s_2)=(t^+_{I_{1j}}\bullet t^+_{I_{2j}})(r_1s_1)(r_2s_2)=t^+_{I_{1j}}(r_1r_2)\wedge t^+_{I_{2j}}(s_1s_2)\\ i^+_{(I_{1j}\bullet I_{2j})}(r_1s_1)(r_2s_2)=(i^+_{I_{1j}}\bullet i^+_{I_{2j}})(r_1s_1)(r_2s_2)=i^+_{I_{1j}}(r_1r_2)\wedge i^+_{I_{2j}}(s_1s_2)\\ f^+_{(I_{1j}\bullet I_{2j})}(r_1s_1)(r_2s_2)=(f^+_{I_{1j}}\bullet f^+_{I_{2j}})(r_1s_1)(r_2s_2)=f^+_{I_{1j}}(r_1r_2)\wedge f^+_{I_{2j}}(s_1s_2)\end{cases}$$

for all $r_1r_2\in U_{1j}$, $s_1s_2\in U_{2j}$.

Example 3.10. *Lexicographic product of IVNGSs* $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$ *shown in Fig. 2 is defined as* $\check{G}_{iv1}\bullet\check{G}_{iv2}=\{I_1\bullet I_2, I_{11}\bullet I_{21}, I_{12}\bullet I_{22}\}$ *and is depicted in Fig. 7 and Fig. 8.*

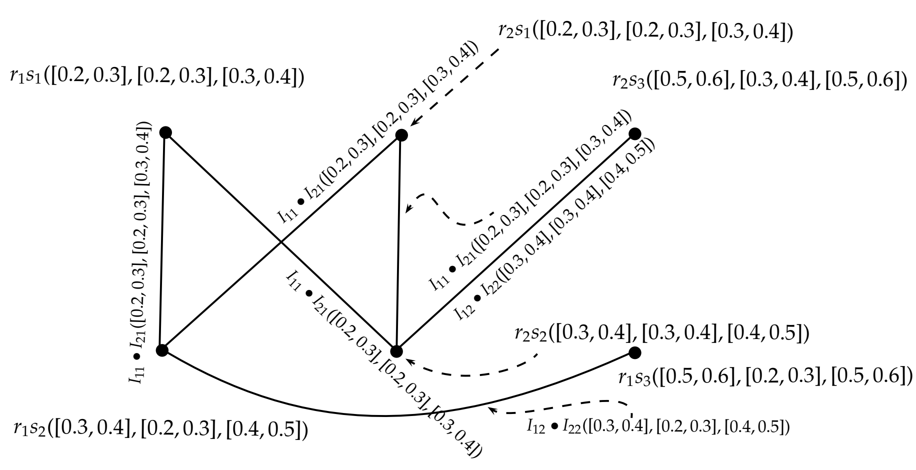

Figure 7: $\check{G}_{iv1}\bullet\check{G}_{iv2}$

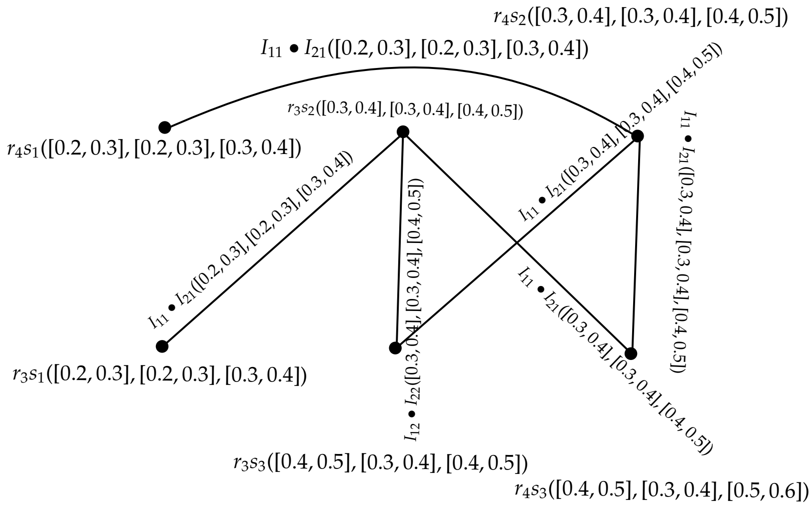

Figure 8: $\check{G}_{iv1} \bullet \check{G}_{iv2}$

Theorem 3.11. *Lexicographic product $\check{G}_{iv1} \bullet \check{G}_{iv2} = (I_1 \bullet I_2, I_{11} \bullet I_{21}, I_{12} \bullet I_{22}, \ldots, I_{1t} \bullet I_{2t})$ of two IVNGSs of graph structures G_{s1} and G_{s2} is an IVNGS of $G_{s1} \bullet G_{s2}$.*

Proof. Consider two cases:

Case 1. For $r \in U_1$, $s_1s_2 \in U_{2j}$,

$$\begin{aligned} t^-_{(I_{1j}\bullet I_{2j})}((rs_1)(rs_2)) &= t^-_{I_1}(r) \wedge t^-_{I_{2j}}(s_1s_2) \\ &\leq t^-_{I_1}(r) \wedge [t^-_{I_2}(s_1) \wedge t^-_{I_2}(s_2)] \\ &= [t^-_{I_1}(r) \wedge t^-_{I_2}(s_1)] \wedge [t^-_{I_1}(r) \wedge t^-_{I_2}(s_2)] \\ &= t^-_{(I_1\bullet I_2)}(rs_1) \wedge t^-_{(I_1\bullet I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} t^+_{(I_{1j}\bullet I_{2j})}((rs_1)(rs_2)) &= t^+_{I_1}(r) \wedge t^+_{I_{2j}}(s_1s_2) \\ &\leq t^+_{I_1}(r) \wedge [t^+_{I_2}(s_1) \wedge t^+_{I_2}(s_2)] \\ &= [t^+_{I_1}(r) \wedge t^+_{I_2}(s_1)] \wedge [t^+_{I_1}(r) \wedge t^+_{I_2}(s_2)] \\ &= t^+_{(I_1\bullet I_2)}(rs_1) \wedge t^+_{(I_1\bullet I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} i^-_{(I_{1j}\bullet I_{2j})}((rs_1)(rs_2)) &= i^-_{I_1}(r) \wedge i^-_{I_{2j}}(s_1s_2) \\ &\leq i^-_{I_1}(r) \wedge [i^-_{I_2}(s_1) \wedge i^-_{I_2}(s_2)] \\ &= [i^-_{I_1}(r) \wedge i^-_{I_2}(s_1)] \wedge [i^-_{I_1}(r) \wedge i^-_{I_2}(s_2)] \\ &= i^-_{(I_1\bullet I_2)}(rs_1) \wedge i^-_{(I_1\bullet I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} i^+_{(I_{1j}\bullet I_{2j})}((rs_1)(rs_2)) &= i^+_{I_1}(r) \wedge i^+_{I_{2j}}(s_1s_2) \\ &\leq i^+_{I_1}(r) \wedge [i^+_{I_2}(s_1) \wedge i^+_{I_2}(s_2)] \\ &= [i^+_{I_1}(r) \wedge i^+_{I_2}(s_1)] \wedge [i^+_{I_1}(r) \wedge i^+_{I_2}(s_2)] \\ &= i^+_{(I_1\bullet I_2)}(rs_1) \wedge i^+_{(I_1\bullet I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} f^-_{(I_{1j}\bullet I_{2j})}((rs_1)(rs_2)) &= f^-_{I_1}(r) \wedge f^-_{I_{2j}}(s_1s_2) \\ &\leq f^-_{I_1}(r) \wedge [f^-_{I_2}(s_1) \wedge f^-_{I_2}(s_2)] \\ &= [f^-_{I_1}(r) \wedge f^-_{I_2}(s_1)] \wedge [f^-_{I_1}(r) \wedge f^-_{I_2}(s_2)] \\ &= f^-_{(I_1\bullet I_2)}(rs_1) \wedge f^-_{(I_1\bullet I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} f^+_{(I_{1j}\bullet I_{2j})}((rs_1)(rs_2)) &= f^+_{I_1}(r) \wedge f^+_{I_{2j}}(s_1s_2) \\ &\leq f^+_{I_1}(r) \wedge [f^+_{I_2}(s_1) \wedge f^+_{I_2}(s_2)] \\ &= [f^+_{I_1}(r) \wedge f^+_{I_2}(s_1)] \wedge [f^+_{I_1}(r) \wedge f^+_{I_2}(s_2)] \\ &= f^+_{(I_1\bullet I_2)}(rs_1) \wedge f^+_{(I_1\bullet I_2)}(rs_2), \end{aligned}$$

for $rs_1, rs_2 \in U_1 \bullet U_2$.

Case 2. For $r_1r_2 \in U_{1j}$, $s_1s_2 \in U_{2j}$,

$$\begin{aligned} t^-_{(I_{1j}\bullet I_{2j})}((r_1s_1)(r_2s_2)) &= t^-_{I_{1j}}(r_1r_2) \wedge t^-_{I_{2j}}(s_1s_2) \\ &\leq [t^-_{I_1}(r_1) \wedge t^-_{I_1}(r_2] \wedge [t^-_{I_2}(s_1) \wedge t^-_{I_2}(s_2)] \\ &= [t^-_{I_1}(r_1) \wedge t^-_{I_2}(s_1)] \wedge [t^-_{I_1}(r_2) \wedge t^-_{I_2}(s_2)] \\ &= t^-_{(I_1\bullet I_2)}(r_1s_1) \wedge t^-_{(I_1\bullet I_2)}(r_2s_2), \end{aligned}$$

$$\begin{aligned} t^+_{(I_{1j}\bullet I_{2j})}((r_1s_1)(r_2s_2)) &= t^+_{I_{1j}}(r_1r_2) \wedge t^+_{I_{2j}}(s_1s_2) \\ &\leq [t^+_{I_1}(r_1) \wedge t^+_{I_1}(r_2] \wedge [t^+_{I_2}(s_1) \wedge t^+_{I_2}(s_2)] \\ &= [t^+_{I_1}(r_1) \wedge t^+_{I_2}(s_1)] \wedge [t^+_{I_1}(r_2) \wedge t^+_{I_2}(s_2)] \\ &= t^+_{(I_1\bullet I_2)}(r_1s_1) \wedge t^+_{(I_1\bullet I_2)}(r_2s_2), \end{aligned}$$

$$\begin{aligned} i^-_{(I_{1j}\bullet I_{2j})}((r_1s_1)(r_2s_2)) &= i^-_{I_{1j}}(r_1r_2) \wedge i^-_{I_{2j}}(s_1s_2) \\ &\leq [i^-_{I_1}(r_1) \wedge i^-_{I_1}(r_2] \wedge [i^-_{I_2}(s_1) \wedge i^-_{I_2}(s_2)] \\ &= [i^-_{I_1}(r_1) \wedge i^-_{I_2}(s_1)] \wedge [i^-_{I_1}(r_2) \wedge i^-_{I_2}(s_2)] \\ &= i^-_{(I_1\bullet I_2)}(r_1s_1) \wedge i^-_{(I_1\bullet I_2)}(r_2s_2), \end{aligned}$$

