Quantum Theory of Individual Electron and Photon Interactions: Geodesic Lorentz Motion, Electromagnetic Time Dilation, the Hyper-Canonical Dirac Equation, and Magnetic Moment Anomalies without Renormalization

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Abstract: Dirac's seminal 1928 paper "The Quantum Theory of the Electron" is the foundation of how we presently understand the behavior of fermions in electromagnetic fields, including their magnetic moments. In sum, it is, as titled, a quantum theory of individual electrons, but in classical electromagnetic fields comprising innumerable photons. Based on the electrodynamic time dilations which the author has previously presented and which arise by geometrizing the Lorentz Force motion, there arises an even-richer "hyper-canonical" variant of the Dirac equation which reduces to the ordinary Dirac equation in the linear limits. This advanced Dirac theory naturally enables the magnetic moment anomaly to be entirely explained without resort to renormalization and other ad hoc add-ons, and it also permits a detailed, granular understanding of how individual fermions interact with individual photons strictly on the quantum level. In sum, it advances Dirac theory to a quantum theory of the electron and the photon and their one-on-one interactions. Six distinct types of experimental tests are proposed.

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PART I: GAUGE SYMMETRY, TIME DILATION, AND THE ENERGY CONTENT OF MATERIAL BODIES IN CLASSICAL ELECTRODYNAMICS

1. From Minkowski Spacetime to Electromagnetic Interactions using Weyl's Local U(1) Gauge Symmetry: A Compact Review of the Known Physics

The modern concept of spacetime originated when Hermann Minkowski in his seminal paper [1] based on the Special Theory of Relativity [2], famously proclaimed that "from now onwards space by itself and time by itself will recede completely to become mere shadows and only a type of union of the two will still stand independently on its own." Following the advent of General Theory in [3], the invariant interval $c^2t^2-x^2-y^2-z^2$ Minkowski discovered became expressed via an infinitesimal metric line element $c^2d\tau^2=\eta_{\mu\nu}dx^\mu dx^\nu$ with a metric tensor $\mathrm{diag}\left(\eta_{\mu\nu}\right)=\left(1,-1,-1,-1\right)$ named for him. Moreover, it became understood that gravitational fields reside in a curved spacetime metric tensor $g_{\mu\nu}$ to which $\eta_{\mu\nu}$ defines the tangent space at each spacetime event, with a line element $c^2d\tau^2=g_{\mu\nu}dx^\mu dx^\nu$ specified according to Riemannian geometry which one of Gauss' preeminent students had been developed half a century earlier.

The equation $c^2d\tau^2=g_{\mu\nu}dx^\mu dx^\nu$ for the proper time line element $d\tau$ is often written in a number of different, albeit mathematically equivalent ways. For example, if one divides through by $d\tau^2$ and defines ("$\equiv$") a four-velocity $u^\mu \equiv dx^\mu / d\tau$ this equation becomes $c^2=g_{\mu\nu}u^\mu u^\nu$. By absorbing the spacetime indices into these vectors and writing $c^2=u_\sigma u^\sigma$, we see that the squared four-velocity is equal to the squared speed of light. Further, if we postulate some material mass m and multiply the foregoing through by m^2, also defining an energy-momentum vector $p^\mu=mu^\mu=mdx^\mu/d\tau=\left(E/c,\mathbf{p}\right)$, we arrive at $m^2c^2=g_{\mu\nu}p^\mu p^\nu=p_\sigma p^\sigma$, well-known as the relativistic energy momentum relation.

A next step often taken is to write down a complex function $\phi=s\exp\left(-ip_\sigma x^\sigma/\hbar\right)$ where $s\left(p^\nu\right)$ is a function of energy-momentum and $\exp\left(-ip_\sigma x^\sigma\right)$ is the kernel used in Fourier transforms between momentum space and configuration space. Using being the spacetime gradient operator $\partial_\mu=\left(\partial/c\partial t,\partial/\partial\mathbf{x}\right)=\left(\partial_t/c,\nabla\right)$ it is easy to see that $i\hbar\partial_\mu\phi=p_\mu\phi$. As a result, starting with $m^2c^2=p_\sigma p^\sigma$ and multiplying through from the right by ϕ, it is straightforward to form the operator equation $0=\left(\hbar^2\partial_\sigma\partial^\sigma+m^2c^2\right)\phi$, better-known as the Klein-Gordon equation for a free (non-interacting) particle.

It is also easy to see that by taking a simple scalar square root one can obtain the linear energy-momentum relation $mc=\pm\sqrt{g_{\mu\nu}p^\mu p^\nu}$, or $mc=\pm\sqrt{\eta_{\mu\nu}p^\mu p^\nu}$ in flat spacetime. But Dirac found in [4] that there exists an operator equation in flat spacetime – essentially a square-root of the Klein-Gordon equation – that uses a set of 4x4 matrices γ^μ *defined such that* $\tfrac{1}{2}\{\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\}\equiv\eta^{\mu\nu}$. First we write $m^2c^2=\eta^{\mu\nu}p_\mu p_\nu=\tfrac{1}{2}\{\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\}p^\mu p^\nu$. Then we observe that $(\gamma^\mu p_\mu)^2=(\gamma^\mu p_\mu)(\gamma^\nu p_\nu)=(\gamma^\nu p_\nu)(\gamma^\mu p_\mu)=\tfrac{1}{2}\{\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\}p_\mu p_\nu=\eta^{\mu\nu}p_\mu p_\nu$. Therefore, $\pm\sqrt{\eta^{\mu\nu}p_\mu p_\nu}=\gamma^\mu p_\mu$. However, in order to connect this with $mc=\pm\sqrt{\eta_{\mu\nu}p^\mu p^\nu}$ two adjustments are required. First, because $\gamma^\mu p_\mu$ is a 4x4 matrix, the mass term mc needs to be formed into mc times a 4x4 identity matrix $I_{4\times4}$, which is implicitly understood, not explicitly shown. Second, because $mcI_{4\times4}$ is a diagonal matrix while $\gamma^\mu p_\mu$ cannot be diagonalized, simply equating $\gamma^\mu p_\mu=mc$ is mathematically nonsensical. Instead, we form a four-component Dirac spinor $u(p^\nu)$ and multiply from the right to obtain $(\gamma^\mu p_\mu-mc)u=0$. This makes mathematical sense as an operator equation with eigenvectors and eigenvalues. Note also that the $\pm$ sign, which results whenever a square-root is taken, gets absorbed into the components of γ^μ, all of which are ±1or $\pm i$ with an balanced number of positive and negative entries. Further, similar to Klein-Gordon equation above, we write down a four-component spinor function $\psi=u\exp(-ip_\sigma x^\sigma/\hbar)$, deduce that $i\hbar\partial_\mu\psi=p_\mu\psi$, and so may write $(i\hbar\gamma^\mu\partial_\mu-mc)\psi=0$ which is Dirac's equation for a non-interacting fermion, e.g. electron in a configuration space.

