A brief investigation into two sets of elliptic curves

Lulu Karami
lulu.karami82@gmail.com

Abstract

This submission is more or less an amateur exposition on a specific elliptic curve, discussing counting points over finite fields as well as constructing an associated L-function and pinning down the affiliated special value $L(E, 1)$ for the elliptic curve E primarily discussed throughout this piece. The techniques and tools presented can be carried over to infinitely many elliptic curves partitioned into 2 sets depending on 'twists' of 2 specific curves; one of which happens to be the curve previously, and vaguely, mentioned.

1 Counting Points

Let ${}_1E_m(X,Y)$ and ${}_2E_n(X,Y)$ be the elliptic curves defined by the equations

$$\begin{aligned} {}_1E_m(X,Y) =& Y^2 - (X^3 + m) \\ {}_2E_n(X,Y) =& Y^2 - (X^3 - nX) \end{aligned}$$

where m and n are **non-zero** integers. They will simply be referred to as E when defining stuff about elliptic curves and labelled without their corresponding subscripts given the right context. The objective in this section "Counting Points" is to find the number of integer solution pairs (a,b), called "points", satisfying

$${}_1E(X,Y)_1 \equiv 0 \pmod p$$

for every prime p. The techniques and tools used for this specific curve can be carried over to ${}_1E_m$ as well as ${}_2E_n$ for infinitely many m, n. From here until the concluding remarks, E will refer to an elliptic curve of the former kind.
The solutions are taken from $\mathbb{Z}^2$ after reduction mod p. To elaborate in detail, if the pairs

$$\begin{aligned} &(a,b) \\ &(c,d) \end{aligned}$$

are entry equivalent mod p, then they are considered identical points. The notation $E(q)$ will be used in place of

$$\begin{aligned} E(q) =& E(a,b) \\ q =& (?_1, ?_2) \end{aligned}$$

where q refers to an arbitrary pair in $\mathbb{F}_p^2$. The number of distinct solutions will be denoted as the set cardinality

$$\mid E/p \mid = \mid \{(X,Y) \big| E \equiv 0 \pmod p) \wedge (X,Y) \in \mathbb{F}_p^2\} \mid$$

And finally, p will be a prime. Sometimes, stated somewhere right below the title of a section, p will be a specific kind of prime.

1.1 Solution for E_1

Take m to be 1 for now. The general case will reveal after the instance $m = 1$ is solved.

The primes $p = 2, 3$ are small enough to compute E_1/p by hand. It's a good idea to compute E_1/p by hand for a couple p before trying to tackle the general prime p. 2 and 3 are small enough to do this with. Since this is all done in a sub-section titled "Solutions for E_1", E will mean E_1 while we're here.
For $p = 2$, the possible solutions to $E = 0$ must be among the set

$$\{(0,0),(0,1),(1,0),(1,1)\}$$

A quick computation shows that there only 2 solutions;

$$E/2 = \{(0,1),(1,0)\}$$

For $p = 3$, the possible solutions must be among the set

$$\{(0,0),(1,0),(-1,0),(0,1),(1,1),(-1,1),(0,-1),(1,-1),(-1,-1)\}$$

Before going ahead and putting these through E and seeing if they satisfy the equation consider how the polynomial $X^3 + 1$ factors in $\mathbb{F}_p[X]$;

$$\begin{aligned} X^3 + 1 =& X^3 + 3X^2 + 3X + 1 - 3(X^2 + X) \\ \equiv& (X + 1)^3 \pmod 3 \end{aligned}$$

Since X will be an integer, we can reduce the equation to

$$Y^2 - (X + 1) \equiv 0 \pmod 3$$

simplifying things even further. When $Y = 0$, the only solution for X is -1 and when $Y = \pm 1$, the only solution for X is 0. Thus the elements of $E/3$ are

$$E/3 = \{(0,-1),(1,0),(-1,0)\}$$

So far, it seems like

$$\mid E/p \mid = p$$

is the rule of thumb (purposefully misleading).