$$\begin{aligned} i^+_{(I_{1j}\bullet I_{2j})}((r_1s_1)(r_2s_2)) &= i^+_{I_{1j}}(r_1r_2) \wedge i^+_{I_{2j}}(s_1s_2) \\ &\leq [i^+_{I_1}(r_1) \wedge i^+_{I_1}(r_2] \wedge [i^+_{I_2}(s_1) \wedge i^+_{I_2}(s_2)] \\ &= [i^+_{I_1}(r_1) \wedge i^+_{I_2}(s_1)] \wedge [i^+_{I_1}(r_2) \wedge i^+_{I_2}(s_2)] \\ &= i^+_{(I_1\bullet I_2)}(r_1s_1) \wedge i^+_{(I_1\bullet I_2)}(r_2s_2), \end{aligned}$$

$$\begin{aligned} f^-_{(I_{1j}\bullet I_{2j})}((r_1s_1)(r_2s_2)) &= f^-_{I_{1j}}(r_1r_2) \wedge f^-_{I_{2j}}(s_1s_2) \\ &\leq [f^-_{I_1}(r_1) \wedge f^-_{I_1}(r_2] \wedge [f^-_{I_2}(s_1) \wedge f^-_{I_2}(s_2)] \\ &= [f^-_{I_1}(r_1) \wedge f^-_{I_2}(s_1)] \wedge [f^-_{I_1}(r_2) \wedge f^-_{I_2}(s_2)] \\ &= f^-_{(I_1\bullet I_2)}(r_1s_1) \wedge f^-_{(I_1\bullet I_2)}(r_2s_2), \end{aligned}$$

$$\begin{aligned} f^{+}_{(I_{1j}\bullet I_{2j})}((r_1s_1)(r_2s_2)) &= f^{+}_{I_{1j}}(r_1r_2) \wedge f^{+}_{I_{2j}}(s_1s_2) \\ &\leq [f^{+}_{I_1}(r_1) \wedge f^{+}_{I_1}(r_2] \wedge [f^{+}_{I_2}(s_1) \wedge f^{+}_{I_2}(s_2)] \\ &= [f^{+}_{I_1}(r_1) \wedge f^{+}_{I_2}(s_1)] \wedge [f^{+}_{I_1}(r_2) \wedge f^{+}_{I_2}(s_2)] \\ &= f^{+}_{(I_1\bullet I_2)}(r_1s_1) \wedge f^{+}_{(I_1\bullet I_2)}(r_2s_2), \end{aligned}$$

$r_1s_1, r_2s_2 \in U_1 \bullet U_2$ and $j \in \{1, 2, \ldots, t\}$. This completes the proof.

□

Definition 3.12. *Let $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ and $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ be two IVNGSs. Strong product of $\check{G}_{iv1}$ and $\check{G}_{iv2}$, denoted by*

$$\check{G}_{iv1} \boxtimes \check{G}_{iv2} = (I_1 \boxtimes I_2, I_{11} \boxtimes I_{21}, I_{12} \boxtimes I_{22}, \ldots, I_{1t} \boxtimes I_{2t}),$$

is defined as:

$$\text{(i)} \begin{cases} t^{-}_{(I_1\boxtimes I_2)}(rs) = (t^{-}_{I_1} \boxtimes t^{-}_{I_2})(rs) = t^{-}_{I_1}(r) \wedge t^{-}_{I_2}(s) \\ i^{-}_{(I_1\boxtimes I_2)}(rs) = (i^{-}_{I_1} \boxtimes i^{-}_{I_2})(rs) = i^{-}_{I_1}(r) \wedge i^{-}_{I_2}(s) \\ f^{-}_{(I_1\boxtimes I_2)}(rs) = (f^{-}_{I_1} \boxtimes f^{-}_{I_2})(rs) = f^{-}_{I_1}(r) \wedge f^{-}_{I_2}(s) \end{cases}$$

$$\text{(ii)} \begin{cases} t^{+}_{(I_1\boxtimes I_2)}(rs) = (t^{+}_{I_1} \boxtimes t^{+}_{I_2})(rs) = t^{+}_{I_1}(r) \wedge t^{+}_{I_2}(s) \\ i^{+}_{(I_1\boxtimes I_2)}(rs) = (i^{+}_{I_1} \boxtimes i^{+}_{I_2})(rs) = i^{+}_{I_1}(r) \wedge i^{+}_{I_2}(s) \\ f^{+}_{(I_1\boxtimes I_2)}(rs) = (f^{+}_{I_1} \boxtimes f^{+}_{I_2})(rs) = f^{+}_{I_1}(r) \wedge f^{-}_{I_2}(s) \end{cases}$$

for all $rs \in U_1 \times U_2$,

$$\text{(iii)} \begin{cases} t^{-}_{(I_{1j}\boxtimes I_{2j})}(rs_1)(rs_2) = (t^{-}_{I_{1j}} \boxtimes t^{-}_{I_{2j}})(rs_1)(rs_2) = t^{-}_{I_1}(r) \wedge t^{-}_{I_{2j}}(s_1s_2) \\ i^{-}_{(I_{1j}\boxtimes I_{2j})}(rs_1)(rs_2) = (i^{-}_{I_{1j}} \boxtimes i^{-}_{I_{2j}})(rs_1)(rs_2) = i^{-}_{I_1}(r) \wedge i^{-}_{I_{2j}}(s_1s_2) \\ f^{-}_{(I_{1j}\boxtimes I_{2j})}(rs_1)(rs_2) = (f^{-}_{I_{1j}} \boxtimes f^{-}_{I_{2j}})(rs_1)(rs_2) = f^{-}_{I_1}(r) \wedge f^{-}_{I_{2j}}(s_1s_2) \end{cases}$$

$$\text{(iv)} \begin{cases} t^{+}_{(I_{1j}\boxtimes I_{2j})}(rs_1)(rs_2) = (t^{+}_{I_{1j}} \boxtimes t^{+}_{I_{2j}})(rs_1)(rs_2) = t^{+}_{I_1}(r) \wedge t^{+}_{I_{2j}}(s_1s_2) \\ i^{+}_{(I_{1j}\boxtimes I_{2j})}(rs_1)(rs_2) = (i^{+}_{I_{1j}} \boxtimes i^{+}_{I_{2j}})(rs_1)(rs_2) = i^{+}_{I_1}(r) \wedge i^{+}_{I_{2j}}(s_1s_2) \\ f^{+}_{(I_{1j}\boxtimes I_{2j})}(rs_1)(rs_2) = (f^{+}_{I_{1j}} \boxtimes f^{+}_{I_{2j}})(rs_1)(rs_2) = f^{+}_{I_1}(r) \wedge f^{+}_{I_{2j}}(s_1s_2) \end{cases}$$

for all $r \in U_1$, $s_1s_2 \in U_{2j}$,

$$\text{(v)} \begin{cases} t^{-}_{(I_{1j}\boxtimes I_{2j})}(r_1s)(r_2s) = (t^{-}_{I_{1j}} \boxtimes t^{-}_{I_{2j}})(r_1s)(r_2s) = t^{-}_{I_2}(s) \wedge t^{-}_{I_{1j}}(r_1r_2) \\ i^{-}_{(I_{1j}\boxtimes I_{2j})}(r_1s)(r_2s) = (i^{-}_{I_{1j}} \boxtimes i^{-}_{I_{2j}})(r_1s)(r_2s) = i^{-}_{I_2}(s) \wedge i^{-}_{I_{2j}}(r_1r_2) \\ f^{-}_{(I_{1j}\boxtimes I_{2j})}(r_1s)(r_2s) = (f^{-}_{I_{1j}} \boxtimes f^{-}_{I_{2j}})(r_1s)(r_2s) = f^{-}_{I_2}(s) \wedge f^{-}_{I_{2j}}(r_1r_2) \end{cases}$$

$$\text{(vi)} \begin{cases} t^{+}_{(I_{1j}\boxtimes I_{2j})}(r_1s)(r_2s) = (t^{+}_{I_{1j}} \boxtimes t^{+}_{I_{2j}})(r_1s)(r_2s) = t^{+}_{I_2}(s) \wedge t^{+}_{I_{1j}}(r_1r_2) \\ i^{+}_{(I_{1j}\boxtimes I_{2j})}(r_1s)(r_2s) = (i^{+}_{I_{1j}} \boxtimes i^{+}_{I_{2j}})(r_1s)(r_2s) = i^{+}_{I_2}(s) \wedge i^{+}_{I_{2j}}(r_1r_2) \\ f^{+}_{(I_{1j}\boxtimes I_{2j})}(r_1s)(r_2s) = (f^{+}_{I_{1j}} \boxtimes f^{+}_{I_{2j}})(r_1s)(r_2s) = f^{+}_{I_2}(s) \wedge f^{+}_{I_{2j}}(r_1r_2) \end{cases}$$

for all $s \in U_2$, $r_1r_2 \in U_{1j}$.

$$\text{(vii)} \begin{cases} t^{-}_{(I_{1j}\boxtimes I_{2j})}(r_1s_1)(r_2s_2) = (t^{-}_{I_{1j}} \boxtimes t^{-}_{I_{2j}})(r_1s_1)(r_2s_2) = t^{-}_{I_{1j}}(r_1r_2) \wedge t^{-}_{I_{2j}}(s_1s_2) \\ i^{-}_{(I_{1j}\boxtimes I_{2j})}(r_1s_1)(r_2s_2) = (i^{-}_{I_{1j}} \boxtimes i^{-}_{I_{2j}})(r_1s_1)(r_2s_2) = i^{-}_{I_{1j}}(r_1r_2) \wedge i^{-}_{I_{2j}}(s_1s_2) \\ f^{-}_{(I_{1j}\boxtimes I_{2j})}(r_1s_1)(r_2s_2) = (f^{-}_{I_{1j}} \boxtimes f^{-}_{I_{2j}})(r_1s_1)(r_2s_2) = f^{-}_{I_{1j}}(r_1r_2) \wedge f^{-}_{I_{2j}}(s_1s_2) \end{cases}$$

$$\text{(viii)} \begin{cases} t^{+}_{(I_{1j}\boxtimes I_{2j})}(r_1s_1)(r_2s_2) = (t^{+}_{I_{1j}} \boxtimes t^{+}_{I_{2j}})(r_1s_1)(r_2s_2) = t^{+}_{I_{1j}}(r_1r_2) \wedge t^{+}_{I_{2j}}(s_1s_2) \\ i^{+}_{(I_{1j}\boxtimes I_{2j})}(r_1s_1)(r_2s_2) = (i^{+}_{I_{1j}} \boxtimes i^{+}_{I_{2j}})(r_1s_1)(r_2s_2) = i^{+}_{I_{1j}}(r_1r_2) \wedge i^{+}_{I_{2j}}(s_1s_2) \\ f^{+}_{(I_{1j}\boxtimes I_{2j})}(r_1s_1)(r_2s_2) = (f^{+}_{I_{1j}} \boxtimes f^{+}_{I_{2j}})(r_1s_1)(r_2s_2) = f^{+}_{I_{1j}}(r_1r_2) \wedge f^{+}_{I_{2j}}(s_1s_2) \end{cases}$$

for all $r_1r_2 \in U_{1j}$, $s_1s_2 \in U_{2j}$.