Dirac's equation as developed above applies within a flat spacetime. To generalize to curved spacetime, thus to gravitation, we first define a set of Γ^μ having a parallel definition $\tfrac{1}{2}\{\Gamma^\mu\Gamma^\nu+\Gamma^\nu\Gamma^\mu\}\equiv g^{\mu\nu}$. We also establish a vierbein, a.k.a. tetrad e_a^μ, with both a superscripted Greek "spacetime/world" index and an early-in-the-alphabet subscripted Latin "Lorentz/ Minkowski" index, and define the tetrad by the relation $e_a^\mu\gamma^a\equiv\Gamma^\mu$. Consequently we deduce that $g^{\mu\nu}=\tfrac{1}{2}\{\gamma^a\gamma^b+\gamma^b\gamma^a\}e_a^\mu e_b^\nu=\eta^{ab}e_a^\mu e_b^\nu$. It is readily seen that the flat spacetime $g^{\mu\nu}=\eta^{\mu\nu}$ and $\Gamma^\mu=\gamma^\mu$ are obtained when $e_a^\mu=1$ along the $\mu=a$ diagonal and zero otherwise, i.e., when e_a^μ is a 4x4 unit matrix. Then, starting with $mc=\pm\sqrt{g_{\mu\nu}p^\mu p^\nu}$ we follow the exact same steps as in the previous paragraph, ending up with $(\Gamma^\mu p_\mu-mc)u=0$ in momentum space and $(i\hbar\Gamma^\mu\partial_\mu-mc)\psi=0$ in configuration space.

However, in curved spacetime, in order to couple the spinor fields ψ to gravity in a generally-covariant manner, we must also advance ∂_μ to a spin-covariant derivative $\partial_\mu\mapsto\nabla_\mu\equiv\partial_\mu-\tfrac{i}{4}\omega_\mu^{ab}\sigma_{ab}$, where a spin connection $\omega_\beta^{ab}=-\omega_\beta^{ba}$ which is antisymmetric in the Lorentz indexes a, b is defined using the gravitational-covariant derivative of $e^{\nu b}$ by $\omega_\mu^{ab}\equiv e_\nu{}^a\partial_{;\mu}e^{\nu b}=e_\nu{}^a(\partial_\mu e^{\nu b}+\Gamma^\nu_{\sigma\mu}e^{\sigma b})$, and where $\sigma_{ab}\equiv\tfrac{i}{2}[\gamma_a\gamma_b-\gamma_b\gamma_a]$ are the bilinear covariants which in the form of $\bar\psi\sigma^{\mu\nu}\psi$ contain the fermion polarization and magnetization bivectors. The extra term $-\tfrac{i}{4}\omega_\mu^{ab}\sigma_{ab}$ also makes its way back into the momentum space Dirac equation which thereby becomes $\left(\Gamma^{\mu}\left(p_{\mu}-\tfrac{i}{4}\omega_{\mu}^{ab}\sigma_{ab}\right)-mc\right)u=0$. The foregoing may all be thought of as equivalent albeit progressively-richer and more-revealing ways of writing the spacetime geometry metric interval $c^2 d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$.

In §9 of [3], one of the most important findings was not only that gravitation could be reduced to pure geometry based on a spacetime metric, but, in a phrase later coined by Wheeler [5], that the resulting theory was a theory of "geometro*dynamics*." Specifically, for a finite proper time $\tau=\int_A^B d\tau$ between any two events *A* and *B*, the lines $0=\delta\int_A^B d\tau$ of minimized variation are the geodesics of motion. Moreover, this equation of motion has been shown for over a century without empirical contradiction to describe gravitational motion. This calculation again begins with $c^2 d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$, now divided through by $c^2 d\tau^2$ and turned into the number:

$$1 = g_{\mu\nu}\frac{dx^{\mu}}{cd\tau}\frac{dx^{\nu}}{cd\tau}. \tag{1.1}$$

Next, taking the scalar square root of this "1" enables us to write the variational equation as:

$$0=\delta\int_A^B d\tau=\delta\int_A^B (1)\,d\tau=\delta\int_A^B d\tau\sqrt{g_{\mu\nu}\frac{dx^{\mu}}{cd\tau}\frac{dx^{\nu}}{cd\tau}}, \tag{1.2}$$

where the $\pm$ sign which attends to taking a square root may be discarded because of the zero on the left-hand side above. Then, using a well-known calculation reviewed in Appendix A because we shall shortly derive the Lorentz Force motion of classical electrodynamics in a similar way, one is able to derive the equation of motion (A.14), reproduced below:

$$\frac{d^2x^{\beta}}{d\tau^2} = -\Gamma^{\beta}{}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}. \tag{1.3}$$

Given that (1.3) is derived when (1.2) is applied to the spacetime metric $c^2 d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ merely divided through by $c^2 d\tau^2$ in the form of (1.1), it is not uncommon to regard $c^2 d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ as the first integral of this equation of motion. So once again, we arrive at an even-richer understanding of the simple metric $c^2 d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ for curved spacetime geometry. And this now brings us to electrodynamics.

During the course of just over a decade, Hermann Weyl in [6], [7], [8] convincingly demonstrated that electromagnetism is a gauge theory based on a *local* U(1) internal symmetry group. The underlying principle of gauge symmetry is that the equations of physics – such as the Dirac equation or the Klein-Gordon equation or their respective Lagrangian densities – must remain invariant under transformations in a complex phase space defined by $\exp(i\Lambda)=\cos\Lambda+i\sin\Lambda$ where $\Lambda(t,\mathbf{x})$ is a *locally-variable* phase angle. Specifically, we require any physics equations containing a generalized function φ to be symmetric under a local transformation $\varphi \to \varphi' \equiv \exp(i\Lambda)\varphi$ which changes the direction but not the magnitude of the function in the phase space. However, because $\partial_\mu \varphi \to \partial_\mu \varphi' = \exp(i\Lambda)(\partial_\mu + i\partial_\mu \Lambda)\varphi$ violates this symmetry, we are required to define a gauge-covariant derivative $\mathcal{D}_\mu$ which likewise transforms as $\mathcal{D}_\mu \to \mathcal{D}'_\mu \equiv \exp(i\Lambda)\mathcal{D}_\mu$. So we introduce a vector gauge field A_μ and a charge q fashioned into $\mathfrak{D}_\mu \equiv \partial_\mu - iqA_\mu / \hbar c$. Now $\mathfrak{D}_\mu \varphi \to \mathfrak{D}_\mu \varphi' = \exp(i\Lambda)\left[\partial_\mu - i\left(qA_\mu / \hbar c - \partial_\mu \Lambda\right)\right]\varphi$. Along with this, if we define $qA_\mu \to qA'_\mu \equiv qA_\mu + \hbar c \partial_\mu \Lambda$ as the transformation for the gauge field, then the $\partial_\mu \Lambda$ terms will cancel, so $\mathfrak{D}_\mu \varphi \to \mathfrak{D}_\mu \varphi' = \exp(i\Lambda)\left[\partial_\mu - iqA_\mu / \hbar c\right]\varphi = \exp(i\Lambda)\mathfrak{D}_\mu \varphi$ is also redirected in the phase space just like $\varphi \to \varphi' \equiv \exp(i\Lambda)\varphi$, exactly as required. Note, in the above we adopt a convention where q is a positive charge. So for an electron, for example, we would set $q = -e$.