1.1.1 About half of all the primes

Consider all the primes of the form $3k + 2$. The multiplicative group

$$\begin{aligned} (\mathbb{Z}/p\mathbb{Z})^\times \simeq& \mathrm{Z}_{p-1} \\ =& \mathrm{Z}_{3k+1} \end{aligned}$$

where Z_l is the cyclic group of order l. Mapping every element to it's cube in this group is an **automorphism**. Translations in $\mathbb{F}_p$,

$$x \mapsto x + 1$$

are **isomorphisms**. This implies E can be rewritten as

$$E = Y^2 - Z$$

where Z, used as a variable and not alluding to a cyclic group, can truly be seen as a linear term in this context of X being an integer and E under reduction mod p; an implication that every Y has a unique solution Z. Since there are p residue classes, there are exactly p pairs of solutions. This leads to the conclusion

$$\big(p = 3k + 2\big) \implies (\mid E/p \mid = p)$$

1.1.2 The other half of all the primes

When p is of the form $3k + 1$, which will always be the case for this sub-sub-section, cubing **shrinks** the field $\mathbb{F}_p$. This can be seen by inspecting the multiplicative group

$$\begin{aligned} \big(\mathbb{Z}/p\mathbb{Z}\big)^\times &\simeq \mathrm{Z}_{p-1} \\ &= \mathrm{Z}_{3k} \end{aligned}$$

For example, take w to be a generator for the group. The endomorphism ϕ, defined as

$$\phi : \alpha \mapsto \alpha^3$$

map the elements $1, w^k$, and w^{2k} all to 1, where k is coming from $p = 3k + 1$. Quick reminder; k will always come from a quotient.
Rather than relying on an approach using transformations of the curve, it is a good idea to lay down some facts;

$$(E(q) \equiv 0) \implies \Big((Y^2 - 1 \text{ is a perfect cube}) \wedge (X^3 + 1 \text{ is a perfect square})\Big)$$

Although both of the nested statements are simple transformations, it is important to take note of their significance.

By introducing a character χ on $\mathbb{Z}/p\mathbb{Z}$

$$\begin{aligned}
\chi : \mathbb{Z}/p\mathbb{Z} \to& \{1, \zeta_3, \zeta_3^2\} \\
\chi : w \mapsto& g(w)^k \\
g : w \mapsto& \zeta_{p-1} \\
\zeta_l =& e^{\frac{2\pi\imath}{l}}
\end{aligned}$$

it is possible to determine whether the equation

$$x^3 \equiv t \pmod p$$

is solvable and how many solutions it has using χ. Indeed, if t is of the form x^3 modulo p, then the quantity $c(t)$ defined as

$$c(t) = 1 + \chi(t) + \chi^2(t)$$

takes on the values

$$c(t) = \begin{cases} 3 & \text{if } x^3 - t \text{ is solvable in } \mathbb{F}_p \\ 0 & \text{if } x^3 - t \text{ is not solvable in } \mathbb{F}_p \end{cases}$$

Let's attempt to evaluate the quantity $\mid E/p \mid$, defined as

$$\mid E/p \mid = \sum_{y \in \mathbb{F}_p} \left(1 + \chi(y^2 - 1) + \chi^2(y^2 - 1)\right)$$

and briefly compare it to what $\mid E/p \mid$ would look like for the general m

$$\mid E/p \mid = \sum_{y \in \mathbb{F}_p} \left(1 + \chi(y^2 - m) + \chi^2(y^2 - m)\right)$$

to eliminate the need of doing this all over again. **Keep the second one in mind while working on** $m = 1$.

The value of the character$\chi(r)$, for an integer r, can be given as

$$\chi(r) = \tau(\bar{\chi})^{-1} \cdot \sum_{l \in \mathbb{F}_p} \bar{\chi}(l)\zeta^{l \cdot r}$$

where the quantity dividing the sum is defined as

$$\tau(\psi) = \sum_{l \in \mathbb{F}_p} \psi(l)\zeta_p^l$$

for a multiplicative character ψ. Going back to $\mid E/p \mid$, we have

$$\begin{aligned} \mid E/p \mid =& \Big(\sum_{y \in \mathbb{F}_p} 1 \Big) + \Big(\sum_{y \in \mathbb{F}_p} \chi(y^2 - 1) \Big) + \Big(\sum_{y \in \mathbb{F}_p} \chi^2(y^2 - 1) \Big) \\ =& p + \Big(\tau(\bar{\chi})^{-1} \cdot \sum_{(y,l) \in \mathbb{F}_p^2} \bar{\chi}(l)\zeta_p^{l \cdot (y^2 - 1)} \Big) + \Big(\tau(\chi)^{-1} \cdot \sum_{(y,l) \in \mathbb{F}_p^2} \chi(l)\zeta_p^{l \cdot (y^2 - 1)} \Big) \end{aligned}$$