Example 3.13. *Strong product of IVNGSs $\check{G}_{iv1}$ and $\check{G}_{iv2}$ shown in Fig. 2 is defined as $\check{G}_{iv1} \boxtimes \check{G}_{iv2} = \{I_1 \boxtimes I_2, I_{11} \boxtimes I_{21}, I_{12} \boxtimes I_{22}\}$ and is represented in Fig. 9 and Fig. 10.*

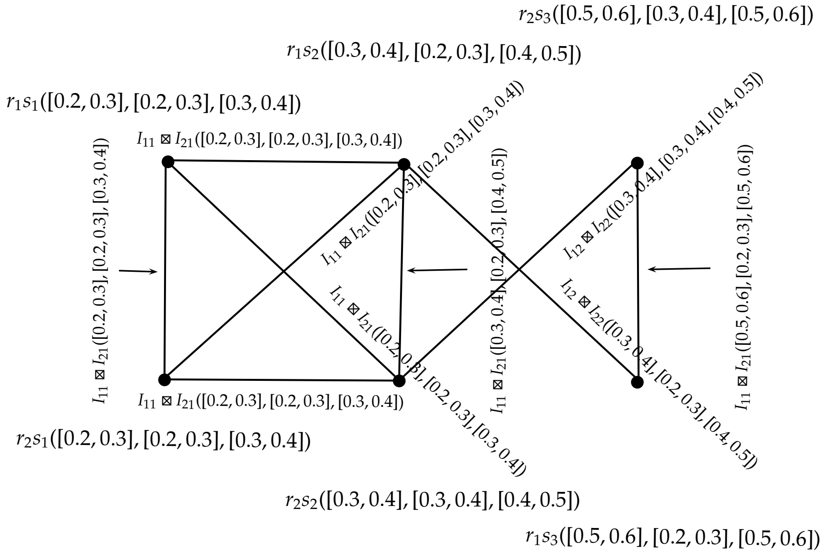

Figure 9: $\check{G}_{iv1} \boxtimes \check{G}_{iv2}$

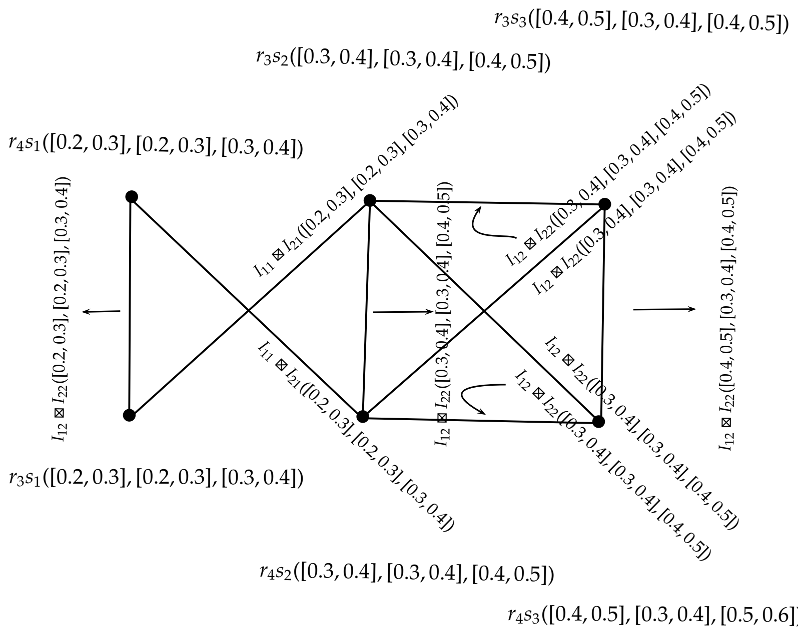

Figure 10: $\check{G}_{iv1} \boxtimes \check{G}_{iv2}$

Theorem 3.14. *Strong product $\check{G}_{iv1} \boxtimes \check{G}_{iv2} = (I_1 \boxtimes I_2, I_{11} \boxtimes I_{21}, I_{12} \boxtimes I_{22}, \dots, I_{1t} \boxtimes I_{2t})$ of two IVNGSs $\check{G}_{iv1}$ and $\check{G}_{iv2}$ of graph structures G_{s1} and G_{s2}, respectively is an IVNGS of $G_{s1} \boxtimes G_{s2}$.*

Proof. Consider three cases:

Case 1. For $r \in U_1$, $s_1s_2 \in U_{2j}$,

$$\begin{aligned} t^-_{(I_{1j}\boxtimes I_{2j})}((rs_1)(rs_2)) &= t^-_{I_1}(r) \wedge t^-_{I_{2j}}(s_1s_2) \\ &\leq t^-_{I_1}(r) \wedge [t^-_{I_2}(s_1) \wedge t^-_{I_2}(s_2)] \\ &= [t^-_{I_1}(r) \wedge t^-_{I_2}(s_1)] \wedge [t^-_{I_1}(r) \wedge t^-_{I_2}(s_2)] \\ &= t^-_{(I_1\boxtimes I_2)}(rs_1) \wedge t^-_{(I_1\boxtimes I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} t^+_{(I_{1j}\boxtimes I_{2j})}((rs_1)(rs_2)) &= t^+_{I_1}(r) \wedge t^+_{I_{2j}}(s_1s_2) \\ &\leq t^+_{I_1}(r) \wedge [t^+_{I_2}(s_1) \wedge t^+_{I_2}(s_2)] \\ &= [t^+_{I_1}(r) \wedge t^+_{I_2}(s_1)] \wedge [t^+_{I_1}(r) \wedge t^+_{I_2}(s_2)] \\ &= t^+_{(I_1\boxtimes I_2)}(rs_1) \wedge t^+_{(I_1\boxtimes I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} i^-_{(I_{1j}\boxtimes I_{2j})}((rs_1)(rs_2)) &= i^-_{I_1}(r) \wedge i^-_{I_{2j}}(s_1s_2) \\ &\leq i^-_{I_1}(r) \wedge [i^-_{I_2}(s_1) \wedge i^-_{I_2}(s_2)] \\ &= [i^-_{I_1}(r) \wedge i^-_{I_2}(s_1)] \wedge [i^-_{I_1}(r) \wedge i^-_{I_2}(s_2)] \\ &= i^-_{(I_1\boxtimes I_2)}(rs_1) \wedge i^-_{(I_1\boxtimes I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} i^+_{(I_{1j}\boxtimes I_{2j})}((rs_1)(rs_2)) &= i^+_{I_1}(r) \wedge i^+_{I_{2j}}(s_1s_2) \\ &\leq i^+_{I_1}(r) \wedge [i^+_{I_2}(s_1) \wedge i^+_{I_2}(s_2)] \\ &= [i^+_{I_1}(r) \wedge i^+_{I_2}(s_1)] \wedge [i^+_{I_1}(r) \wedge i^+_{I_2}(s_2)] \\ &= i^+_{(I_1\boxtimes I_2)}(rs_1) \wedge i^+_{(I_1\boxtimes I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} f^-_{(I_{1j}\boxtimes I_{2j})}((rs_1)(rs_2)) &= f^-_{I_1}(r) \wedge f^-_{I_{2j}}(s_1s_2) \\ &\leq f^-_{I_1}(r) \wedge [f^-_{I_2}(s_1) \wedge f^-_{I_2}(s_2)] \\ &= [f^-_{I_1}(r) \wedge f^-_{I_2}(s_1)] \wedge [f^-_{I_1}(r) \wedge f^-_{I_2}(s_2)] \\ &= f^-_{(I_1\boxtimes I_2)}(rs_1) \wedge f^-_{(I_1\boxtimes I_2)}(rs_2), \end{aligned}$$

$$\begin{aligned} f^+_{(I_{1j}\boxtimes I_{2j})}((rs_1)(rs_2)) &= f^+_{I_1}(r) \wedge f^+_{I_{2j}}(s_1s_2) \\ &\leq f^+_{I_1}(r) \wedge [f^+_{I_2}(s_1) \wedge f^+_{I_2}(s_2)] \\ &= [f^+_{I_1}(r) \wedge f^+_{I_2}(s_1)] \wedge [f^+_{I_1}(r) \wedge f^+_{I_2}(s_2)] \\ &= f^+_{(I_1\boxtimes I_2)}(rs_1) \wedge f^+_{(I_1\boxtimes I_2)}(rs_2), \end{aligned}$$

for $rs_1, rs_2 \in U_1 \boxtimes U_2$.

Case 2. For $r \in U_2$, $s_1s_2 \in U_{1j}$,

$$\begin{aligned} t^-_{(I_{1j}\boxtimes B_{2j})}((s_1r)(s_2r)) &= t^-_{I_2}(r) \wedge t^-_{I_{1j}}(s_1s_2) \\ &\leq t^-_{I_2}(r) \wedge [t^-_{I_1}(s_1) \wedge t^-_{I_1}(s_2)] \\ &= [t^-_{I_2}(r) \wedge t^-_{I_1}(s_1)] \wedge [t^-_{I_2}(r) \wedge t^-_{I_1}(s_2)] \\ &= t^-_{(I_1\boxtimes I_2)}(s_1r) \wedge t^-_{(I_1\boxtimes I_2)}(s_2r), \end{aligned}$$

$$\begin{aligned} t^+_{(I_{1j}\boxtimes I_{2j})}((s_1r)(s_2r)) &= t^+_{I_2}(r) \wedge t^+_{I_{1j}}(s_1s_2) \\ &\leq t^+_{I_2}(r) \wedge [t^+_{I_1}(s_1) \wedge t^+_{I_1}(s_2)] \\ &= [t^+_{I_2}(r) \wedge t^+_{I_1}(s_1)] \wedge [t^+_{I_2}(r) \wedge t^+_{I_1}(s_2)] \\ &= t^+_{(I_1\boxtimes I_2)}(s_1r) \wedge t^+_{(I_1\boxtimes I_2)}(s_2r), \end{aligned}$$

$$\begin{aligned} i^-_{(I_{1j}\boxtimes I_{2j})}((s_1r)(s_2r)) &= i^-_{I_2}(r) \wedge i^-_{I_{1j}}(s_1s_2) \\ &\leq i^-_{I_2}(r) \wedge [i^-_{I_1}(s_1) \wedge i^-_{I_1}(s_2)] \\ &= [i^-_{I_2}(r) \wedge i^-_{I_1}(s_1)] \wedge [i^-_{I_2}(r) \wedge i^-_{I_1}(s_2)] \\ &= i^-_{(I_1\boxtimes I_2)}(s_1r) \wedge i^-_{(I_1\boxtimes I_2)}(s_2r), \end{aligned}$$

$$\begin{aligned} i^+_{(I_{1j}\boxtimes I_{2j})}((s_1r)(s_2r)) &= i^+_{I_2}(r) \wedge i^+_{I_{1j}}(s_1s_2) \\ &\leq i^+_{I_2}(r) \wedge [i^+_{I_1}(s_1) \wedge i^+_{I_1}(s_2)] \\ &= [i^+_{I_2}(r) \wedge i^+_{I_1}(s_1)] \wedge [i^+_{I_2}(r) \wedge i^+_{I_1}(s_2)] \\ &= i^+_{(I_1\boxtimes I_2)}(s_1r) \wedge i^+_{(I_1\boxtimes I_2)}(s_2r), \end{aligned}$$

$$\begin{aligned} f^-_{(I_{1j}\boxtimes I_{2j})}((s_1r)(s_2r)) &= f^-_{I_2}(r) \wedge f^-_{I_{1j}}(s_1s_2) \\ &\leq f^-_{I_2}(r) \wedge [f^-_{I_1}(s_1) \wedge f^-_{I_1}(s_2)] \\ &= [f^-_{I_2}(r) \wedge f^-_{I_1}(s_1)] \wedge [f^-_{I_2}(r) \wedge f^-_{I_1}(s_2)] \\ &= f^-_{(I_1\boxtimes I_2)}(s_1r) \wedge f^-_{(I_1\boxtimes I_2)}(s_2r), \end{aligned}$$

$$\begin{aligned} f^+_{(I_{1j}\boxtimes I_{2j})}((s_1r)(s_2r)) &= f^+_{I_2}(r) \wedge f^+_{I_{1j}}(s_1s_2) \\ &\leq f^+_{I_2}(r) \wedge [f^+_{I_1}(s_1) \wedge f^+_{I_1}(s_2)] \\ &= [f^+_{I_2}(r) \wedge f^+_{I_1}(s_1)] \wedge [f^+_{I_2}(r) \wedge f^+_{I_1}(s_2)] \\ &= f^+_{(I_1\boxtimes I_2)}(s_1r) \wedge f^+_{(I_1\boxtimes I_2)}(s_2r), \end{aligned}$$

for $s_1r, s_2r \in U_1 \boxtimes U_2$.

