Three Limits

Edgar Valdebenito

abstract

This note presents three limits for $1/\pi$.

1. Introduction

There are many formulas of π of many types. Among others , these include series, products, continued fractions, geometric constructions, pi iterations, special values, integrals, and limits.

An interesting infinite product is

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\ldots \tag{1}$$

A related formula is given by

$$\pi = \lim_{n\to\infty} 2^{n+1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}} \tag{2}$$

In this note we remenber three limits for $1/\pi$.

$$\frac{1}{\pi} = 0.318309886183\ldots \tag{3}$$

2. Three Limits

$$\frac{1}{\pi} = \lim_{n\to\infty} 2^{-n-1}\sum_{k=1}^{2^n}\sin\left(\frac{k\pi}{2^n}\right) \tag{4}$$

$$\frac{1}{\pi}=\lim_{n\to\infty} 2^{-n-1}\sum_{k=1}^{2^n}\sin\left(\frac{(2k-1)\pi}{2^{n+1}}\right) \tag{5}$$

$$\frac{1}{\pi}=\lim_{n\to\infty} 2^{-n-1}\sum_{k=1}^{2^n}\cos\left(\frac{(2k-1)\pi}{2^{n+2}}\right) \tag{6}$$

3. Graphics

$$f(n)=2^{-n-1}\sum_{k=1}^{2^n}\sin\left(\frac{k\pi}{2^n}\right) \quad ,n=0,1,2,3,\ldots \tag{7}$$

$$g(n)=2^{-n-1}\sum_{k=1}^{2^n}\sin\left(\frac{(2k-1)\pi}{2^{n+1}}\right) \quad ,n=0,1,2,3,\ldots \tag{8}$$

$$h(n)=2^{-n-1}\sum_{k=1}^{2^n}\cos\left(\frac{(2k-1)\pi}{2^{n+2}}\right) \quad ,n=0,1,2,3,\ldots \tag{9}$$

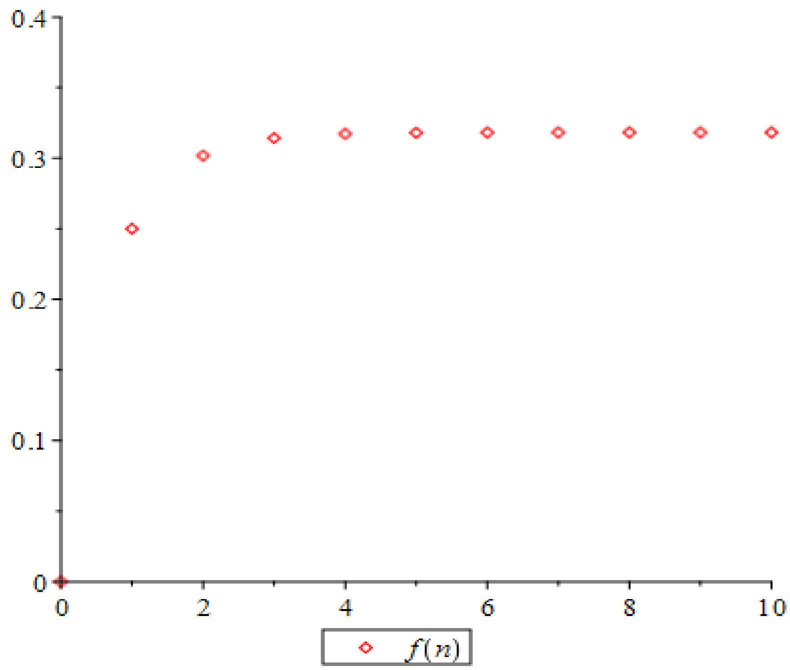

Figure 1.

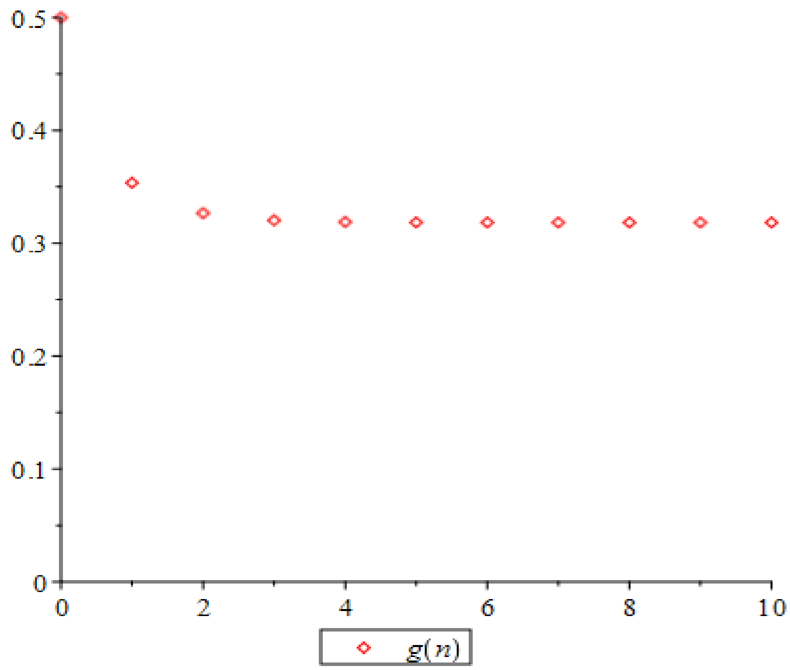

Figure 2.