Let's slowly dissect this together; the sum

$$\sum_{(y,l) \in \mathbb{F}_p^2} f(l, y)$$

is interpreted as the double sum

$$\sum_{1 \le y \le p} \sum_{1 \le l \le p} f(l, y)$$

which is **finite**. Rather than sum over l in the interior sum then proceed to y, a reversal of the order for the sums lastly defining $\mid E/p \mid$ gives

$$\begin{aligned} \mid E/p \mid =& p + \Big(\tau(\bar{\chi})^{-1} \cdot \sum_{1 \le l \le p} \bar{\chi}(l)\zeta_p^{-l} \sum_{1 \le y \le p} \zeta^{l \cdot y^2} \Big) + \Big(\tau(\chi)^{-1} \cdot \sum_{1 \le l \le p} \chi(l)\zeta_p^{-l} \sum_{1 \le y \le p} \zeta^{l \cdot y^2} \Big) \\ =& p + \Big(\tau(\bar{\chi})^{-1} \sum_{1 \le l \le p} \bar{\chi}(l)\zeta_p^{-l} \cdot J(l) \Big) + \Big(\tau(\chi)^{-1} \sum_{1 \le l \le p} \chi(l)\zeta_p^{-l} \cdot J(l) \Big) \end{aligned}$$

where, $J(l)$ is the sum

$$\begin{aligned} J(l) =& \sum_{1 \le r \le p} \zeta_p^{r^2 \cdot l} \\ =& 1 + \sum_{\substack{1 \le r \le p \\ \text{r is square}}} 2\zeta_p^r \end{aligned}$$

Note that r is a perfect square **modulo** p, the appearance of the factor of 2 stems from $(-r)^2 = r^2$, and the quantity 1 added to the sum is from the p^2. This can be rewritten as

$$\begin{aligned} J(l) = &\sum_{1\leq r\leq p} (1+\psi(r))\zeta_p^{r\cdot l} \\ =&\Big(\sum_{1\leq r\leq p} \zeta_p^{r\cdot l}\Big) + \Big(\sum_{1\leq r\leq p} \psi(r)\zeta_p^{r\cdot l}\Big) \end{aligned}$$

where ψ is the non-trivial quadratic character of modulus p. The usage of $(1+\psi(t))$ counts the solutions in characteristic p to the equation $x^2 = r$, similarly to the quantity $c(t)$ defined earlier.
Before moving ahead, recall the appearance of J in the sum

$$\Big(\tau(\bar{\chi})^{-1} \sum_{1\leq l\leq p} \bar{\chi}(l)\zeta_p^{-l}\cdot J(l)\Big)$$

The last term, when $l = p$, is zero. There is no need to worry about when l is an integer multiple of p. This allows us to remove the first sum

$$\sum_{1\leq r\leq p} \zeta_p^{r\cdot l}$$

from the definition of J. The substitution

$$\begin{aligned} J(l) = &\sum_{1\leq l\leq p} \psi(r)\zeta_p^{r\cdot l} \\ =&\psi(l)\cdot\tau(\psi) \end{aligned}$$

can be safely used without changing the desired value $\mid E/p \mid$. Substituting m back into summation and sompleting everything gives a closed form for our desired quantity;

$$\mid E/p \mid = p + \psi(-m)\chi(-m)\gamma + \psi(-m)\bar{\chi}(-m)\bar{\gamma}$$

where the quantity γ is given by

$$\gamma = \frac{\tau(\psi)\tau(\psi\cdot\bar{\chi})}{\tau(\bar{\chi})}$$

1.1.3 What is γ?

Given that the absolute value of a Gauss sum of modulus p and primitive character is $\sqrt{p}$, the absolute value

$$\begin{aligned} \mid \gamma \mid = \mid & \frac{\tau(\psi)\tau(\psi \cdot \bar{\chi})}{\tau(\bar{\chi})} \mid \\ = & \sqrt{p} \end{aligned}$$

along with the fact that γ is a sum with elements from the set

$$\{1, \zeta_3, \zeta_3^2\}$$

is indicative that γ is an integer in the ring $\mathbb{Z}(\zeta_3)$ as well as a prime with field norm p. Squaring γ in quotient rings gives

$$\begin{aligned} \gamma^2 \equiv & \frac{\tau(\psi^2)\tau(\psi^2 \cdot \bar{\chi}^2)}{\tau(\bar{\chi}^2)} \pmod 2 \\ \equiv & -\frac{\tau(\chi)}{\tau(\chi)} \pmod 2 \\ \equiv & 1 \pmod 2 \end{aligned}$$