Case 3. For $r_1r_2 \in U_{1j}$, $s_1s_2 \in U_{2j}$,

$$\begin{aligned} t^-_{(I_{1j}\boxtimes I_{2j})}((r_1s_1)(r_2s_2)) &= t^-_{I_{1j}}(r_1r_2) \wedge t^-_{I_{2j}}(s_1s_2) \\ &\leq [t^-_{I_1}(r_1) \wedge t^-_{I_1}(r_2)] \wedge [t^-_{I_2}(s_1) \wedge t^-_{I_2}(s_2)] \\ &= [t^-_{I_1}(r_1) \wedge t^-_{I_2}(s_1)] \wedge [t^-_{I_1}(r_2) \wedge t^-_{I_2}(s_2)] \\ &= t^-_{(I_1\boxtimes I_2)}(r_1s_1) \wedge t^-_{(I_1\boxtimes I_2)}(r_2s_2), \end{aligned}$$

$$\begin{aligned} t^+_{(I_{1j}\boxtimes I_{2j})}((r_1s_1)(r_2s_2)) &= t^+_{I_{1j}}(r_1r_2) \wedge t^+_{I_{2j}}(s_1s_2) \\ &\leq [t^+_{I_1}(r_1) \wedge t^+_{I_1}(r_2)] \wedge [t^+_{I_2}(s_1) \wedge t^+_{I_2}(s_2)] \\ &= [t^+_{I_1}(r_1) \wedge t^+_{I_2}(s_1)] \wedge [t^+_{I_1}(r_2) \wedge t^+_{I_2}(s_2)] \\ &= t^+_{(I_1\boxtimes I_2)}(r_1s_1) \wedge t^+_{(I_1\boxtimes I_2)}(r_2s_2), \end{aligned}$$

$$\begin{aligned}
i^{-}_{(I_{1j}\boxtimes I_{2j})}((r_1s_1)(r_2s_2)) &= i^{-}_{I_{1j}}(r_1r_2) \wedge i^{-}_{I_{2j}}(s_1s_2)\\
&\leq [i^{-}_{I_1}(r_1) \wedge i^{-}_{I_1}(r_2)] \wedge [i^{-}_{I_2}(s_1) \wedge i^{-}_{I_2}(s_2)]\\
&= [i^{-}_{I_1}(r_1) \wedge i^{-}_{I_2}(s_1)] \wedge [i^{-}_{I_1}(r_2) \wedge i^{-}_{I_2}(s_2)]\\
&= i^{-}_{(I_1\boxtimes I_2)}(r_1s_1) \wedge i^{-}_{(I_1\boxtimes I_2)}(r_2s_2),
\end{aligned}$$

$$\begin{aligned}
i^{+}_{(I_{1j}\boxtimes I_{2j})}((r_1s_1)(r_2s_2)) &= i^{+}_{I_{1j}}(r_1r_2) \wedge i^{+}_{I_{2j}}(s_1s_2)\\
&\leq [i^{+}_{I_1}(r_1) \wedge i^{+}_{I_1}(r_2)] \wedge [i^{+}_{I_2}(s_1) \wedge i^{+}_{I_2}(s_2)]\\
&= [i^{+}_{I_1}(r_1) \wedge i^{+}_{I_2}(s_1)] \wedge [i^{+}_{I_1}(r_2) \wedge i^{+}_{I_2}(s_2)]\\
&= i^{+}_{(I_1\boxtimes I_2)}(r_1s_1) \wedge i^{+}_{(I_1\boxtimes I_2)}(r_2s_2),
\end{aligned}$$

$$\begin{aligned}
f^{-}_{(I_{1j}\boxtimes I_{2j})}((r_1s_1)(r_2s_2)) &= f^{-}_{I_{1j}}(r_1r_2) \wedge f^{-}_{I_{2j}}(s_1s_2)\\
&\leq [f^{-}_{I_1}(r_1) \wedge f^{-}_{I_1}(r_2)] \wedge [f^{-}_{I_2}(s_1) \wedge f^{-}_{I_2}(s_2)]\\
&= [f^{-}_{I_1}(r_1) \wedge f^{-}_{I_2}(s_1)] \wedge [f^{-}_{I_1}(r_2) \wedge f^{-}_{I_2}(s_2)]\\
&= f^{P}_{(I_1\boxtimes I_2)}(r_1s_1) \wedge f^{-}_{(I_1\boxtimes I_2)}(r_2s_2),
\end{aligned}$$

$$\begin{aligned}
f^{+}_{(I_{1j}\boxtimes I_{2j})}((r_1s_1)(r_2s_2)) &= f^{+}_{I_{1j}}(r_1r_2) \wedge f^{+}_{I_{2j}}(s_1s_2)\\
&\leq [f^{+}_{I_1}(r_1) \wedge f^{+}_{I_1}(r_2)] \wedge [f^{+}_{I_2}(s_1) \wedge f^{+}_{I_2}(s_2)]\\
&= [f^{+}_{I_1}(r_1) \wedge f^{+}_{I_2}(s_1)] \wedge [f^{+}_{I_1}(r_2) \wedge f^{+}_{I_2}(s_2)]\\
&= f^{+}_{(I_1\boxtimes I_2)}(r_1s_1) \wedge f^{+}_{(I_1\boxtimes I_2)}(r_2s_2),
\end{aligned}$$

$r_1s_1, r_2s_2 \in U_1 \boxtimes U_2$.

All cases hold for all $j \in \{1, 2, \ldots, t\}$. This completes the proof. □

Definition 3.15. *Let $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ and $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ be two IVNGSs. Composition of $\check{G}_{iv1}$ and $\check{G}_{iv2}$, denoted by*

$$\check{G}_{iv1} \circ \check{G}_{iv2} = (I_1 \circ I_2, I_{11} \circ I_{21}, I_{12} \circ I_{22}, \ldots, I_{1t} \circ I_{2t}),$$

is defined as:

$$\text{(i)} \begin{cases} t^{-}_{(I_1\circ I_2)}(rs) = (t^{-}_{I_1} \circ t^{-}_{I_2})(rs) = t^{-}_{I_1}(r) \wedge t^{-}_{I_2}(s)\\ i^{-}_{(I_1\circ I_2)}(rs) = (i^{-}_{I_1} \circ i^{-}_{I_2})(rs) = i^{-}_{I_1}(r) \wedge i^{-}_{I_2}(s)\\ f^{-}_{(I_1\circ I_2)}(rs) = (f^{-}_{I_1} \circ f^{-}_{I_2})(rs) = f^{-}_{I_1}(r) \wedge f^{-}_{I_2}(s) \end{cases}$$

$$\text{(ii)} \begin{cases} t^{+}_{(I_1\circ I_2)}(rs) = (t^{+}_{I_1} \circ t^{+}_{I_2})(rs) = t^{+}_{I_1}(r) \wedge t^{+}_{I_2}(s)\\ i^{+}_{(I_1\circ I_2)}(rs) = (i^{+}_{I_1} \circ i^{+}_{I_2})(rs) = i^{+}_{I_1}(r) \wedge i^{+}_{I_2}(s)\\ f^{+}_{(I_1\circ I_2)}(rs) = (f^{+}_{I_1} \circ f^{+}_{I_2})(rs) = f^{+}_{I_1}(r) \wedge f^{+}_{I_2}(s) \end{cases}$$

for all $rs \in U_1 \times U_2$,

$$\text{(iii)} \begin{cases} t^{-}_{(I_{1j}\circ I_{2j})}(rs_1)(rs_2) = (t^{-}_{I_{1j}} \circ t^{-}_{I_{2j}})(rs_1)(rs_2) = t^{-}_{I_1}(r) \wedge t^{-}_{I_{2j}}(s_1s_2)\\ i^{-}_{(I_{1j}\circ I_{2j})}(rs_1)(rs_2) = (i^{-}_{I_{1j}} \circ i^{-}_{I_{2j}})(rs_1)(rs_2) = i^{-}_{I_1}(r) \wedge i^{-}_{I_{2j}}(s_1s_2)\\ f^{-}_{(I_{1j}\circ I_{2j})}(rs_1)(rs_2) = (f^{-}_{I_{1j}} \circ f^{-}_{I_{2j}})(rs_1)(rs_2) = f^{-}_{I_1}(r) \wedge f^{-}_{I_{2j}}(s_1s_2) \end{cases}$$

$$\text{(iv)}\begin{cases} t^{+}_{(I_{1j}\circ I_{2j})}(rs_1)(rs_2) = (t^{+}_{I_{1j}} \circ t^{+}_{I_{2j}})(rs_1)(rs_2) = t^{+}_{I_1}(r) \wedge t^{+}_{I_{2j}}(s_1s_2) \\ i^{+}_{(I_{1j}\circ I_{2j})}(rs_1)(rs_2) = (i^{+}_{I_{1j}} \circ i^{+}_{I_{2j}})(rs_1)(rs_2) = i^{+}_{I_1}(r) \wedge i^{+}_{I_{2j}}(s_1s_2) \\ f^{+}_{(I_{1j}\circ I_{2j})}(rs_1)(rs_2) = (f^{+}_{I_{1j}} \circ f^{+}_{I_{2j}})(rs_1)(rs_2) = f^{+}_{I_1}(r) \wedge f^{+}_{I_{2j}}(s_1s_2) \end{cases}$$

for all $r \in U_1$, $s_1s_2 \in U_{2j}$,

$$\text{(v)}\begin{cases} t^{-}_{(I_{1j}\circ I_{2j})}(r_1s)(r_2s) = (t^{-}_{I_{1j}} \circ t^{-}_{I_{2j}})(r_1s)(r_2s) = t^{-}_{I_2}(s) \wedge t^{-}_{I_{1j}}(r_1r_2) \\ i^{-}_{(I_{1j}\circ I_{2j})}(r_1s)(r_2s) = (i^{-}_{I_{1j}} \circ i^{-}_{I_{2j}})(r_1s)(r_2s) = i^{-}_{I_2}(s) \wedge i^{-}_{I_{1j}}(r_1r_2) \\ f^{-}_{(I_{1j}\circ I_{2j})}(r_1s)(r_2s) = (f^{-}_{I_{1j}} \circ f^{-}_{I_{2j}})(r_1s)(r_2s) = f^{-}_{I_2}(s) \wedge f^{-}_{I_{1j}}(r_1r_2) \end{cases}$$

$$\text{(vi)}\begin{cases} t^{+}_{(I_{1j}\circ I_{2j})}(r_1s)(r_2s) = (t^{+}_{I_{1j}} \circ t^{+}_{I_{2j}})(r_1s)(r_2s) = t^{+}_{I_2}(s) \wedge t^{+}_{I_{1j}}(r_1r_2) \\ i^{+}_{(I_{1j}\circ I_{2j})}(r_1s)(r_2s) = (i^{+}_{I_{1j}} \circ i^{+}_{I_{2j}})(r_1s)(r_2s) = i^{+}_{I_2}(s) \wedge i^{+}_{I_{1j}}(r_1r_2) \\ f^{+}_{(I_{1j}\circ I_{2j})}(r_1s)(r_2s) = (f^{+}_{I_{1j}} \circ f^{+}_{I_{2j}})(r_1s)(r_2s) = f^{+}_{I_2}(s) \wedge f^{+}_{I_{1j}}(r_1r_2) \end{cases}$$

for all $s \in U_2$, $r_1r_2 \in U_{1j}$.