Then, armed with $\mathfrak{D}_\mu \equiv \partial_\mu - iqA_\mu / \hbar c$, we merely substitute $\partial_\mu \mapsto \mathfrak{D}_\mu$ into any physics equation containing ∂_μ operating on a general function φ, and are assured this equation will have a local U(1) gauge symmetry. So for Dirac's equation operating on $\varphi = \psi$, in flat spacetime where $\partial_{;\mu} e^{\nu b} = 0$ thus $\omega_\mu^{ab} = 0$ and $\nabla_\mu = \partial_\mu$ we substitute $\partial_\mu \mapsto \mathfrak{D}_\mu \equiv \partial_\mu - iqA_\mu / \hbar c$ for the spin-covariant derivative to obtain $0 = \left(i\hbar \gamma^\mu \mathfrak{D}_\mu - mc\right)\psi = \left(\gamma^\mu \left(i\hbar \partial_\mu + qA_\mu / c\right) - mc\right)\psi$. For the Klein-Gordon equation we obtain $0 = \left(\hbar^2 \mathfrak{D}_\sigma \mathfrak{D}^\sigma + m^2 c^2\right)\phi = \left(\hbar^2 \left(\partial_\sigma - iqA_\sigma / \hbar c\right)\left(\partial^\sigma - iqA^\sigma / \hbar c\right) + m^2 c^2\right)\phi$ by doing the same with $\varphi = \phi$. Empirical evidence for almost a century has established these to be correct equations for interacting fermions and bosons, with q being a physical electric charge and A^μ being a physical electromagnetic vector potential. In fact, if we subject a generalized gauge potential G^μ with related charges g to a gauge transformation $G^\mu \to G'^\mu \equiv \exp(i\Lambda)G^\mu$ and likewise require invariance of the field strength $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu$ under this transformation, we can even obtain $F^{\mu\nu} = \mathfrak{D}^\mu G^\nu - \mathfrak{D}^\nu G^\mu = \partial^{[\mu} G^{\nu]} - ig\left[G^\mu, G^\nu\right] / \hbar c$ using this heuristic prescription $\partial_\mu \mapsto \mathfrak{D}_\mu$ with $\mathfrak{D}_\mu \equiv \partial_\mu - igG_\mu / \hbar c$. This application of local gauge symmetry to gauge fields themselves, will be recognized to now yield a non-Abelian Yang-Mills [9] field strength such as that of $SU(2)_L$ weak and $SU(3)_{QCD}$ strong interactions.

From here we backtrack from configuration to momentum space via the relation $i\hbar \partial_\mu \varphi = p_\mu \varphi$ for an ordinary derivative operating on a function φ containing the Fourier kernel $\exp\left(-ip_\sigma x^\sigma\right)$. Consequently, using $\psi = u \exp\left(-ip_\sigma x^\sigma / \hbar\right)$ then removing the kernel, Dirac's equation becomes $\left(\gamma^\mu \left(p_\mu + qA_\mu / c\right) - mc\right)u = 0$ in flat spacetime, we reveals the electron magnetic moment, see, e.g., section 2.6 of [10]. Likewise, using $\phi = s \exp\left(-ip_\sigma x^\sigma / \hbar\right)$ and then removing the kernel, the Klein-Gordon equation becomes $0 = \left(\left(p_\sigma + qA_\sigma / c\right)\left(p^\sigma + qA^\sigma / c\right) - m^2 c^2\right)s$. Here, however, because there are no γ^μ matrices,

$s(p^\nu)$ may be removed, and we end up with a mathematically perfectly sensible equation $m^2c^2 = (p_\sigma + qA_\sigma / c)(p^\sigma + qA^\sigma / c)$. Defining a gauge-covariant or "canonical" momentum $\pi^\mu \equiv p^\mu + qA^\mu / c$, this is compactly written as $m^2c^2 = \pi_\sigma \pi^\sigma$, and is simply the relativistic energy-momentum relation $m^2c^2 = p_\sigma p^\sigma$ generalized via local U(1) gauge symmetry to encompass a test charge q with mass m within a vector potential A^σ. From this we see that in momentum space in flat spacetime, requiring local U(1) gauge symmetry leads to a prescription $p^\mu \mapsto \pi^\mu$, which is the momentum-space parallel to the configuration space prescription $\partial_\mu \mapsto \mathfrak{D}_\mu$. So in momentum space Dirac's flat spacetime equation becomes $(\gamma^\mu \pi_\mu - mc)u = 0$ and the relativistic energy momentum relation underpinning the Klein-Gordon equation becomes $m^2c^2 = \pi_\sigma \pi^\sigma$.

Taking a closer look at the relation $m^2c^2 = \pi_\sigma \pi^\sigma$ with $\pi^\mu \equiv p^\mu + qA^\mu / c$, we may write:

$$m^2c^2 = \pi_\sigma \pi^\sigma = \left(p_\sigma + \frac{qA_\sigma}{c} \right)\left(p^\sigma + \frac{qA^\sigma}{c} \right) = p_\sigma p^\sigma + \frac{q}{c}\left(A_\sigma p^\sigma + p_\sigma A^\sigma \right) + \frac{q^2}{c^2} A_\sigma A^\sigma . \tag{1.4}$$

In the above we have avoided commuting p^σ with A^σ to combine the mixed terms $A_\sigma p^\sigma + p_\sigma A^\sigma$ into $2A_\sigma p^\sigma$ or $2p_\sigma A^\sigma$. This is because $A^\sigma = (\phi, \mathbf{A})$ is a function of the spacetime coordinates $x^\mu = (ct, \mathbf{x})$ while $p^\sigma = (E/c, \mathbf{p})$ is an energy momentum vector. So when we treat position and momentum as Heisenberg operator matrices we cannot commute **x** and **p** without exercising care, because of the canonical relation $[x_i, p_j] = i\hbar\delta_{ij}$. Likewise, because the Hamiltonian operator H has energy eigenvalues $H|s\rangle = (E - mc^2)|s\rangle$ when operating on a state vector $|s\rangle$, the Heisenberg Equation of motion $[H, A^\nu] = -i\hbar d_t A^\nu + i\hbar \partial_t A^\nu$ (take careful note of the total versus partial derivatives) also requires us to exercise care when we commute $cp^0 = E$ with $A^0 = \phi$ whenever fixed-basis state vectors $|s\rangle$ and field operators ϕ are involved. So to combine terms in (1.4) to show, say, $2A_\sigma p^\sigma$ while not ignoring Heisenberg commutation, we may make use of the commutator $[p_\sigma, A^\sigma] = p_\sigma A^\sigma - A_\sigma p^\sigma$ to identically rewrite (1.4) as:

$$m^2c^2 = \pi_\sigma \pi^\sigma = \left(p_\sigma + \frac{qA_\sigma}{c} \right)\left(p^\sigma + \frac{qA^\sigma}{c} \right) = p_\sigma p^\sigma + 2\frac{q}{c} A_\sigma p^\sigma + \frac{q}{c}\left[p_\sigma, A^\sigma \right] + \frac{q^2}{c^2} A_\sigma A^\sigma . \tag{1.5}$$