Since the ring

$$\mathbb{Z}(\zeta_3)/\big(2\mathbb{Z}(\zeta_3)\big)$$

is a field with 4 elements, it can be concluded that γ is of the form

$$\gamma = a + 2b\zeta_3$$

for a, b integers and a odd since squaring is an automorphism in this ring which fixes only 0 and 1 modulo 2. Furthermore, in the ring

$$\mathbb{Z}(\zeta_3)/\big(\sqrt{-3}\mathbb{Z}(\zeta_3)\big)$$

Cubing γ gives

$$\begin{aligned} \gamma^3 \equiv & \frac{\tau(\psi^3)\tau(\psi^3 \cdot \bar{\chi}^3)}{\tau(\bar{\chi}^3)} \pmod{\sqrt{-3}} \\ \equiv & \frac{\tau(\psi)^2}{\tau(\chi^3)} \pmod{\sqrt{-3}} \\ \equiv & -(-1)^{\frac{p-1}{2}} p \pmod{\sqrt{-3}} \\ \equiv & -(-1)^{\frac{p-1}{2}} \pmod{\sqrt{-3}} \end{aligned}$$

which implies γ is of the form

$$\gamma = -\,\chi_4(p)\chi_3(a+2b)\cdot(a+2b\zeta_3)$$

where χ_4 and χ_3 are the primitive Dirichlet characters of modulus 4 and 3 respectively. Plugging this back in gives

$$\mid E/p \mid = p - \psi(m)\chi(m)\chi_3(a+2b)(a+2b\zeta_3) - \psi(m)\bar{\chi}(m)\chi_3(a+2b)(a+2b\zeta_3^2)$$

2 L-function and cusp form of E_1

It is interesting to see what happens if one takes the quantity

$$a_p = p- \mid E/p \mid$$

and weave it into an L-function in the following fashion;

$$L(E,s) = \prod_{\text{good} p} L_p(p^{-s})^{-1}$$

where $L_p(X)$ is the polynomial

$$L_p(X) = 1 - a_p X + pX^2$$

and 'good' primes are primes not equal to $p = 2, 3$ for most E_1's, depending on m. For now, $m = 1$ will be the case and $p = 2, 3$ will be completely ignored in the Euler Product for $L(E, s)$. For other m's not a 6^{th} power, this corresponds to either a **quadratic** twist, a **cubic** twist, or a combination of both.
The Euler product for $L(E)$ can then be simplified down to

$$\prod_{\pi} \left(1 - \frac{\chi_3(\pi)}{(\mathrm{N}\pi)^s}\right)^{-1}$$

where the product runs over all Eisenstein Primes not equal to $\pi = 2, \sqrt{-3}$ and the primes are 1 modulo 2. In other words, they are of the form

$$\pi = a + b\sqrt{-3}$$

Expanding the L-function as a Dirichlet series gives

$$\begin{aligned} 2L(E,s) = &\sum_{a+b\sqrt{3}\neq 0} \chi(a) \frac{a+b\sqrt{-3}}{(a^2+3b^2)^s} \\ =&2\sum_{n\geq 1} \frac{c_n}{n^s} \end{aligned}$$

Coincidentally, there happens to exist a peculiar generating function;

$$\begin{aligned} f(q) = &\sum_{(a,b)\in\mathbb{Z}^2} \chi_3(a)(a+b\sqrt{-3})q^{a^2+3b^2} \\ =&2\sum_{n\geq 1} c_n q^n \end{aligned}$$