$$\text{(vii)}\begin{cases} t^{-}_{(I_{1j}\circ I_{2j})}(r_1s_1)(r_2s_2) = (t^{-}_{I_{1j}} \circ t^{-}_{I_{2j}})(r_1s_1)(r_2s_2) = t^{-}_{I_{1j}}(r_1r_2) \wedge t^{-}_{I_2}(s_1) \wedge t^{-}_{I_2}(s_2) \\ i^{-}_{(I_{1j}\circ I_{2j})}(r_1s_1)(r_2s_2) = (i^{-}_{I_{1j}} \circ i^{-}_{I_{2j}})(r_1s_1)(r_2s_2) = i^{-}_{I_{1j}}(r_1r_2) \wedge i^{-}_{I_2}(s_1) \wedge i^{-}_{I_2}(s_2) \\ f^{-}_{(I_{1j}\circ I_{2j})}(r_1s_1)(r_2s_2) = (f^{-}_{I_{1j}} \circ f^{-}_{I_{2j}})(r_1s_1)(r_2s_2) = f^{P}_{I_{1j}}(r_1r_2) \wedge f^{-}_{I_2}(s_1) \wedge f^{-}_{I_2}(s_2) \end{cases}$$

$$\text{(viii)}\begin{cases} t^{+}_{(I_{1j}\circ I_{2j})}(r_1s_1)(r_2s_2) = (t^{+}_{I_{1j}} \circ t^{+}_{I_{2j}})(r_1s_1)(r_2s_2) = t^{+}_{I_{1j}}(r_1r_2) \wedge t^{+}_{I_2}(s_1) \wedge t^{+}_{I_2}(s_2) \\ i^{+}_{(I_{1j}\circ I_{2j})}(r_1s_1)(r_2s_2) = (i^{+}_{I_{1j}} \circ i^{+}_{I_{2j}})(r_1s_1)(r_2s_2) = i^{+}_{I_{1j}}(r_1r_2) \wedge i^{+}_{I_2}(s_1) \wedge i^{+}_{I_2}(s_2) \\ f^{+}_{(I_{1j}\circ I_{2j})}(r_1s_1)(r_2s_2) = (f^{+}_{I_{1j}} \circ f^{+}_{I_{2j}})(r_1s_1)(r_2s_2) = f^{+}_{I_{1j}}(r_1r_2) \wedge f^{+}_{I_2}(s_1) \wedge f^{+}_{I_2}(s_2) \end{cases}$$

for all $r_1r_2 \in U_{1j}$, $s_1s_2 \in U_{2j}$ *such that* $s_1 \neq s_2$.

Example 3.16. *Composition of IVNGSs* $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$ *shown in Fig. 2 is defined as* $\check{G}_{iv1} \circ \check{G}_{iv2} = \{I_1 \circ I_2, I_{11} \circ I_{21}, I_{12} \circ I_{22}\}$ *and is depicted in Fig. 11 and Fig. 12.*

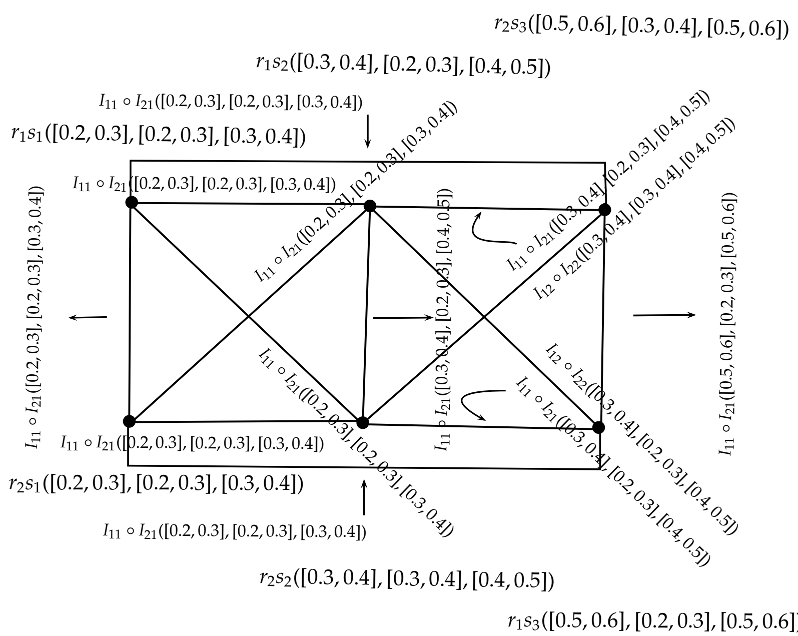

Figure 11: $\check{G}_{iv1} \circ \check{G}_{iv2}$

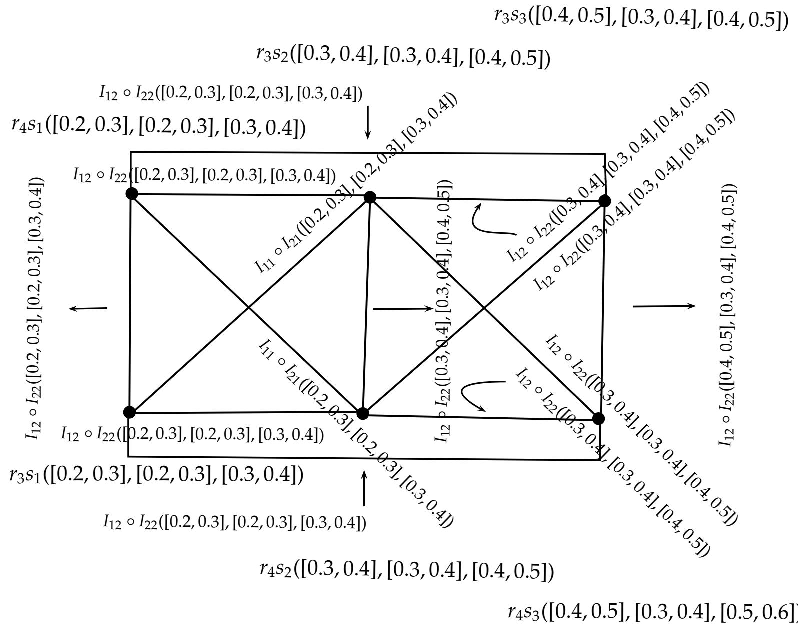

Figure 12: $\check{G}_{iv1} \circ \check{G}_{iv2}$

Theorem 3.17. *Composition $\check{G}_{iv1} \circ \check{G}_{iv2} = (I_1 \circ I_2, I_{11} \circ I_{21}, I_{12} \circ I_{22}, \ldots, I_{1t} \circ I_{2t})$ of two IVNGSs $\check{G}_{iv1}$ and $\check{G}_{iv2}$ of graph structures G_{s1} and G_{s2}, respectively is an IVNGS of $G_{s1} \circ G_{s2}$.*

Proof. Consider three cases:

Case 1. For $r \in U_1$, $s_1 s_2 \in U_{2j}$

$$\begin{aligned}
t^-_{(I_{1j} \circ I_{2j})}((rs_1)(rs_2)) &= t^-_{I_1}(r) \wedge t^-_{I_{2j}}(s_1 s_2) \\
&\leq t^-_{I_1}(r) \wedge [t^-_{I_2}(s_1) \wedge t^-_{I_2}(s_2)] \\
&= [t^-_{I_1}(r) \wedge t^-_{I_2}(s_1)] \wedge [t^-_{I_1}(r) \wedge t^-_{I_2}(s_2)] \\
&= t^-_{(I_1 \circ I_2)}(rs_1) \wedge t^-_{(I_1 \circ I_2)}(rs_2),
\end{aligned}$$

$$\begin{aligned}
t^+_{(I_{1j} \circ I_{2j})}((rs_1)(rs_2)) &= t^+_{I_1}(r) \wedge t^+_{I_{2j}}(s_1 s_2) \\
&\leq t^+_{I_1}(r) \wedge [t^+_{I_2}(s_1) \wedge t^+_{I_2}(s_2)] \\
&= [t^+_{I_1}(r) \wedge t^+_{I_2}(s_1)] \wedge [t^+_{I_1}(r) \wedge t^+_{I_2}(s_2)] \\
&= t^+_{(I_1 \circ I_2)}(rs_1) \wedge t^+_{(I_1 \circ I_2)}(rs_2),
\end{aligned}$$

$$\begin{aligned}
i^-_{(I_{1j} \circ I_{2j})}((rs_1)(rs_2)) &= i^-_{I_1}(r) \wedge i^-_{I_{2j}}(s_1 s_2) \\
&\leq i^-_{I_1}(r) \wedge [i^-_{I_2}(s_1) \wedge i^-_{I_2}(s_2)] \\
&= [i^-_{I_1}(r) \wedge i^-_{I_2}(s_1)] \wedge [i^-_{I_1}(r) \wedge i^-_{I_2}(s_2)] \\
&= i^-_{(I_1 \circ I_2)}(rs_1) \wedge i^-_{(I_1 \circ I_2)}(rs_2),
\end{aligned}$$

$$\begin{aligned}
i^{+}_{(I_{1j}\circ I_{2j})}((rs_1)(rs_2)) &= i^{+}_{I_1}(r) \wedge i^{+}_{I_{2j}}(s_1s_2)\\
&\leq i^{+}_{I_1}(r) \wedge [i^{+}_{I_2}(s_1) \wedge i^{+}_{I_2}(s_2)]\\
&= [i^{+}_{I_1}(r) \wedge i^{+}_{I_2}(s_1)] \wedge [i^{+}_{I_1}(r) \wedge i^{+}_{I_2}(s_2)]\\
&= i^{+}_{(I_1\circ I_2)}(rs_1) \wedge i^{+}_{(I_1\circ I_2)}(rs_2),
\end{aligned}$$

$$\begin{aligned}
f^{-}_{(I_{1j}\circ I_{2j})}((rs_1)(rs_2)) &= f^{-}_{I_1}(r) \wedge f^{-}_{I_{2j}}(s_1s_2)\\
&\leq f^{-}_{I_1}(r) \wedge [f^{-}_{I_2}(s_1) \wedge f^{-}_{I_2}(s_2)]\\
&= [f^{-}_{I_1}(r) \wedge f^{-}_{I_2}(s_1)] \wedge [f^{-}_{I_1}(r) \wedge f^{-}_{I_2}(s_2)]\\
&= f^{-}_{(I_1\circ I_2)}(rs_1) \wedge f^{-}_{(I_1\circ I_2)}(rs_2),
\end{aligned}$$

$$\begin{aligned}
f^{+}_{(I_{1j}\circ B_{2j})}((rs_1)(rs_2)) &= f^{+}_{I_1}(r) \wedge f^{+}_{I_{2j}}(s_1s_2)\\
&\leq f^{+}_{I_1}(r) \wedge [f^{+}_{I_2}(s_1) \wedge f^{+}_{I_2}(s_2)]\\
&= [f^{+}_{I_1}(r) \wedge f^{+}_{I_2}(s_1)] \wedge [f^{+}_{I_1}(r) \wedge f^{+}_{I_2}(s_2)]\\
&= f^{+}_{(I_1\circ I_2)}(rs_1) \wedge f^{+}_{(I_1\circ I_2)}(rs_2),
\end{aligned}$$

$rs_1, rs_2 \in U_1 \circ U_2$.