Then, if we choose to approximate around these commutation issues and thereby set $[p_\sigma, A^\sigma] = 0$ which amounts to taking a classical $\hbar \to 0$ limit, (1.5) easily reduces to:

$$m^2c^2 = \pi_\sigma \pi^\sigma = \left(p_\sigma + \frac{qA_\sigma}{c} \right)\left(p^\sigma + \frac{qA^\sigma}{c} \right) = p_\sigma p^\sigma + 2\frac{q}{c} A_\sigma p^\sigma + \frac{q^2}{c^2} A_\sigma A^\sigma . \tag{1.6}$$

All of the foregoing is well-known, well-established, empirically-validated physics. Now, however, continuing deductively from the above, we shall uncover some equally-valid new relations and new physics which do not appear to be known to date. At the outset we will work from the classical approximation (1.6) in which we have set $\left[p_\sigma, A^\sigma\right]=0$ and thus effectively set $\hbar = 0$. Later, after sufficient development in section 7, we will shift over and work from (1.5) to fully account for the quantum mechanics of the commutation $\left[p_\sigma, A^\sigma\right]$, and thereby will be able to see precisely how quantum mechanics alters the classical results we shall obtain from (1.6).

2. Derivation of Geodesic Lorentz Force Motion from Local U(1) Gauge Symmetry

Starting with the classical $\hbar \to 0$ relation (1.6), let us use the definitions $p^\mu \equiv m u^\mu$ for the ordinary energy-momentum and $u^\mu \equiv dx^\mu / d\tau$ for the 4-velocity to write (1.6) as:

$$m^2c^2 = \pi_\sigma \pi^\sigma = m^2 u_\sigma u^\sigma + 2\frac{qm}{c} A_\sigma u^\sigma + \frac{q^2}{c^2} A_\sigma A^\sigma = m^2 \frac{dx_\sigma}{d\tau}\frac{dx^\sigma}{d\tau} + 2\frac{qm}{c} A_\sigma \frac{dx^\sigma}{d\tau} + \frac{q^2}{c^2} A_\sigma A^\sigma . \qquad (2.1)$$

Then, continuing to backtrack, we divide the above through by m^2c^2 and also raise an index to show the metric tensor in the first term after the final equality. We thereby obtain:

$$1 = \frac{\pi_\sigma \pi^\sigma}{mc^2} = g_{\mu\nu} \frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + 2\frac{q}{mc^2} A_\sigma \frac{dx^\sigma}{cd\tau} + \frac{q^2}{m^2c^4} A_\sigma A^\sigma . \qquad (2.2)$$

The above is identical to (1.1) unless both $q \neq 0$ and $A^\sigma \neq 0$. That is, unless we have *both* a test charge with a charge-to-mass ratio q/m, and *also* a potential A^σ with which that test charge is interacting, (2.2) is the same as (1.1). This using (2.2) with *either* $q=0$ *or* $A^\sigma = 0$ in the variational equation (1.2) will produce the gravitational geodesic motion of (1.3).

This raises the question whether using (2.2) with both $q \neq 0$ and $A^\sigma \neq 0$ in the variation $0 = \delta \int_A^B d\tau$ as in (1.2) might produce *the Lorentz Force motion of electrodynamics together with the gravitational motion.* In other words, (2.2) raises the question whether the combined classical gravitational and electromagnetic motions can *both* be derived as geodesic motions from a variation using (2.2), which, as is easily seen, is just $m^2c^2 = \pi_\sigma \pi^\sigma$ from (1.6) divided through by through by m^2c^2. And (1.6) of course, is in turn merely the relativistic energy-momentum relation $m^2c^2 = p_\sigma p^\sigma$ following application of the $p^\mu \mapsto \pi^\mu$ prescription which comes about by requiring Weyl's local U(1) gauge symmetry. And $m^2c^2 = p_\sigma p^\sigma$ is in turn just another way of representing the metric $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ once a rest mass m has been postulated and the metric multiplied through by $m^2 / d\tau^2$ while lowering an index. So all roads lead back to $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$.

To prove that the electrodynamic Lorentz Force motion can be understood as geodesic motion just like gravitational motion, as we did at (1.1) to (1.3), we first take the square root of the "1" in (2.2) and use it in the variational equation, to write the following, in contrast to (1.2):

$$0=\delta\int_A^B d\tau=\delta\int_A^B (1)\,d\tau=\delta\int_A^B d\tau\sqrt{g_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+2\frac{q}{mc^2}A_\sigma\frac{dx^\sigma}{cd\tau}+\frac{q^2}{m^2c^4}A_\sigma A^\sigma}\;. \tag{2.3}$$

We then apply δ to the integrand and use (2.2) to remove the denominator, obtaining:

$$0=\delta\int_A^B d\tau=\frac{1}{2}\int_A^B d\tau\delta\left(g_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+2\frac{q}{mc^2}A_\sigma\frac{dx^\sigma}{cd\tau}+\frac{q^2}{m^2c^4}A_\sigma A^\sigma\right). \tag{2.4}$$

The first of the three terms corresponds with (A.1) which leads to gravitational motion. So we segregate that term right away, then apply (A.12) which is directly derived from (A.1), to obtain:

$$\begin{aligned}0=\delta\int_A^B d\tau=&\int_A^B \delta x^\alpha d\tau\left(\frac{1}{2}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-g_{\alpha\nu}\frac{d^2x^\nu}{c^2d\tau^2}\right)\\&+\int_A^B d\tau\delta\left(\frac{q}{mc^2}A_\sigma\frac{dx^\sigma}{cd\tau}+\frac{q^2}{2m^2c^4}A_\sigma A^\sigma\right)\end{aligned}. \tag{2.5}$$

Because $-\Gamma^\beta{}_{\mu\nu}=\frac{1}{2}g^{\beta\alpha}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)$, we see that the gravitational motion (A.14) i.e. (1.3) is already contained in the top line above. So now let's develop the bottom line which contains the additional electrodynamic terms added by the U(1) gauge symmetry via the parallel configuration and momentum space rules $\partial_\mu \mapsto \mathfrak{D}_\mu$ and $p^\mu \mapsto \pi^\mu$ reviewed in section 1.