$f(q)$ happens to satisfy

$$f(\frac{-1}{\tau}) = -\tau^2 f(\tau)$$

where q is interpreted to be

$$q = e^{\frac{\pi \imath \tau}{3}}$$

When talking about quantities such as $f(\sqrt{-3})$, it is implied that $f(\tau)$ is being used rather than $f(q)$. Rewriting $f(q)$ and reexpressing it as a product of two functions reveals why;

$$\begin{aligned} f(q) &= \sum_{(a,b)\in\mathbb{Z}^2} \chi_3(a)(a)q^{a^2+3b^2} \\ &= \Big(\sum_{a\in\mathbb{Z}} a\chi_3(a)q^{a^2}\Big) \cdot \Big(\sum_{b\in\mathbb{Z}} q^{3b^2}\Big) \\ &= 2\theta(\chi_3,\tau)\theta(\tau) \end{aligned}$$

Where each θ-function is given respectively by the product of the two series. The product of the two **Theta functions** can be expressed as an η-quotient;

$$\begin{aligned} \theta(\chi_3,\tau)\cdot\theta(\tau) &= \Big(\frac{\eta^2(\frac{\tau}{2})\cdot\eta^2(2\tau)}{\eta(\tau)}\Big) \cdot \Big(\frac{\eta^5(\tau)}{\eta^2(\frac{\tau}{2})\cdot\eta^2(2\tau)}\Big) \\ &= \eta^4(\tau) \end{aligned}$$

Both expressions can be derived by massaging the **Jacobi triple product** in the right way.
$\eta(\tau)$ can also be given by the product

$$\eta(\tau) = e^{\frac{\pi\imath\tau}{12}} \cdot \prod_{n\geq 1} \big(1 - e^{2\pi\imath n\tau}\big)$$

The existence of a modular form $f(\tau)$ implies the existence of a functional equation on $L(E)$. The completed L-function

$$\big(\frac{\pi}{3}\big)^{-s}\Gamma(s)\cdot L(E,s) = \Lambda(s)$$

satisfies the reflection formula

$$\Lambda(s) = \Lambda(2-s)$$

which can be proven by the mellin transform of $f(\imath\tau)$.

3 A related Eisenstein Series

For reasons which will be revealed in the next section, consider the weight 1 Eisenstein series

$$g(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{\chi_3(m)}{m+n\tau}$$

Although conditional convergence is an issue here, it is possible to make the sum converge for all τ in the upper half plane.

3.1 Fourier series of $g(\tau)$

To start off with finding the Fourier series of $g(\tau)$, it is best to begin with the function

$$g_0(z) = \sum_{m\in\mathbb{Z}} \frac{\chi_3(m)}{m+z}$$

g_0's Fourier series can be found by treating the sum in the following way;

$$\sum_{m\in\mathbb{Z}} \frac{\chi_3(m)}{m+z} = \sum_{l=1}^{2} \chi_3(l) \sum_{k\in\mathbb{Z}} \frac{1}{3k+l+z}$$

From here, the goal is to use the formula

$$\sum_{m\in\mathbb{Z}} \frac{1}{m+z} = \pi\cot(\pi\cdot z)$$

and cotangent's Fourier series

$$\pi\cot(\pi\cdot z) = -\pi\imath - 2\pi\imath\cdot\sum_{r\geq 1} e^{2\pi\imath rz}$$

to make sense of this series;

$$\begin{aligned}\sum_{l=1}^{2} \chi_3(l) \sum_{k\in\mathbb{Z}} \frac{1}{3k+l+z} &= \sum_{l=1}^{2} \frac{\chi_3(l)}{3} \sum_{k\in\mathbb{Z}} \frac{1}{k+\frac{l+z}{3}} \\ &= \sum_{l=1}^{2} \Big(\frac{\chi_3(l)}{3}\Big)\cdot\pi\cot\big(\pi\cdot(\frac{l+z}{3})\big)\end{aligned}$$

The last expression can be used to give a Fourier series for g_0 easily by using the fourier series of cotangent;

$$g_0(z) = \frac{2\pi}{\sqrt{3}} \cdot \sum_{r\geq 1} \chi_3(r) e^{\frac{2\pi \imath r z}{3}}$$

This series for g_0 can be used to find a series for g by considering the sum

$$g(\tau) = \Big(\sum_{m\in\mathbb{Z}\setminus 0} \frac{\chi_3(m)}{m} \Big) + \Big(\sum_{n\in\mathbb{Z}\setminus 0} g_0(n\tau) \Big)$$