Case 2. For $r \in U_2$, $s_1s_2 \in U_{1j}$

$$\begin{aligned}
t^{-}_{(I_{1j}\circ I_{2j})}((s_1r)(s_2r)) &= t^{-}_{I_2}(r) \wedge t^{-}_{I_{1j}}(s_1s_2)\\
&\leq t^{-}_{I_2}(r) \wedge [t^{-}_{I_1}(s_1) \wedge t^{-}_{I_1}(s_2)]\\
&= [t^{-}_{I_2}(r) \wedge t^{-}_{I_1}(s_1)] \wedge [t^{-}_{I_2}(r) \wedge t^{-}_{I_1}(s_2)]\\
&= t^{-}_{(I_1\circ I_2)}(s_1r) \wedge t^{-}_{(I_1\circ I_2)}(s_2r),
\end{aligned}$$

$$\begin{aligned}
t^{+}_{(I_{1j}\circ I_{2j})}((s_1r)(s_2r)) &= t^{+}_{I_2}(r) \wedge t^{+}_{I_{1j}}(s_1s_2)\\
&\leq t^{+}_{I_2}(r) \wedge [t^{+}_{I_1}(s_1) \wedge t^{+}_{I_1}(s_2)]\\
&= [t^{+}_{I_2}(r) \wedge t^{+}_{I_1}(s_1)] \wedge [t^{+}_{I_2}(r) \wedge t^{+}_{I_1}(s_2)]\\
&= t^{+}_{(I_1\circ I_2)}(s_1r) \wedge t^{+}_{(I_1\circ I_2)}(s_2r),
\end{aligned}$$

$$\begin{aligned}
i^{-}_{(I_{1j}\circ I_{2j})}((s_1r)(s_2r)) &= i^{-}_{I_2}(r) \wedge i^{-}_{I_{1j}}(s_1s_2)\\
&\leq i^{-}_{I_2}(r) \wedge [i^{-}_{I_1}(s_1) \wedge i^{-}_{I_1}(s_2)]\\
&= [i^{-}_{I_2}(r) \wedge i^{-}_{I_1}(s_1)] \wedge [i^{-}_{I_2}(r) \wedge i^{-}_{I_1}(s_2)]\\
&= i^{-}_{(I_1\circ I_2)}(s_1r) \wedge i^{-}_{(I_1\circ I_2)}(s_2r),
\end{aligned}$$

$$\begin{aligned}
i^{+}_{(I_{1j}\circ I_{2j})}((s_1r)(s_2r)) &= i^{+}_{I_2}(r) \wedge i^{+}_{I_{1j}}(s_1s_2)\\
&\leq i^{+}_{I_2}(r) \wedge [i^{+}_{I_1}(s_1) \wedge i^{+}_{I_1}(s_2)]\\
&= [i^{+}_{I_2}(r) \wedge i^{+}_{I_1}(s_1)] \wedge [i^{+}_{I_2}(r) \wedge i^{+}_{I_1}(s_2)]\\
&= i^{+}_{(I_1\circ I_2)}(s_1r) \wedge i^{+}_{(I_1\circ I_2)}(s_2r),
\end{aligned}$$

$$\begin{aligned}
f^-_{(I_{1j}\circ I_{2j})}((s_1r)(s_2r)) &= f^-_{I_2}(r) \wedge f^-_{I_{1j}}(s_1s_2)\\
&\leq f^-_{I_2}(r) \wedge [f^-_{I_1}(s_1) \wedge f^-_{I_1}(s_2)]\\
&= [f^-_{I_2}(r) \wedge f^-_{I_1}(s_1)] \wedge [f^-_{I_2}(r) \wedge f^-_{I_1}(s_2)]\\
&= f^-_{(I_1\circ I_2)}(s_1r) \wedge f^-_{(I_1\circ I_2)}(s_2r),
\end{aligned}$$

$$\begin{aligned}
f^+_{(I_{1j}\circ I_{2j})}((s_1r)(s_2r)) &= f^+_{I_2}(r) \wedge f^+_{I_{1j}}(s_1s_2)\\
&\leq f^+_{I_2}(r) \wedge [f^+_{I_1}(s_1) \wedge f^+_{I_1}(s_2)]\\
&= [f^+_{I_2}(r) \wedge f^+_{I_1}(s_1)] \wedge [f^+_{I_2}(r) \wedge f^+_{I_1}(s_2)]\\
&= f^+_{(I_1\circ I_2)}(s_1r) \wedge f^+_{(I_1\circ I_2)}(s_2r),
\end{aligned}$$

$s_1r, s_2r \in U_1 \circ U_2$.

Case 3. For $r_1r_2 \in U_{1j}$, $s_1s_2 \in U_{2j}$ such that $s_1 \neq s_2$

$$\begin{aligned}
t^-_{(I_{1j}\circ I_{2j})}((r_1s_1)(r_2s_2)) &= t^-_{I_{1j}}(r_1r_2) \wedge t^-_{I_2}(s_1) \wedge t^-_{I_2}(s_2)\\
&\leq [t^-_{I_1}(r_1) \wedge t^-_{I_1}(r_2)] \wedge [t^-_{I_2}(s_1) \wedge t^-_{I_2}(s_2)]\\
&= [t^-_{I_1}(r_1) \wedge t^-_{I_2}(s_1)] \wedge [t^-_{I_1}(r_2) \wedge t^-_{I_2}(s_2)]\\
&= t^-_{(I_1\circ I_2)}(r_1s_1) \wedge t^-_{(I_1\circ I_2)}(r_2s_2),
\end{aligned}$$

$$\begin{aligned}
t^+_{(I_{1j}\circ I_{2j})}((r_1s_1)(r_2s_2)) &= t^+_{I_{1j}}(r_1r_2) \wedge t^+_{I_2}(s_1) \wedge t^+_{I_2}(s_2)\\
&\leq [t^+_{I_1}(r_1) \wedge t^+_{I_1}(r_2)] \wedge [t^+_{I_2}(s_1) \wedge t^+_{I_2}(s_2)]\\
&= [t^+_{I_1}(r_1) \wedge t^+_{I_2}(s_1)] \wedge [t^+_{I_1}(r_2) \wedge t^+_{I_2}(s_2)]\\
&= t^+_{(I_1\circ I_2)}(r_1s_1) \wedge t^+_{(I_1\circ I_2)}(r_2s_2),
\end{aligned}$$

$$\begin{aligned}
i^-_{(I_{1j}\circ I_{2j})}((r_1s_1)(r_2s_2)) &= i^-_{I_{1j}}(r_1r_2) \wedge i^-_{I_2}(s_1) \wedge i^-_{I_2}(s_2)\\
&\leq [i^-_{I_1}(r_1) \wedge i^-_{I_1}(r_2)] \wedge [i^-_{I_2}(s_1) \wedge i^-_{I_2}(s_2)]\\
&= [i^-_{I_1}(r_1) \wedge i^-_{I_2}(s_1)] \wedge [i^-_{I_1}(r_2) \wedge i^-_{I_2}(s_2)]\\
&= i^-_{(I_1\circ I_2)}(r_1s_1) \wedge i^-_{(I_1\circ I_2)}(r_2s_2),
\end{aligned}$$

$$\begin{aligned}
i^+_{(I_{1j}\circ I_{2j})}((r_1s_1)(r_2s_2)) &= i^+_{I_{1j}}(r_1r_2) \wedge i^+_{I_2}(s_1) \wedge i^+_{I_2}(s_2)\\
&\leq [i^+_{I_1}(r_1) \wedge i^+_{I_1}(r_2)] \wedge [i^+_{I_2}(s_1) \wedge i^+_{I_2}(s_2)]\\
&= [i^+_{I_1}(r_1) \wedge i^+_{I_2}(s_1)] \wedge [i^+_{I_1}(r_2) \wedge i^+_{I_2}(s_2)]\\
&= i^+_{(I_1\circ I_2)}(r_1s_1) \wedge i^+_{(I_1\circ I_2)}(r_2s_2),
\end{aligned}$$

$$\begin{aligned}
f^-_{(I_{1j}\circ I_{2j})}((r_1s_1)(r_2s_2)) &= f^-_{I_{1j}}(r_1r_2) \wedge f^-_{I_2}(s_1) \wedge f^-_{I_2}(s_2)\\
&\leq [f^-_{I_1}(r_1) \wedge f^-_{I_1}(r_2)] \wedge [f^-_{I_2}(s_1) \wedge f^-_{I_2}(s_2)]\\
&= [f^-_{I_1}(r_1) \wedge f^-_{I_2}(s_1)] \wedge [f^-_{I_1}(r_2) \wedge f^-_{I_2}(s_2)]\\
&= f^-_{(I_1\circ I_2)}(r_1s_1) \wedge f^-_{(I_1\circ I_2)}(r_2s_2),
\end{aligned}$$

$$\begin{aligned} f^{+}_{(I_{1j}\circ I_{2j})}((r_1s_1)(r_2s_2)) &= f^{+}_{I_{1j}}(r_1r_2) \wedge f^{+}_{I_2}(s_1) \wedge f^{+}_{I_2}(s_2) \\ &\leq [f^{+}_{I_1}(r_1) \wedge f^{+}_{I_1}(r_2)] \wedge [f^{+}_{I_2}(s_1) \wedge f^{+}_{I_2}(s_2)] \\ &= [f^{+}_{I_1}(r_1) \wedge f^{+}_{I_2}(s_1)] \wedge [f^{+}_{I_1}(r_2) \wedge f^{+}_{I_2}(s_2)] \\ &= f^{+}_{(I_1\circ I_2)}(r_1s_1) \wedge f^{+}_{(I_1\circ I_2)}(r_2s_2), \end{aligned}$$

$r_1s_1, r_2s_2 \in U_1 \circ U_2$.