For the bottom line of (2.5) we first distribute δ using the product rule, and assume no variation in the charge-to-mass ratio i.e. that $\delta(q/m)=0$ over the path from *A* to *B*, thus finding:

$$\begin{aligned}0=\delta\int_A^B d\tau=&\int_A^B \delta x^\alpha d\tau\left(\frac{1}{2}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-g_{\alpha\nu}\frac{d^2x^\nu}{c^2d\tau^2}\right)\\&+\int_A^B d\tau\left(\frac{q}{mc^2}\delta A_\sigma\frac{dx^\sigma}{cd\tau}+\frac{q}{mc^2}A_\sigma\delta\frac{dx^\sigma}{cd\tau}+\frac{q^2}{2m^2c^4}\delta\left(A_\sigma A^\sigma\right)\right)\end{aligned}. \tag{2.6}$$

From (A.3) we may deduce that $\delta A_\sigma=\delta x^\alpha\partial_\alpha A_\sigma$ and $\delta\left(A_\sigma A^\sigma\right)=\delta x^\alpha\partial_\alpha\left(A_\sigma A^\sigma\right)$. We use these as well as $\delta d=d\delta$ employed for (A.2) to advance the above to:

$$0=\delta\int_A^B d\tau=\int_A^B \delta x^\alpha d\tau\left(\frac{1}{2}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-g_{\alpha\nu}\frac{d^2x^\nu}{c^2d\tau^2}\right)$$
$$+\int_A^B d\tau\left(\frac{q}{mc^2}\delta x^\alpha\partial_\alpha A_\sigma\frac{dx^\sigma}{cd\tau}+\frac{q}{mc^2}A_\sigma\frac{d\delta x^\sigma}{cd\tau}+\frac{q^2}{2m^2c^4}\delta x^\alpha\partial_\alpha\left(A_\sigma A^\sigma\right)\right). \tag{2.7}$$

We next use (A.10) to obtain $dA_\sigma / cd\tau=\partial_\alpha A_\sigma dx^\alpha / cd\tau$. Then, for the second term on the bottom line above, to set up an integration-by-parts, we use this with the product rule to form:

$$\frac{d}{cd\tau}\left(A_\sigma\delta x^\sigma\right)=\delta x^\sigma\frac{dA_\sigma}{cd\tau}+A_\sigma\frac{d\delta x^\sigma}{cd\tau}=\delta x^\sigma\partial_\alpha A_\sigma\frac{dx^\alpha}{cd\tau}+A_\sigma\frac{d\delta x^\sigma}{cd\tau}. \tag{2.8}$$

Using (2.8) in (2.7) then produces:

$$0=\delta\int_A^B d\tau=\int_A^B \delta x^\alpha d\tau\left(\frac{1}{2}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-g_{\alpha\nu}\frac{d^2x^\nu}{c^2d\tau^2}\right)$$
$$+\int_A^B d\tau\left(\frac{q}{mc^2}\delta x^\alpha\partial_\alpha A_\sigma\frac{dx^\sigma}{cd\tau}+\frac{q}{mc^2}\left(\frac{d}{cd\tau}\left(A_\sigma\delta x^\sigma\right)-\delta x^\sigma\partial_\alpha A_\sigma\frac{dx^\alpha}{cd\tau}\right)+\frac{q^2}{2m^2c^4}\delta x^\alpha\partial_\alpha\left(A_\sigma A^\sigma\right)\right). \tag{2.9}$$

The term containing total integral in the above is equal to zero because of the boundary conditions on the definite integral in the variation. Specifically, in the above:

$$\int_A^B d\tau\frac{d}{d\tau}\left(A_\sigma\delta x^\sigma\right)=\int_A^B d\left(A_\sigma\delta x^\sigma\right)=\left(A_\sigma\delta x^\sigma\right)\Big|_A^B=0\,, \tag{2.10}$$

This is zero for the same reasons that (A.7) is zero when calculating the gravitational geodesics. Consequently, using (2.10) in (2.9) and with a renaming of summed indexes so there is a δx^α with a common α index in all terms, then factoring this out, (2.9) becomes:

$$0=\delta\int_A^B d\tau=\int_A^B \delta x^\alpha d\tau\begin{pmatrix}\frac{1}{2}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-g_{\alpha\nu}\frac{d^2x^\nu}{c^2d\tau^2}\\ +\frac{q}{mc^2}\left(\partial_\alpha A_\sigma-\partial_\sigma A_\alpha\right)\frac{dx^\sigma}{cd\tau}+\frac{q^2}{2m^2c^4}\partial_\alpha\left(A_\sigma A^\sigma\right)\end{pmatrix}. \tag{2.11}$$

It is very important that the integration-by-parts produced both a sign reversal as well as an index reversal, because $F_{\alpha\sigma}=\partial_\alpha A_\sigma-\partial_\sigma A_\alpha$ is the covariant-indexed electromagnetic field strength.

Now we are at (A.12) for the gravitational geodesics, but with some new terms. For the same reasons as at (A.12), the expression inside the large parenthesis above must be zero. So setting this to zero, using $F_{\alpha\sigma}=\partial_\alpha A_\sigma-\partial_\sigma A_\alpha$, multiplying all terms by $g^{\beta\alpha}$ to raise an index, using $-\Gamma^\beta{}_{\mu\nu}=\frac{1}{2}g^{\beta\alpha}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)$, and segregating the acceleration, yields:

$$\frac{d^2x^\beta}{c^2d\tau^2}=-\Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+\frac{q}{mc^2}F^\beta{}_\sigma\frac{dx^\sigma}{cd\tau}+\frac{q^2}{2m^2c^4}\partial^\beta\left(A_\sigma A^\sigma\right). \tag{2.12}$$

So it is possible to derive (2.12) from the variation $0=\delta\int_A^B d\tau$ using $1=\pi_\sigma\pi^\sigma/mc^2$ from (2.2) which simply restates the locally U(1) gauge-symmetric relativistic energy-momentum relation $m^2c^2=\pi_\sigma\pi^\sigma$ of (1.6). Therefore the Lorentz Force motion which has been thoroughly validated empirically over the course of decades can indeed be understood as geodesic motion just like the gravitational motion. This does not appear to have previously been reported in the literature, and so warrants attention at least from viewpoint of at least *mathematical* physics.

However (2.12) also has an extra term $\left(q^2/2m^2c^4\right)\partial^\beta\left(A_\sigma A^\sigma\right)$ which warrants *physical* attention. As we shall later see, this term is naturally removed by a variant of the Lorenz gauge $\partial_\sigma A^\sigma=0$ when (1.5) is applied with the commutator $\left[p_\sigma,A^\sigma\right]\neq 0$ i.e. $\hbar\neq 0$ in accordance with quantum mechanics. In other words, this added term arises precisely because we have neglected quantum mechanics by using (1.6) rather than (1.5) in the variation (2.3), and disappears once quantum mechanics is taken into account and the commutator not approximated to zero.