The trick here to get a Fourier series of the form

$$g(\tau) = \sum_{n\geq 0} b_n \cdot e^{\frac{2\pi \imath n\tau}{3}}$$

is to rewrite the sum over $g_0(n\tau)$ as

$$\sum_{n\in\mathbb{Z}\setminus 0} g_0(n\tau) = \sum_{n\geq 1} \big(g_0(n\tau) + g_0(-n\tau) \big)$$

I'm going to leave out the proof of g_0 being an even function since it is way too elementary to be relevant. But it is true and it implies

$$\begin{aligned} \sum_{n\in\mathbb{Z}\setminus 0} g_0(n\tau) =& 2\sum_{n\geq 1} g_0(n\tau) \\ =& \frac{4\pi}{\sqrt{3}} \cdot \sum_{m,n\geq 1} \chi_3(m) e^{\frac{2\pi \imath m n\tau}{3}} \end{aligned}$$

This gives the Fourier series

$$\begin{aligned} g(\tau) =& 2L(\chi_3, 1) + \frac{4\pi}{\sqrt{3}} \cdot \sum_{m,n\geq 1} \chi_3(m) e^{\frac{2\pi \imath m n\tau}{3}} \\ =& \frac{2\pi}{3\sqrt{3}} + \frac{4\pi}{\sqrt{3}} \cdot \sum_{m,n\geq 1} \chi_3(m) e^{\frac{2\pi \imath m n\tau}{3}} \end{aligned}$$

where the quantity $L(\chi_3, 1)$ is comes from

$$L(\chi_3, s) = \sum_{n\geq 1} \frac{\chi_3(n)}{n^s}$$

3.2 Further inspection of the fourier series

What do the Fourier coefficients of g look like? Or more precisely; What do they mean? Factoring the constant coefficient gives

$$\begin{aligned} g(\tau) =& \frac{2\pi}{3\sqrt{3}} \cdot \left(1 + 6 \sum_{m,n\geq 1} \chi_3(m) e^{\frac{2\pi \imath m n \tau}{3}}\right) \\ =& \frac{2\pi}{3\sqrt{3}} \cdot \left(1 + 6 \sum_{m\geq 1} r_m e^{\frac{2\pi \imath m}{3}}\right) \end{aligned}$$

where r_m is defined as

$$\zeta(s) \cdot L(\chi_3, s) = \sum_{n\geq 1} \frac{r_n}{n^s}$$

Coincidentally, the L-function of the UFD

$$R =\mathbb{Z}(\frac{-1+\sqrt{-3}}{2})$$

happens to coincide with the L-function whose Dirichlet Coefficients are r_n! And since R has unique factorization, we can work our way backwards from this fact, remembering the norm of an element of R and the presence of the 6 units in R, to conclude the following on g;

$$g(\tau) =\frac{2\pi}{3\sqrt{3}} \cdot \sum_{(m,n)\in\mathbb{Z}^2} e^{\frac{2\pi\imath(m^2-mn+n^2)\tau}{3}}$$

4 $L(E,1)$

The reason why g was introduced was to pin down the value of

$$2L(E,1) = \sum_{a,b\in\mathbb{Z}^2\setminus(0,0)} \frac{\chi_3(a)(a+b\sqrt{-3})}{(a^2+3b^2)^1}$$

This can be given as

$$\begin{aligned}2L(E,1) = &\sum_{a,b\in\mathbb{Z}^2\setminus(0,0)} \frac{\chi_3(a)}{a+b\sqrt{-3}}\\ =&g(\sqrt{-3})\end{aligned}$$

If the series

$$\frac{2\pi}{3\sqrt{3}} \sum_{(m,n)\in\mathbb{Z}^2} e^{\frac{-2\pi(m^2-mn+n^2)}{\sqrt{3}}}$$

is used alone, all this can do is yield a series converging somewhat faster than the regular Dirichlet series of $2L(E,1)$. However, the following equality can be shown by carefully breaking up the double sum right above and using products of Jacobi theta functions;

$$g(\tau) = \frac{2\pi}{3\sqrt{3}} \cdot \Big(\frac{\eta^5(\frac{2\tau}{3})\cdot\eta^5(2\tau)}{\eta^2(\frac{\tau}{3})\cdot\eta^2(\frac{4\tau}{3})\cdot\eta^2(\tau)\cdot\eta^2(4\tau)} + 4\frac{\eta^2(\frac{4\tau}{3})\cdot\eta^2(4\tau)}{\eta(\frac{2\tau}{3})\eta(2\tau)}\Big)$$