All cases hold for all $j \in \{1, 2, \ldots, t\}$. This completes the proof. □

Definition 3.18. *Let $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ and $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ be two IVNGSs. Union of $\check{G}_{iv1}$ and $\check{G}_{iv2}$, denoted by*

$$\check{G}_{iv1} \cup \check{G}_{iv2} = (I_1 \cup I_2, I_{11} \cup I_{21}, I_{12} \cup I_{22}, \ldots, I_{1t} \cup I_{2t}),$$

is defined as:

$$\text{(i)} \begin{cases} t^{-}_{(I_1\cup I_2)}(r) = (t^{-}_{I_1} \cup t^{-}_{I_2})(r) = t^{-}_{I_1}(r) \vee t^{-}_{I_2}(r) \\ i^{-}_{(I_1\cup I_2)}(r) = (i^{-}_{I_1} \cup i^{-}_{I_2})(r) = i^{-}_{I_1}(r) \vee i^{-}_{I_2}(r) \\ f^{-}_{(I_1\cup I_2)}(r) = (f^{-}_{I_1} \cup f^{-}_{I_2})(r) = f^{-}_{I_1}(r) \wedge f^{-}_{I_2}(r) \end{cases}$$

$$\text{(ii)} \begin{cases} t^{+}_{(I_1\cup I_2)}(r) = (t^{+}_{I_1} \cup t^{+}_{I_2})(r) = t^{+}_{I_1}(r) \vee t^{+}_{I_2}(r) \\ i^{+}_{(I_1\cup I_2)}(r) = (i^{+}_{I_1} \cup i^{+}_{I_2})(r) = i^{+}_{I_1}(r) \vee i^{+}_{I_2}(r) \\ f^{+}_{(I_1\cup I_2)}(r) = (f^{+}_{I_1} \cup f^{+}_{I_2})(r) = f^{+}_{I_1}(r) \wedge f^{+}_{I_2}(r) \end{cases}$$

for all $r \in U_1 \cup U_2$,

$$\text{(iii)} \begin{cases} t^{-}_{(I_{1j}\cup I_{2j})}(rs) = (t^{-}_{I_{1j}} \cup t^{-}_{I_{2j}})(rs) = t^{-}_{I_{1j}}(rs) \vee t^{-}_{I_{2j}}(rs) \\ i^{-}_{(I_{1j}\cup I_{2j})}(rs) = (i^{-}_{I_{1j}} \cup i^{-}_{I_{2j}})(rs) = i^{-}_{I_{1j}}(rs) \vee i^{-}_{I_{2j}}(rs) \\ f^{-}_{(I_{1j}\cup I_{2j})}(rs) = (f^{-}_{I_{1j}} \cup f^{-}_{I_{2j}})(rs) = f^{-}_{I_{1j}}(rs) \wedge f^{-}_{I_{2j}}(rs) \end{cases}$$

$$\text{(iv)} \begin{cases} t^{+}_{(I_{1j}\cup I_{2j})}(rs) = (t^{+}_{I_{1j}} \cup t^{+}_{I_{2j}})(rs) = t^{+}_{I_{1j}}(rs) \vee t^{+}_{I_{2j}}(rs) \\ i^{+}_{(I_{1j}\cup I_{2j})}(rs) = (i^{+}_{I_{1j}} \cup i^{+}_{I_{2j}})(rs) = i^{+}_{I_{1j}}(rs) \vee i^{+}_{I_{2j}}(rs) \\ f^{+}_{(I_{1j}\cup I_{2j})}(rs) = (f^{+}_{I_{1j}} \cup f^{+}_{I_{2j}})(rs) = f^{+}_{I_{1j}}(rs) \wedge f^{+}_{I_{2j}}(rs) \end{cases}$$

for all $rs \in U_{1j} \cup U_{2j}$.

Example 3.19. *Union of two IVNGSs $\check{G}_{iv1}$ and $\check{G}_{iv2}$ shown in Fig. 2 is defined as $\check{G}_{iv1} \cup \check{G}_{iv2} = \{I_1 \cup I_2, I_{11} \cup I_{21}, I_{12} \cup I_{22}\}$ and is depicted in Fig. 13.*

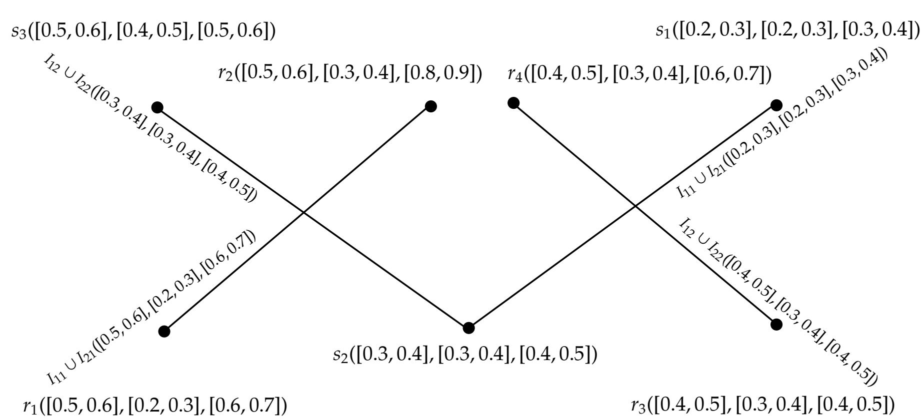

Figure 13: $\check{G}_{iv1} \cup \check{G}_{iv2}$

Theorem 3.20. *Union $\check{G}_{iv1} \cup \check{G}_{iv2} = (I_1 \cup I_2, I_{11} \cup I_{21}, I_{12} \cup I_{22}, \dots, I_{1t} \cup I_{2t})$ of two IVNGSs $\check{G}_{iv1}$ and $\check{G}_{iv2}$ of graph structures G_{s1} and G_{s2} is an IVNGS of $G_{s1} \cup G_{s2}$.*

Theorem 3.21. *If $G_s = (U_1 \cup U_2, U_{11} \cup U_{21}, U_{12} \cup U_{22}, \dots, U_{1t} \cup U_{2t})$ is union of two graph structures $G_{s1} = (U_1, U_{11}, U_{12}, \dots, U_{1t})$ and $G_{s2} = (U_2, U_{21}, U_{22}, \dots, U_{2t})$, then every IVNGS $\check{G}_{iv} = (I, I_1, I_2, \dots, I_t)$ of G_s is union of two IVNGSs $\check{G}_{iv1}$ and $\check{G}_{iv2}$ of graph structures G_{s1} and G_{s2}, respectively.*

Proof. Firstly, we define I_1, I_2, I_{1j} and I_{2j} for $j \in \{1, 2, \dots, t\}$ as:
$t^-_{I_1}(r) = t^-_I(r),\ i^-_{I_1}(r) = i^-_I(r),\ f^-_{I_1}(r) = f^-_I(r),$
$t^+_{I_1}(r) = t^+_I(r),\ i^+_{I_1}(r) = i^+_I(r), f^+_{I_1}(r) = f^+_I(r)$, if $r \in U_1$.

$t^-_{I_2}(r) = t^-_I(r),\ i^-_{I_2}(r) = i^-_I(r),\ f^-_{I_2}(r) = f^-_I(r),$
$t^+_{I_2}(r) = t^+_I(r),\ i^+_{I_2}(r) = i^+_I(r),\ f^+_{I_2}(r) = f^+_I(r)$, if $r \in U_2$.

$t^-_{I_{1j}}(r_1r_2) = t^-_{I_j}(r_1r_2),\ i^-_{I_{1j}}(r_1r_2) = i^-_{I_j}(r_1r_2),\ f^-_{I_{1j}}(r_1r_2) = f^-_{I_j}(r_1r_2),$
$t^+_{I_{1j}}(r_1r_2) = t^+_{I_j}(r_1r_2),\ i^+_{I_{1j}}(r_1r_2) = i^+_{I_j}(r_1r_2),\ f^+_{I_{1j}}(r_1r_2) = f^+_{I_j}(r_1r_2)$, if $r_1r_2 \in U_{1j}$.
$t^-_{I_{2j}}(r_1r_2) = t^-_{I_j}(r_1r_2),\ i^-_{I_{2j}}(r_1r_2) = i^-_{I_j}(r_1r_2),\ f^-_{I_{2j}}(r_1r_2) = f^-_{I_j}(r_1r_2),$
$t^+_{I_{2j}}(r_1r_2) = t^+_{I_j}(r_1r_2),\ i^+_{I_{2j}}(r_1r_2) = i^+_{I_j}(r_1r_2),\ f^+_{I_{2j}}(r_1r_2) = f^+_{I_j}(r_1r_2)$, if $r_1r_2 \in U_{2j}$.

Then $I = I_1 \cup I_2$ and $I_j = I_{1j} \cup I_{2j}$, $j \in \{1, 2, \dots, t\}$. Now for $r_1r_2 \in U_{kj}$,
$t^-_{I_{kj}}(r_1r_2) = t^-_{I_j}(r_1r_2) \le t^-_I(r_1) \wedge t^-_I(r_2) = t^-_{I_k}(r_1) \wedge t^-_{I_k}(r_2),$
$i^-_{I_{kj}}(r_1r_2) = i^-_{I_j}(r_1r_2) \le i^-_I(r_1) \wedge i^-_I(r_2) = i^-_{I_k}(r_1) \wedge i^-_{I_k}(r_2),$
$f^-_{I_{kj}}(r_1r_2) = f^-_{I_j}(r_1r_2) \le f^-_I(r_1) \wedge f^-_I(r_2) = f^-_{I_k}(r_1) \wedge f^-_{I_k}(r_2),$

$t^+_{I_{kj}}(r_1r_2) = t^+_{I_j}(r_1r_2) \le t^+_I(r_1) \wedge t^+_I(r_2) = t^+_{I_k}(r_1) \wedge t^+_{I_k}(r_2),$
$i^+_{I_{kj}}(r_1r_2) = i^+_{I_j}(r_1r_2) \le i^+_I(r_1) \wedge i^+_I(r_2) = i^+_{I_k}(r_1) \wedge i^+_{I_k}(r_2),$
$f^+_{I_{kj}}(r_1r_2) = f^+_{I_j}(r_1r_2) \le f^+_I(r_1) \wedge f^+_I(r_2) = f^+_{I_k}(r_1) \wedge f^+_{I_k}(r_2),$

Hence $\check{G}_{ivk} = (I_k, I_{k1}, I_{k2}, \dots, I_{kt})$ is an IVNGS of graph structure G_{sk}, $k = 1, 2$. Thus $\check{G}_{iv} = (I, I_1, I_2, \dots, I_t)$ an IVNGS of $G_s = G_{s1} \cup G_{s2}$, is union of two IVNGSs $\check{G}_{iv1}$ and $\check{G}_{iv2}$. □

