# THE SET OF STANDARD NUMBERS

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Abstract. The set of standard numbers $\mathbb{D}$ is constructed from an axiom of infinite fraction sum together with the power set of the set of all rational numbers in the form $2^{-n}$. The power set contains all possible infinite binary sequences who represent the fraction part of the standard numbers together with the integers for the whole number part. The set of standard numbers includes the rational numbers and forms a field ($\mathbb{D}$, +, *) similar to, yet distinct from the set of real numbers $\mathbb{R}$.

## 1. Introduction

In the set of real numbers $\mathbb{R}$, the number 0.4999... is the same number as 0.5. By allowing all such numbers to be different, e.g. 0.4999... $\neq$ 0.5, a similar set of numbers can be constructed in a more simple and straightforward way than in the known constructions of the set of real numbers.

## 2. Construction

This new set is in this paper called the set of standard numbers $\mathbb{D}$ and starts with the following axiom:

**Axiom of Infinite Fraction Sum** *The sum of an infinite number of digits in base-b following the radix point in positional notation is a unique standard number in the interval [0, 1) represented by the sequence of digits itself.*

In order for the axiom of infinite fraction sum to have an actual use we need to be able to construct infinite sequences of digits. This is for the set of standard numbers achieved by:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers.
$\mathbb{Q}$ is the set of rational numbers.
$\mathrm{T} = \{1/2, 1/4, 1/8, \ldots\}$ is the set $\{x \mid x = 2^{-n}, n \in \mathbb{N}\}$ which is a subset of $\mathbb{Q}$.

The set T is countably infinite and in order to construct infinite sequences a set V is defined as the power set of T:

$\mathrm{V} = \mathrm{P}(\mathrm{T})$

By containing all subsets of T, set V includes all possible combinations of the rational numbers in T as an uncountably infinite set $\{\{\}, \{\frac{1}{2}\}, \{\frac{1}{4}\}, \{\frac{1}{8}\}, \{\frac{1}{2}, \frac{1}{4}\},$

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$\{\frac{1}{2}, \frac{1}{8}\}$, $\{\frac{1}{4}, \frac{1}{8}\}$, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}, \ldots\}$. This is the same as saying that V contains all possible infinite binary sequences (the finite sequences are followed by an infinite number of zeros).

**Theorem 2.1.** *The members of set V=P(T) satisfy all possible infinite number of digits in base-2 in the Axiom of Infinite Fraction Sum.*

*Proof.* The power set of T by definition contains all possible combinations of the members of T. And the members of T represent all possible positions following the radix point in base-2. □

The set of standard numbers $\mathbb{D}$ is constructed as containing elements in the form z.f as a sequence (z, f) where z is an integer and f is a member of V. And for completeness, union is made with the set of rational numbers.

$\mathbb{D} = \{(z, f) \mid z \in \mathbb{Z}, f \in V\} \cup \mathbb{Q}$, satisfying the following axioms, where x, y, z $\in \mathbb{D}$:

**Equality operator ==**
$\forall x \forall y((\{x\} \cup \{y\} = \{x\} \to x == y) \leftrightarrow y == x)$

**Total order**
$\forall x(x \leqslant x)$
$\forall x \forall y((x \leqslant y \wedge y \leqslant x) \to x == y)$
$\forall x \forall y \forall z((x \leqslant y \wedge y \leqslant z) \to x \leqslant z)$
$\forall x \forall y(x \leqslant y \vee y \leqslant x)$

**Less than < and greater than > operators**
$\forall x \forall y(\neg(x == y) \to (x < y \leftrightarrow y > x))$
$\forall x \forall y((\neg(x == y) \wedge x \leqslant y) \to x < y)$

**Closure under addition**
$\forall x \forall y(s == x + y \to s \in \mathbb{D})$

And including the rest of the axioms (omitted here for brevity) for defining operator + (addition) and operator  (multiplication) on $\mathbb{D}$ as a field. From the addition and multiplication operators, subtraction and division are defined as:

$\forall x \forall y(x - y \leftrightarrow x + (-y))$
$\forall x \forall y(x/y \leftrightarrow x * y^{-1})$

**Theorem 2.2.** *The set of standard numbers $\mathbb{D}$ forms a field under addition and multiplication ($\mathbb{D}$, +, ∗).*

A formal proof for theorem 2.2 is beyond this paper and left for others to examine.

The reason for the union with the set of rational numbers is that some rational numbers, such as 1/3, are in the set of standard numbers different than the decimal expansion 0.333... This means that 1/3+1/3+1/3=1.0 while 0.333...+0.333...+0.333=0.999... $\neq$ 1.0.

In other cases the rational numbers are the same as decimal expansions, such as 5/2=2.5000... and then it is the same number and the same element in $\mathbb{D}$.

Another example is x = 2.0 - 0.999... and y = 2.0 - 1.0. To show that x and y are different numbers the less than operator can be used. The ordering of the integers takes precedence which in this case means that it is enough to look at the fraction part. And for all positive numbers the binary sequences are ordered from .000... to .111...

$\forall x \forall y (\exists i (x_i = 1 \land y_i = 0 \land \forall j (j < i \rightarrow x_j = y_j)) \rightarrow y < x)$ where x, y $\in$ V and i, j $\in \mathbb{N}$

The above formula compares the first bit (in the infinite sequences) that is different in x and y and the fraction with a 0 in that position is less than the fraction with a 1 in the same position.