Which gives the 'closed' form for $2L(E,1)$ as a sum of η quotients evaluated at $\tau = \sqrt{-3}$;

$$2L(E,1) = \frac{2\pi}{3\sqrt{3}} \cdot \Big(\frac{\eta^5(\frac{-2}{\sqrt{-3}})\cdot\eta^5(2\sqrt{-3})}{\eta^2(\frac{-1}{\sqrt{-3}})\cdot\eta^2(\frac{-4}{\sqrt{-3}})\cdot\eta^2(\sqrt{-3})\cdot\eta^2(4\sqrt{-3})} + 4\frac{\eta^2(\frac{-4}{\sqrt{-3}})\cdot\eta^2(4\sqrt{-3})}{\eta(\frac{-2}{\sqrt{-3}})\eta(2\sqrt{-3})}\Big)$$

It is possible to accurately guess what $L(E,1)$ might be exactly from here. The presence of $\sqrt{-3}$ in the η quotients, as well as the fact that the net sum of η powers is 2 in each one implies the quantity

$$\frac{2L(E,1)}{(\frac{2\pi}{3\sqrt{3}})\cdot \mid \eta(\frac{-1+\sqrt{-3}}{2}) \mid^2}$$

is in fact **algebraic**. With the equality,

$$\mid \eta(\frac{-1+\sqrt{-3}}{2}) \mid^2 = \frac{\sqrt[4]{3}\Gamma^3(\frac{1}{3})}{(2\pi)^2}$$

whose derivation can be found using this special case for the previously mentioned ring R,

$$| \, 2\pi\imath \cdot \eta^2(\frac{-1+\sqrt{-3}}{2}) \, |^2 = \exp(-6\zeta_R'(0))$$

Mathematica's **RootApproximant**[ ] makes a very good guess for the algebraic quotient. Putting the hypothesized algebraic quotient and the transcendental factor back together, it is **extremely** likely $2L(E,1)$ can be given in the closed form

$$2L(E,1) = \frac{\Gamma^3\left(\frac{1}{3}\right)}{(2\pi)\sqrt[6]{108}}$$

5 Concluding Remarks for $m \neq 1$ and ${}_2E_n$

When talking of other m, the same can be done as demonstrated in this exposition for plenty of ${}_1E_m$. I haven't worked out any other examples, but it would be very easy to find $L({}_1E_m, 1)$ for m either a prime of the form $3k+1$ or a product of primes of the form $3k+1$. The quantity would look something along the lines of

$$L({}_1E_m, 1) = \epsilon \cdot \sum_{l \in \mathbb{Z}/m'\mathbb{Z}} \chi_{m'}(l) g\left(\frac{\gamma + l}{m'}\right)$$

where m' is the smallest integer such that m is an integer power of m', $\chi_{m'}$ is a non-trivial primitive character of modulus m' whose order divides 6, γ satisfies

$$\gamma \in \mathbb{Q}(\sqrt{-3}) \wedge \mathrm{Im}(\gamma) > 0$$

and ϵ is algebraic. ϵ involves some quotient containing a power of m', $\tau(\chi_{m'})$, and some Eisenstein Integer. They're somewhere in the numerator and/or denominator. There's also the possibility the quantity, defined by a product of Euler factors,

$$\prod_{p \text{ is bad}} L_p(p^{-1})$$

needs to be taken into consideration (cautiously) for ϵ, if needed.

In other words; ϵ is easy to compute and the quantity

$$\frac{L({}_1E_m, 1)}{L(\chi_3, 1)}$$

is a finite linear combination, over a cyclotomic field, of η quotients evaluated at γ.

I'm also too lazy to do anything for ${}_2E_n$ except mention two things, which can be taken however way you, the reader, want; as a pair of breif closing statements or poorly worded exercises:

1)Very similar stuff can be done for whenever n is a prime of the form $4k+1$ or a product of primes of the form $4k+1$.

2) Express $\theta^2(\tau)$ and/or $\theta(\tau)\theta(2\tau)$ as a weight 1 Eisenstein Series. One of these plays the same role as $g(\tau)$. I'm not remotely motivated to confirm which.