Definition 3.22. *Let $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \dots, I_{1t})$ and $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \dots, I_{2t})$ be two IVNGSs and $U_1 \cap U_2 = \emptyset$. Join of $\check{G}_{iv1}$ and $\check{G}_{iv2}$, denoted by*

$$\check{G}_{iv1} + \check{G}_{iv2} = (I_1 + I_2, I_{11} + I_{21}, I_{12} + I_{22}, \dots, I_{1t} + I_{2t}),$$

is defined as:

$$\text{(i)} \begin{cases} t^-_{(I_1+I_2)}(r) = t^-_{(I_1 \cup I_2)}(r) \\ i^-_{(I_1+I_2)}(r) = i^-_{(I_1 \cup I_2)}(r) \\ f^-_{(I_1+I_2)}(r) = f^-_{(I_1 \cup I_2)}(r) \end{cases}$$

$$\text{(ii)} \begin{cases} t^+_{(I_1+I_2)}(r) = t^+_{(I_1 \cup I_2)}(r) \\ i^+_{(I_1+I_2)}(r) = i^+_{(I_1 \cup I_2)}(r) \\ f^+_{(I_1+I_2)}(r) = f^+_{(I_1 \cup I_2)}(r) \end{cases}$$

for all $r \in U_1 \cup U_2$,

$$\text{(iii)} \begin{cases} t^-_{(I_{1j}+I_{2j})}(rs) = t^-_{(I_{1j} \cup I_{2j})}(rs) \\ i^-_{(I_{1j}+I_{2j})}(rs) = i^-_{(I_{1j} \cup I_{2j})}(rs) \\ f^-_{(I_{1j}+I_{2j})}(rs) = f^-_{(I_{1j} \cup I_{2j})}(rs) \end{cases}$$

$$\text{(iv)} \begin{cases} t^{+}_{(I_{1j}+I_{2j})}(rs) = t^{+}_{(I_{1j}\cup I_{2j})}(rs) \\ i^{+}_{(I_{1j}+I_{2j})}(rs) = i^{+}_{(I_{1j}\cup I_{2j})}(rs) \\ f^{+}_{(I_{1j}+I_{2j})}(rs) = f^{+}_{(I_{1j}\cup I_{2j})}(rs) \end{cases}$$
for all $rs \in U_{1j} \cup U_{2j}$,

$$\text{(v)} \begin{cases} t^{-}_{(I_{1j}+I_{2j})}(rs) = (t^{-}_{I_{1j}} + t^{-}_{I_{2j}})(rs) = t^{-}_{I_1}(r) \wedge t^{-}_{I_2}(s) \\ i^{-}_{(I_{1j}+I_{2j})}(rs) = (i^{-}_{I_{1j}} + i^{-}_{I_{2j}})(rs) = i^{-}_{I_1}(r) \wedge i^{-}_{I_2}(s) \\ f^{-}_{(I_{1j}+I_{2j})}(rs) = (f^{-}_{I_{1j}} + f^{-}_{I_{2j}})(rs) = f^{-}_{I_1}(r) \wedge f^{-}_{I_2}(s) \end{cases}$$

$$\text{(vi)} \begin{cases} t^{+}_{(I_{1j}+I_{2j})}(rs) = (t^{+}_{I_{1j}} + t^{+}_{I_{2j}})(rs) = t^{+}_{I_1}(r) \wedge t^{+}_{I_2}(s) \\ i^{+}_{(I_{1j}+I_{2j})}(rs) = (i^{+}_{I_{1j}} + i^{+}_{I_{2j}})(rs) = i^{+}_{I_1}(r) \wedge i^{+}_{I_2}(s) \\ f^{+}_{(I_{1j}+I_{2j})}(rs) = (f^{+}_{I_{1j}} + f^{+}_{I_{2j}})(rs) = f^{+}_{I_1}(r) \wedge f^{+}_{I_2}(s) \end{cases}$$
for all $r \in U_1$, $s \in U_2$.

Example 3.23. *Join of two IVNGSs* $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$ *shown in Fig. 2 is defined as* $\check{G}_{iv1} + \check{G}_{iv2} = \{I_1 + I_2, I_{11} + I_{21}, I_{12} + I_{22}\}$ *and is depicted in Fig. 14.*

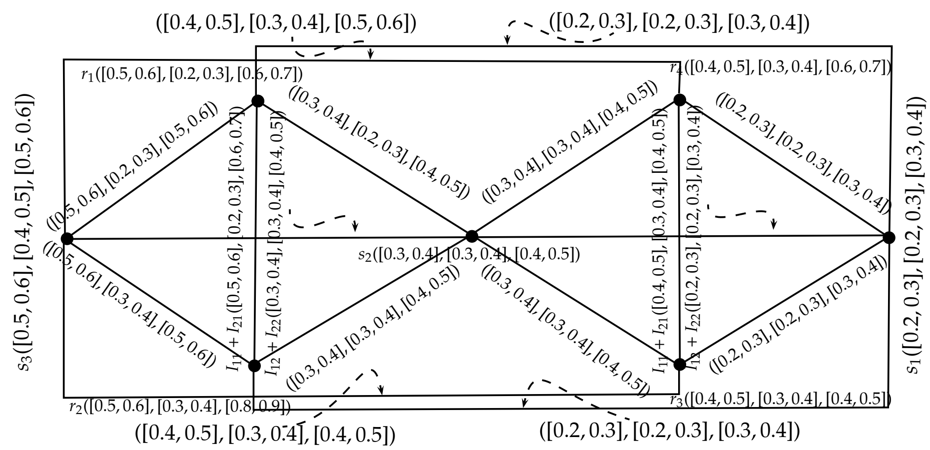

Figure 14: $\check{G}_{iv1} + \check{G}_{iv2}$

Theorem 3.24. *Join* $\check{G}_{iv1} + \check{G}_{iv2} = (I_1 + I_2, I_{11} + I_{21}, I_{12} + I_{22}, \ldots, I_{1t} + I_{2t})$ *of two IVNGSs* $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$ *of graph structures* G_{s1} *and* G_{s2}*, respectively is IVNGS of* $G_{s1} + G_{s2}$.

Theorem 3.25. *If* $G_s = (U_1 + U_2, U_{11} + U_{21}, U_{12} + U_{22}, \ldots, U_{1t} + U_{2t})$ *is join of two GSs* $G_{s1} = (U_1, U_{11}, U_{12}, \ldots, U_{1t})$ *and* $G_{s2} = (U_2, U_{21}, U_{22}, \ldots, U_{2t})$*, then every strong IVNGS* $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ *of* G_s *is join of two strong IVNGSs* $\check{G}_{iv1}$ *and* $\check{G}_{iv2}$ *of graph structures* G_{s1} *and* G_{s2}*, respectively.*

Proof. We define I_k and I_{kj} for $k = 1, 2$ and $j = 1, 2, \ldots, t$ as:
$t^{-}_{I_k}(r) = t^{-}_{I}(r),\ i^{-}_{I_k}(r) = i^{-}_{I}(r),\ f^{-}_{I_k}(r) = f^{-}_{I}(r),$
$t^{+}_{I_k}(r) = t^{+}_{I}(r),\ i^{+}_{I_k}(r) = i^{+}_{I}(r),\ f^{+}_{I_k}(r) = f^{+}_{I}(r)$, if $r \in U_k$

$t^{-}_{I_{kj}}(r_1r_2) = t^{-}_{I_j}(r_1r_2),\ i^{-}_{I_{kj}}(r_1r_2) = i^{-}_{I_{kj}}(r_1r_2),\ f^{-}_{I_{kj}}(r_1r_2) = f^{-}_{I_j}(r_1r_2),$
$t^{+}_{I_{kj}}(r_1r_2) = t^{+}_{I_j}(r_1r_2),\ i^{+}_{I_{kj}}(r_1r_2) = i^{+}_{I_{kj}}(r_1r_2),\ f^{+}_{I_{kj}}(r_1r_2) = f^{+}_{I_j}(r_1r_2)$, if $r_1r_2 \in U_{kj}$.

Now for $r_1r_2 \in U_{kj}$, $k = 1, 2$, $j = 1, 2, \ldots, t$
$t^{-}_{I_{kj}}(r_1r_2) = t^{-}_{I_j}(r_1r_2) = t^{-}_{I}(r_1) \wedge t^{-}_{I}(r_2) = t^{-}_{I_k}(r_1) \wedge t^{-}_{I_k}(r_2),$

$i^-_{I_{kj}}(r_1r_2) = i^-_{I_j}(r_1r_2) = i^-_I(r_1) \wedge i^-_I(r_2) = i^-_{I_k}(r_1) \wedge i^-_{I_k}(r_2),$
$f^-_{I_{kj}}(r_1r_2) = f^-_{I_j}(r_1r_2) = f^-_I(r_1) \wedge f^-_I(r_2) = f^-_{I_k}(r_1) \wedge f^-_{I_k}(r_2),$

$t^+_{I_{kj}}(r_1r_2) = t^+_{I_j}(r_1r_2) = t^+_I(r_1) \wedge t^+_I(r_2) = t^+_{I_k}(r_1) \wedge t^+_{I_k}(r_2),$
$i^+_{I_{kj}}(r_1r_2) = i^+_{I_j}(r_1r_2) = i^+_I(r_1) \wedge i^+_I(r_2) = i^+_{I_k}(r_1) \wedge i^+_{I_k}(r_2),$
$f^+_{I_{kj}}(r_1r_2) = f^+_{I_j}(r_1r_2) = f^+_I(r_1) \wedge f^+_I(r_2) = f^+_{I_k}(r_1) \wedge f^+_{I_k}(r_2),$

Hence $\check{G}_{ivk} = (I_k, I_{k1}, I_{k2}, \dots, I_{kt})$ is a strong IVNGS of graph structure G_{sk}, $k = 1, 2$.
Moreover, we will show that $\check{G}_{iv}$ is join of $\check{G}_{iv1}$ and $\check{G}_{iv2}$. According to the definitions 3.18 and 3.22, $I = I_1 \cup I_2 = I_1 + I_2$ and $I_j = I_{1j} \cup I_{2j} = I_{1j} + I_{2j}$, for all $r_1r_2 \in U_{1j} \cup U_{2j}$.
When $r_1r_2 \in U_{1j} + U_{2j}$ $(U_{1j} \cup U_{2j})$, that is, $r_1 \in U_1$ and $r_2 \in U_2$,

$t^-_{I_j}(r_1r_2) = t^-_I(r_1) \wedge t^-_I(r_2) = t^-_{I_k}(r_1) \wedge t^-_{I_k}(r_2) = t^-_{(I_{1j}+I_{2j})}(r_1r_2),$
$i^-_{I_j}(r_1r_2) = i^-_I(r_1) \wedge i^-_I(r_2) = i^-_{I_k}(r_1) \wedge i^-_{I_k}(r_2) = i^-_{(I_{1j}+I_{2j})}(r_1r_2),$
$f^-_{I_j}(r_1r_2) = f^-_I(r_1) \wedge f^-_I(r_2) = f^-_{I_k}(r_1) \wedge f^-_{I_k}(r_2) = f^-_{(I_{1j}+I_{2j})}(r_1r_2),$

$t^+_{I_j}(r_1r_2) = t^+_I(r_1) \wedge t^+_I(r_2) = t^+_{I_k}(r_1) \wedge t^+_{I_k}(r_2) = t^+_{(I_{1j}+I_{2j})}(r_1r_2),$
$i^+_{I_j}(r_1r_2) = i^+_I(r_1) \wedge i^+_I(r_2) = i^+_{I_k}(r_1) \wedge i^+_{I_k}(r_2) = i^+_{(I_{1j}+I_{2j})}(r_1r_2),$
$f^+_{I_j}(r_1r_2) = f^+_I(r_1) \wedge f^+_I(r_2) = f^+_{I_k}(r_1) \wedge f^+_{I_k}(r_2) = f^+_{(I_{1j}+I_{2j})}(r_1r_2).$

When $r_1 \in U_2$, $r_2 \in U_1$, we get similar calculations. It's true for $j = 1, 2, \dots, t$.
This completes the proof. □

We have defined the concept of interval-valued neutrosophic line graphs in Definition 2.3. Thus, we close this research article with the following open problems:
Problem 1. Prove or disprove that interval-valued neutrosophic line graph structures are generalization of interval-valued neutrosophic line graphs.
Problem 2. Prove or disprove that if f is a weak isomorphism of interval-valued neutrosophic graph structures onto interval-valued neutrosophic line graph structures, then f is an isomorphism.
Acknowledgement. The authors are grateful to the reviewer's valuable comments that improved the quality of our paper.

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