3. The Canonical Relativistic Energy-Momentum Relation, and the Apparently "Peculiar" Quadratic Line Element with which it is Synonymous

At (2.1) we took the relation $m^2c^2=\pi_\sigma\pi^\sigma$ of (1.4) in the classical $\hbar\to 0$ limit and divided through by m^2c^2 to arrive at (2.2) which, when used in the variation (2.3), yielded the geodesic equation (2.12). This includes Lorentz Force motion plus an extra term containing $\partial^\beta\left(A_\sigma A^\sigma\right)$. Let us now take this same $m^2c^2=\pi_\sigma\pi^\sigma$ of (1.4), (1.5) and use $p^\sigma=mdx^\sigma/d\tau$ to obtain:

$$\begin{aligned}m^2c^2=\pi_\sigma\pi^\sigma&=\left(m\frac{dx_\sigma}{d\tau}+\frac{qA_\sigma}{c}\right)\left(m\frac{dx^\sigma}{d\tau}+\frac{qA^\sigma}{c}\right)\\&=m^2\frac{dx_\sigma}{d\tau}\frac{dx^\sigma}{d\tau}+2\frac{qm}{c}A_\sigma\frac{dx^\sigma}{d\tau}+\frac{q}{c}\left[m\frac{dx_\sigma}{d\tau},A^\sigma\right]+\frac{q^2}{c^2}A_\sigma A^\sigma\end{aligned}. \tag{3.1}$$

In the classical $\hbar\to 0$ limit of (1.6) where we neglect commutation by setting $\left[p_\sigma,A^\sigma\right]=0$, using the approximation sign "$\cong$" prior to the final expression as a reminder of this, we obtain:

$$m^2c^2=\pi_\sigma\pi^\sigma=\left(m\frac{dx_\sigma}{d\tau}+\frac{qA_\sigma}{c}\right)\left(m\frac{dx^\sigma}{d\tau}+\frac{qA^\sigma}{c}\right)\cong m^2\frac{dx_\sigma}{d\tau}\frac{dx^\sigma}{d\tau}+2\frac{qm}{c}A_\sigma\frac{dx^\sigma}{d\tau}+\frac{q^2}{c^2}A_\sigma A^\sigma. \tag{3.2}$$

Then, also defining a gauge-covariant coordinate element $\mathcal{D}x^\mu \equiv dx^\mu + \left(q/mc^2\right)A^\mu c d\tau$, we simply multiply through by $d\tau^2/m^2$ and raise some selected indices to obtain:

$$\begin{aligned} c^2 d\tau^2 &= \frac{d\tau^2}{m^2}\pi_\sigma\pi^\sigma = \left(dx_\sigma + \frac{q}{mc^2}A_\sigma c d\tau\right)\left(dx^\sigma + \frac{q}{mc^2}A^\sigma c d\tau\right) = g_{\mu\nu}\mathcal{D}x^\mu \mathcal{D}x^\nu \\ &\cong g_{\mu\nu}dx^\mu dx^\nu + 2\frac{q}{mc^2}A_\sigma dx^\sigma c d\tau + \frac{q^2}{m^2c^4}A_\sigma A^\sigma c^2 d\tau^2 \end{aligned} . \tag{3.3}$$

The above is simply the metric equation $c^2 d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$ supplemented by new terms which come about because of gauge symmetry. These new terms are non-zero whenever there is a test charge with $q/m \neq 0$ situated in a gauge potential $A^\sigma \neq 0$. They arise because of the local U(1) gauge symmetry, and in fact reveal that the momentum space prescription $p^\mu \mapsto \pi^\mu$ and the configuration space prescription $\partial_\mu \mapsto \mathfrak{D}_\mu$ previously reviewed also go hand-in-hand with a parallel prescription $dx^\mu \mapsto \mathcal{D}x^\mu$ for the infinitesimal coordinate interval.

However, this metric (3.3) is unusual because it is *quadratic* in the line element $ds = c d\tau$. This quadratic is seen if we rewrite the bottom line of (3.3) which contains the classical $\hbar \to 0$ line element, with the approximation sign removed, in the form:

$$0 = \left(1 - \frac{q^2}{m^2c^4}A_\sigma A^\sigma\right)c^2 d\tau^2 - 2\frac{q}{mc^2}A_\sigma dx^\sigma c d\tau - g_{\mu\nu}dx^\mu dx^\nu , \tag{3.4}$$

and then use this in the quadratic equation to obtain the solution:

$$c d\tau = \frac{\frac{q}{mc^2}A_\sigma dx^\sigma \pm \sqrt{\left[g_{\mu\nu}\left(1 - \frac{q^2}{m^2c^4}A_\sigma A^\sigma\right) + \frac{q^2}{m^2c^4}A_\mu A_\nu\right]dx^\mu dx^\nu}}{1 - \frac{q^2}{m^2c^4}A_\sigma A^\sigma} . \tag{3.5}$$

Now, on the one hand, the metric (3.3) is just another way of stating the well-established relation $m^2c^2 = \pi_\sigma\pi^\sigma$ which is merely the relativistic energy-momentum relation $m^2c^2 = p_\sigma p^\sigma$ after imposing local U(1) gauge symmetry which causes the momentum space replacement $p^\mu \mapsto \pi^\mu$. In (3.3) that relation is written as $c^2 d\tau^2 = \left(d\tau^2/m^2\right)\pi_\sigma\pi^\sigma$, which is just another variant of $1 = \pi_\sigma\pi^\sigma/mc^2$ which was used in (2.3) to obtain the geodesic motion in (2.12).

On the other hand, when couched in the form of (3.3), and especially after obtaining the quadratic solution (3.5), this metric (3.3) *<u>appears</u> to have some problems*, and certainly, as a quadratic in $d\tau$, it is an unusual line element. One might notice that the metric (3.3), (3.5) is a function $d\tau(q/m)$ of the q/m ratio of a test charge and suppose this to mean that the *invariant* line element $ds = cd\tau$ and the background fields A^μ and $g_{\mu\nu}$ are actually not invariant when q/m is changed, which would not be permitted by field theory. And, one may notice that the term $A_\sigma A^\sigma$ is not invariant under a local U(1) gauge transformation, giving the line element a gauge-dependency. One might even go so far as to believe that this is a "peculiar" or even "aberrant" line element that cannot be associated to a Riemannian geometry, and moreover, that geodesics calculated starting with this line element are strongly non-linear involving irrational functions of electromagnetic potential. And one might then conclude that any development based on (3.3) can lead to no more than a chain of allegations and mistakes.

At the same time, however, (3.3) is simply (2.2) multiplied through by $c^2 d\tau^2$. When (2.2) is used in the variation (2.3) the resulting geodesics are given by (2.12) which does contain both the gravitational motion *and the Lorentz Force motion*, differing only by the final $\partial^\beta \left(A_\sigma A^\sigma\right)$ term which is a non-linear function of the electromagnetic potential, and which we still need to attend to. So to dismiss (3.3) out of hand because of its unusual form or the foregoing conceptual challenges would be a mistake. This is because if $c^2 d\tau^2 = d\tau^2 \pi_\sigma \pi^\sigma / m^2$ in (3.3) is a wrong equation then so too is $m^2c^2 = \pi_\sigma \pi^\sigma$ in (1.6), given that *these are the very same equation* obtained from one another by the elementary algebra of multiplying both sides of an equation by the same objects. And if $m^2c^2 = \pi_\sigma \pi^\sigma$ is a wrong equation, this would precipitate an unwarranted crisis in gauge theory itself, because the prescription to go from $m^2c^2 = p_\sigma p^\sigma$ to $m^2c^2 = \pi_\sigma \pi^\sigma$ via $p^\mu \mapsto \pi^\mu$ would also be wrong, yet this prescription is fundamental to local gauge theory as reviewed between (1.3) and (1.4). Or, $m^2c^2 = p_\sigma p^\sigma$ would have to be wrong, which would be in collision with all the relativistic physics we know. Therefore, we have little choice but to adopt the view that (3.3) though peculiar in appearance is actually just as correct as $m^2c^2 = \pi_\sigma \pi^\sigma$ with which it is synonymous. And we now also know that the $1 = \pi_\sigma \pi^\sigma / mc^2$ variant of (3.3) which is (2.2) produces the well-established geodesic motion contain in (2.12), plus an extra term still to be studied. Consequently, taking (3.3) as a challenge not than a mistake, we must find out more about the heretofore undiscovered physics which arises when the metric (3.3) is carefully studied in depth to all it its logical conclusions. This study will now become the focus of the rest of this paper.

