Question 406: Mathematical Formulas Involving Pi

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abstract

This note presents some integrals involving pi

1. Introduction

A Philosophical Introduction:

I could never resist an integral ,G.H. Hardy (1877-1947)

Remark: $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.1415926535...$

2. Formulas

$$\pi = 4\int_0^{\pi/2}\frac{1}{3-2\sin x+\cos x}dx \tag{1}$$

$$\pi = 4\int_0^{\pi/2}\frac{1}{3+\sin x-2\cos x}dx \tag{2}$$

$$\pi = 3\int_0^1 \sin^{-1}\left(\frac{2\sqrt{1+12x-9x^2}+18x-2}{5+15x}\right)dx \tag{3}$$

$$\frac{\pi}{30\sqrt{3}} = \int_0^{\pi/2}\frac{1}{19+5\sin x+6\cos x}dx \tag{4}$$

$$\frac{\pi}{30\sqrt{3}} = \int_0^{\pi/2} \frac{1}{19 + 6\sin x + 5\cos x} dx \tag{5}$$

$$\pi = \int_0^{\pi/2} \frac{2n(n+1)}{n^2 + n + 1 - (n+1)\sin x + n\cos x} dx \quad , n \in \mathbb{N} \tag{6}$$

$$\pi \sum_{n=0}^{\infty} \left(\frac{2^{2n}}{(2n)!} - \frac{1}{(n!)^2} \right) = \int_{e^{-2}}^{e^2} \sin^{-1}\left(\frac{\ln x}{2} \right) dx \tag{7}$$

$$\frac{\pi}{256}\left(\sqrt{452 + 254\sqrt{2}} - \sqrt{452 - 254\sqrt{2}} \right) = \int_0^1 \frac{1 + x^6}{1 + 28x^2 + 70x^4 + 28x^6 + x^8} dx \tag{8}$$

$$\frac{\pi}{256}\left(\sqrt{4 + 2\sqrt{2}} - \sqrt{4 - 2\sqrt{2}} \right) = \int_0^1 \frac{x^2 + x^4}{1 + 28x^2 + 70x^4 + 28x^6 + x^8} dx \tag{9}$$

$$\pi = 8\sqrt{2 - \sqrt{2}}\left(2 - \sqrt{2 + \sqrt{2}} \right) \int_0^1 \frac{\left(2 + \sqrt{2}\right)\left(1 + x^6\right) + \left(14 - \sqrt{2}\right)\left(x^2 + x^4\right)}{1 + 28x^2 + 70x^4 + 28x^6 + x^8} dx \tag{10}$$

$$\pi = 2(n+2)\int_0^1 \sin^{-1}\left(\sqrt[n+2]{x} \right) dx - 2(n+1)\int_0^1 \sin^{-1}\left(\sqrt[n]{x} \right) dx \quad , n > 0 \tag{11}$$

$$\pi = 6 - 8\int_0^1 \sqrt{\frac{x}{x + \sqrt{8 + x^2}}} dx \tag{12}$$

$$\pi = \frac{21\sqrt{3}}{4} - 9\sqrt{6}\int_0^1 \sqrt{\frac{x}{3x + \sqrt{16 + x^2}}} dx \tag{13}$$

$$\pi = \frac{5\sqrt{3}}{2} - 2\sqrt{6}\int_0^1 \sqrt{\frac{x}{x+\sqrt{48+x^2}}}\,dx \tag{14}$$

$$\pi = 14\sqrt{2} - 16 - 8\sqrt{2}\int_0^{\sqrt{10-7\sqrt{2}}} \sqrt{\frac{x}{x+\sqrt{16+x^2}}}\,dx \tag{15}$$

$$\pi = 4\int_0^1 \sin^{-1}\left(\frac{x}{\sqrt{x^2+(1-x)^2}}\right)dx \tag{16}$$

$$\pi = -3\left(2+\sqrt{3}\right)\ln\left(\frac{\sqrt{3}+1}{2}\right) + 6\left(2+\sqrt{3}\right)\int_0^{\left(3-\sqrt{3}\right)/2} \sin^{-1}\left(\frac{x}{\sqrt{x^2+(1-x)^2}}\right)dx \tag{17}$$

$$\pi = 4\sqrt{2-\sqrt{2}} + 4\sqrt{2-\sqrt{2}}\,a\int_0^1 \sqrt{\frac{2b}{1+ax}-c}\,dx \tag{18}$$

$$\begin{cases} a = 7+4\sqrt{2}-2\left(2+\sqrt{2}\right)^{3/2} \\ b = \left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right) \\ c = 7+4\sqrt{2}+2\left(2+\sqrt{2}\right)^{3/2} \end{cases} \tag{19}$$

$$\pi = 6\ln 2 + 3\ln 3 + 57\sqrt{3} - 12\ln\left(\sqrt{3}+1\right) - 93 + \int_{97\sqrt{3}-168}^{1} \sqrt{\sqrt[3]{\frac{12}{x}+\frac{1}{9}\sqrt{\frac{11664}{x^2}-3}} + \sqrt[3]{\frac{12}{x}-\frac{1}{9}\sqrt{\frac{11664}{x^2}-3}}}\,dx \tag{20}$$

$$\pi = \frac{45\sqrt{3}}{8} - \frac{9}{2}\int_0^1 \sqrt{\sqrt{2\sqrt{16\left(f(x)\right)^2+81x^2}-4f(x)}-2\sqrt{f(x)}}\,dx \tag{21}$$

$$f\left(x\right)=\frac{3x}{16}\left(\sqrt[3]{\frac{3\sqrt{49152+6561x^2}+243x}{2}}-\sqrt[3]{\frac{3\sqrt{49152+6561x^2}-243x}{2}}\right) \tag{22}$$

$$\pi=\frac{15\sqrt{3}}{4}-6\int_0^1\sqrt{1-2\sin\left(\frac{1}{3}\sin^{-1}\left(1-\frac{81x^2}{128}\right)\right)}\,dx \tag{23}$$

$$\pi=\frac{422}{315}+2\int_0^1\sqrt[10]{\frac{x}{2}+\frac{x}{2}\sqrt[5]{\frac{x}{2}+\frac{x}{2}\sqrt[5]{\frac{x}{2}+\frac{x}{2}\sqrt[5]{\frac{x}{2}+\ldots}}}}\,dx \tag{24}$$

References

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2. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals, Series, and Products. seventh edition. Edited by Alan Jeffrey and Daniel Zwillinger. Academic Press, 2007.
3. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., and Clark, C.W.: NIST Handbook of Mathematical Functions. Cambridge University Press,2010.