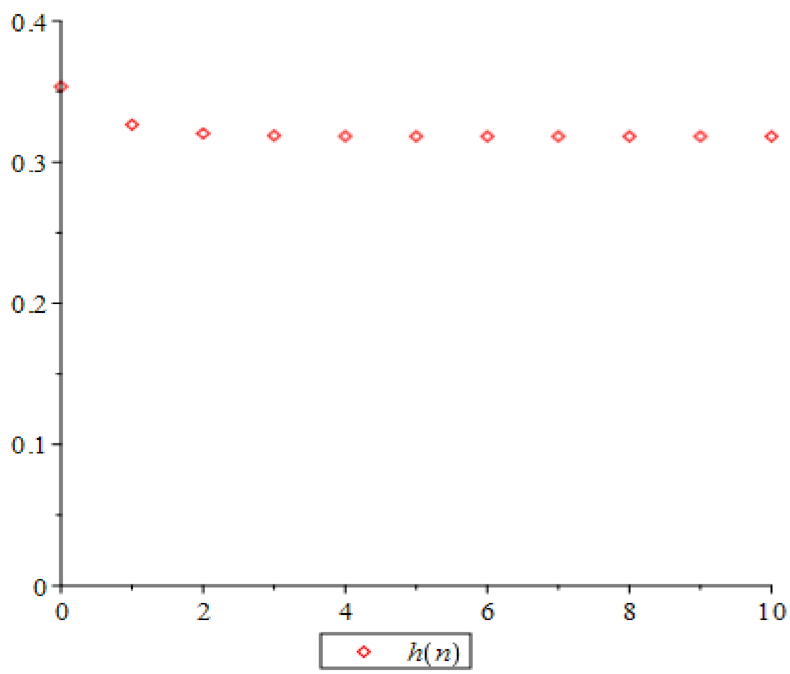

Figure 3.

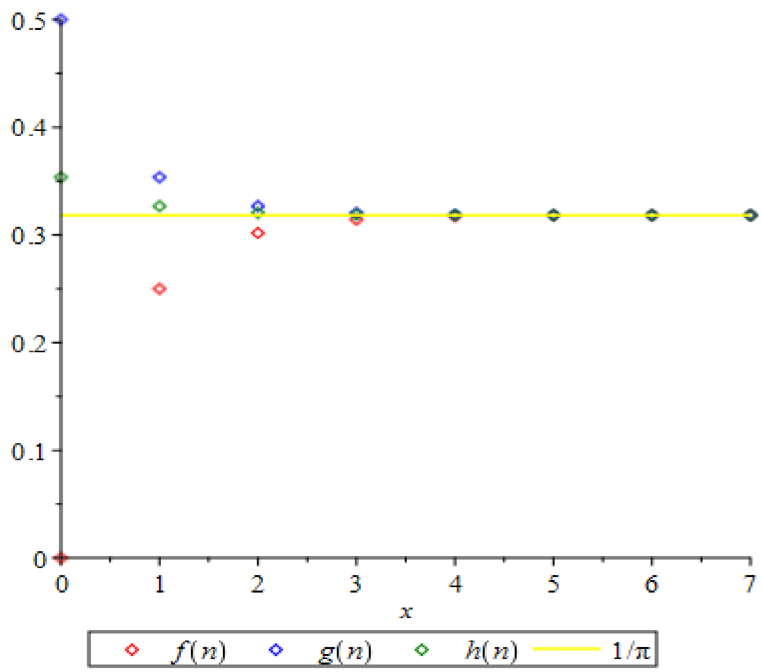

Figure 4.

4. Relations

$$f(n) = 2^{-n-1}\left(\sin\left(\frac{\pi}{2^{n+1}}\right)\right)^{-1}\cos\left(\frac{\pi}{2^{n+1}}\right) \quad ,n = 0,1,2,3,\ldots \tag{10}$$

$$g(n) = 2^{-n-1}\left(\sin\left(\frac{\pi}{2^{n+1}}\right)\right)^{-1} \quad ,n = 0,1,2,3,\ldots \tag{11}$$

$$h(n) = g(n+1) \quad ,n = 0,1,2,3,\ldots \tag{12}$$

$$f(n) = g(n)\cos\left(\frac{\pi}{2^{n+1}}\right) \quad ,n = 0,1,2,3,\ldots \tag{13}$$

$$g(n) > g(n+1) \quad ,n = 0,1,2,3,\ldots \tag{14}$$

$$h(n) > h(n+1) \quad ,n = 0,1,2,3,\ldots \tag{15}$$

$$g(n) > h(n) \quad ,n = 0,1,2,3,\ldots \tag{16}$$

$$f(n) < g(n) \quad , n = 0,1,2,3,\ldots \tag{17}$$

$$g(n) > \frac{1}{\pi} \quad , n = 0,1,2,3,\ldots \tag{18}$$

$$f(n) < \frac{1}{\pi} \quad , n = 0,1,2,3,\ldots \tag{19}$$

For $n = 0,1,2,3,\ldots$:

$$f(n) = \frac{2^{-n-1}}{\sqrt{2 - \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}}} \sqrt{2 + \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}} \tag{20}$$

$$g(n) = \frac{2^{-n-1}}{\sqrt{2 - \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}}} \tag{21}$$

$$h(n) = \frac{2^{-n-2}}{\sqrt{2 - \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1}}} \tag{22}$$

References

1. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals, Series, and Products.seventh edition. Edited by Alan Jeffrey and Daniel Zwillinger. Academic Press, 2007.
2. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., and Clark, C.W.: NIST Handbook of Mathematical Functions. Cambridge University Press, 2010.
3. Vieta, F.: Uriorun de rebus mathematicis responsorun. Liber VII. 1593. Reprinted in New York: Georg Olms, pp. 398-400 and 436-446, 1970.
4. Wolfram Math World: Pi Formulas. http://mathworld.wolfram.com/PiFormulas.html