4. The Quadratic Line Element at Rest with no Gravitation

The metric (3.3) is unusual in appearance for the several reasons laid out above, and yet it is not incorrect unless $m^2c^2 = \pi_\sigma \pi^\sigma$ is incorrect, which it is not. To make better sense of (3.3), it is helpful to place the vector potential and the test charge into a rest frame thus placing the test charge and the source of the potential at rest relative to one another, and to work in flat spacetime. To do so we take a classical vector potential $A^\mu = \left(\phi, \mathbf{A}\right)$ and transform this to a rest frame so that $A^\mu = \left(\phi_0, \mathbf{0}\right)$ where ϕ_0 is the proper scalar potential. Additionally, starting with the coordinate element $dx^\mu = \left(cdt, d\mathbf{x}\right)$ we set $dx^\mu = \left(cdt, \mathbf{0}\right)$ to place the test particle in the same rest frame. Then then set $g_{\mu\nu} = \eta_{\mu\nu}$ to work in flat spacetime. Thus, at rest without gravitation, the classical $\hbar \to 0$ metric (3.3) becomes:

$$d\tau^2 = dt^2 + 2\frac{q\phi_0}{mc^2}dtd\tau + \frac{q^2{\phi_0}^2}{m^2c^4}d\tau^2 . \tag{4.1}$$

It will be seen that this is quadratic in both $d\tau$ and dt, so we can solve this equation either way and obtain the same result. Choosing to write the quadratic in dt we have:

$$0 = dt^2 + 2\frac{q\phi_0}{mc^2}d\tau dt - \left(1 - \frac{q^2{\phi_0}^2}{m^2c^4}\right)d\tau^2 . \tag{4.2}$$

Via the quadratic equation this solves to:

$$dt = -\frac{q\phi_0}{mc^2}d\tau \pm \sqrt{\frac{q^2{\phi_0}^2}{m^2c^4}d\tau^2 + \left(1 - \frac{q^2{\phi_0}^2}{m^2c^4}\right)d\tau^2} = -\frac{q\phi_0}{mc^2}d\tau \pm d\tau = \left(\pm 1 - \frac{q\phi_0}{mc^2}\right)d\tau . \tag{4.3}$$

Then, imposing the condition that when $q = 0$ or $\phi_0 = 0$ we must have $dt = d\tau$ so that in the absence of any electromagnetic interaction (or motion or gravitation) the coordinate time flows at the same rate as the proper time, we can discard the minus sign in (4.3), obtaining the simplified:

$$\frac{dt}{d\tau} = 1 - \frac{q\phi_0}{mc^2} . \tag{4.4}$$

With $d\tau$ segregated this is alternatively written as:

$$d\tau = \frac{1}{1 - \frac{q\phi_0}{mc^2}}dt . \tag{4.5}$$

The above (4.5) is the exact quadratic solution for the "peculiar" line element (3.5) at rest and absent gravitation. So (3.5) is the general case of (4.5), obtained by restoring motion via a Lorentz transform and gravitational fields by curving the spacetime. And (3.3) to which (4.4), (4.5) is the at rest solution absent gravitation, is just an algebraic variant of the well-established $m^2c^2 = \pi_\sigma \pi^\sigma$ which in turn is merely the relativistic relation $m^2c^2 = p_\sigma p^\sigma$ with local U(1) gauge symmetry.

Now, it is well-established from Special and General Relativity that when two clocks are in relative motion and / or are differently-situated in a gravitational potential, the ratio of the time coordinate element to the proper time element $dt / d\tau \neq 1$. This is time dilation, and when multiplied through by mc^2 to obtain $E = p^0 = mc^2 \cdot dt / d\tau$ this also gives us the total energy content of the material body with mass *m*. Yet (4.4) and (4.5) indicate that *even at rest and absent gravitation*, whenever there is a test charge with $q / m \neq 0$ in a proper scalar potential $\phi_0 \neq 0$ we continue to have $dt / d\tau \neq 1$. This result – which is brand new physics – teaches *that there are also time dilations which occur whenever there are electromagnetic interactions*. So we now must study these electromagnetic time dilations and come to understand their operational meaning and how they are observed in the natural world.

5. Derivation of Electromagnetic Interaction Time Dilations using an Inequivalence Principle

We observed earlier following (3.5) that one of the perplexing features of (3.3) and (3.5) is that they are functions $d\tau(q/m)$ of the q/m ratio of a test charge. But of course, the line element $ds = cd\tau$ cannot change when q/m changes, but must be invariant under such changes. So too, field theory mandates that the background fields A^μ and $g_{\mu\nu}$ also be invariant when q/m changes. So the question now arises, how do we ensure that (3.3) and (3.5) adhere to this mandate?

Ever since Galileo's legendary Pisa experiment it has been known that if two different masses m and $m' \neq m$ are dropped under the very same circumstances in the very same gravitational field, the motion will be exactly the same for each mass. This came to be understood as signifying an experimental equality between gravitational and inertial mass. By elevating this to the equivalence principle, Einstein was able to find a geometric way of formulating gravity. This is seen by the absence of the mass *m* in the gravitational motion that is part of (2.12). But for electromagnetism – in fundamental contrast to gravitation – two different test charges with q/m and $q'/m' \neq q/m$ do *not* exhibit identical motions even in identical electromagnetic fields under identical circumstances, as seen by the presence of this q/m ratio in the Lorenz Force motion of (2.12). This is understood to signify an experimental *inequality* between electrical mass a.k.a. charge and inertial mass. So now, we formally elevate this to an *inequivalence principle* which plays the same role in electrodynamics that the equivalence principle plays in gravitation, by taking the affirmative step of postulating a brand new symmetry principle which mandates as follows:

> **Charge-to-Mass Ratio Gauge Symmetry Postulate:** The metric interval $d\tau$ and background fields A^μ and $g_{\mu\nu}$, and by implication $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, must remain invariant under any and all transformations which re-scale, i.e. *re-gauge* the charge-to-mass ratio via a re-gauging transformation $q/m \to q'/m' \neq q/m$.

To implement this principle, we first inventory all of the physical numbers and objects appearing in the "peculiar" quadratic metric (3.3). These are the speed of light *c*, the line element $d\tau$, the metric tensor $g_{\mu\nu}$ containing the gravitational field, the gauge field A^μ which is the electromagnetic potential, the q/m ratio, and the coordinate elements dx^μ. So, under a re-gauging $q/m \to q'/m' \neq q/m$ of the charge-to-mass ratio, we of course require the speed of light to remain invariant, $c \to c' \equiv c$. But we also require, by the above symmetry principle, that $d\tau \to d\tau' \equiv d\tau$, $g_{\mu\nu} \to g'_{\mu\nu} \equiv g_{\mu\nu}$ and $A^\mu \to A'^\mu \equiv A^\mu$ also remain invariant. So the only objects remaining which may be transformed when we re-gauge $q/m \to q'/m' \neq q/m$ are the coordinate elements dx^μ. We know very well from the Special and General Theories of Relativity that the observed $dx^\mu \to dx'^\mu \neq dx^\mu$ do in fact change when two different observers are in relative motion or have different placements in a gravitational field. And (4.4), (4.5) already indicate that this is also true of at least the time element $dx^0 = cdt$ when there are electrodynamic interactions.

So now we work from (3.3) to *define* a coordinate transformation $dx^\mu \to dx'^\mu \neq dx^\mu$ which occurs whenever we transform $q/m \to q'/m' \neq q/m$ in accordance with these symmetries, via:

$$\begin{aligned} c^2 d\tau^2 &= \frac{d\tau^2}{m^2}\pi_\sigma \pi^\sigma = \left(dx_\sigma + \frac{q}{mc^2}A_\sigma c d\tau\right)\left(dx^\sigma + \frac{q}{mc^2}A^\sigma c d\tau\right) = g_{\mu\nu}\mathcal{D}x^\mu \mathcal{D}x^\nu \\ \to c^2 d\tau'^2 &= \frac{d\tau^2}{m^2}\pi'_\sigma \pi'^\sigma \equiv c^2 d\tau^2 = \left(dx'_\sigma + \frac{q'}{m'c^2}A_\sigma c d\tau\right)\left(dx'^\sigma + \frac{q'}{m'c^2}A^\sigma c d\tau\right) = g_{\mu\nu}\mathcal{D}x'^\mu \mathcal{D}x'^\nu \end{aligned} . \tag{5.1}$$

Note that $\mathcal{D}x^\mu = dx^\mu + \left(q/mc^2\right)A^\mu c d\tau \to \mathcal{D}x'^\mu = dx'^\mu + \left(q'/m'c^2\right)A^\mu c d\tau$ is the transformation for the gauge-covariant coordinate elements $\mathcal{D}x^\mu$. If we then apply the $\hbar = 0$ classical approximation from (1.6) which sets $\left[p_\sigma, A^\sigma\right] = 0$, the above transformation $dx^\mu \to dx'^\mu \neq dx^\mu$ becomes:

$$\begin{aligned} c^2 d\tau^2 &= g_{\mu\nu}dx^\mu dx^\nu + 2\frac{q}{mc^2}A_\sigma dx^\sigma c d\tau + \frac{q^2}{m^2c^4}A_\sigma A^\sigma c^2 d\tau^2 \\ \to c^2 d\tau'^2 \equiv c^2 d\tau^2 &= g_{\mu\nu}dx'^\mu dx'^\nu + 2\frac{q'}{m'c^2}A_\sigma dx'^\sigma c d\tau + \frac{q'^2}{m'^2c^4}A_\sigma A^\sigma c^2 d\tau^2 \end{aligned} . \tag{5.2}$$

Now we move to a rest frame and remove all gravitation to directly deduce what happens to the time coordinate when we re-gauge $q/m \to q'/m' \neq q/m$. This is the exact same calculation we did from (4.1) to (4.5), except now we have some transformed objects annotated with "primes." So with $A^\mu = \left(\phi_0, \mathbf{0}\right)$ and $dx^\mu = \left(cdt, \mathbf{0}\right)$ and $g_{\mu\nu} = \eta_{\mu\nu}$ the above becomes (contrast (4.1)):

$$d\tau^2 = dt^2 + 2\frac{q\phi_0}{mc^2}dt d\tau + \frac{q^2{\phi_0}^2}{m^2c^4}d\tau^2 = dt'^2 + 2\frac{q'\phi_0}{m'c^2}dt' d\tau + \frac{q'^2{\phi_0}^2}{m'^2c^4}d\tau^2 . \tag{5.3}$$

This contains a first quadratic for dt and a second quadratic for dt'. We already have the solution for dt, which is (4.4). So the solution for dt', shown together with (4.4) for dt, is:

$$\frac{dt}{d\tau} = 1 - \frac{q\phi_0}{mc^2}; \qquad \frac{dt'}{d\tau} = 1 - \frac{q'\phi_0}{m'c^2} . \tag{5.4}$$

Now, because of the above symmetry postulate, $d\tau$ is the same invariant object in each of $dt/d\tau$ and $dt'/d\tau$ above. Likewise, c and ϕ_0 are also the same. And we used the same $\eta_{\mu\nu}$ to derive each of (5.4). Therefore, with two different massive charged bodies both at rest *in the same proper potential* ϕ_0, one with q/m and the other with q'/m', we deduce from (5.4) that the ratio:

$$\frac{dt}{dt'} = \frac{1 - q\phi_0 / mc^2}{1 - q'\phi_0 / m'c^2}. \tag{5.5}$$

Because the above compares measurements of time, we should be more specific about what is meant by the rate at which time flows for various charged bodies. The meaning and construction of so-called "geometrodynamic clocks" has been widely developed in the literature, see, e.g. section 5.2 of Ohanian's [11]. What (5.5) tells us is that if we start with an electrically-neutral material body which qualifies as a true geometrodynamic clock (g-clock), for example, a cesium oscillator through which a second is defined in the International System of Units (SI) by the standard of 9,192,631,770 oscillation "ticks," then if that clock is charged and placed into an electromagnetic proper potential ϕ_0, the rate of time signaling will be altered based on (5.5). So suppose that we wish to measure the ratio (5.5). One experiment we might do is to start with two identical, electrically-neutral g-clocks. We leave the first g-clock neutral so it maintains $q = 0$. We then charge the second g-clock to $q' \neq 0$. We then use the neutral $q = 0$ g-clock as a laboratory clock to measure the laboratory time element dt, and compare this to the dt' element measured by oscillations of the second $q' \neq 0$ clock. So for this experiment, with $q = 0$ (5.5) becomes:

$$\frac{dt}{dt'} = \frac{1}{1 - \frac{q'\phi_0}{m'c^2}}. \tag{5.6}$$

In Relativity Theory the time dilation factors $\gamma_v \equiv dt / d\tau = 1/\sqrt{1 - v^2 / c^2}$ for motion and $\gamma_g \equiv dt / d\tau = 1/\sqrt{g_{00}}$ for gravitational interaction associate dt with the time ticked off by the laboratory clock of an observer at rest or outside a gravitational field, and $d\tau$ with the proper time ticked off by an observed clock in relative motion or inside the gravitational field. The derivations of these two relativistic relations are reviewed in Appendix B. So in (5.6), we make a parallel association of dt with the neutral laboratory clock resting with an observer. Then, absent any gravitation or motion we now equate dt' with $d\tau$ so that $d\tau \equiv dt'$ becomes the proper time ticked off by the charged q' / m' clock being observed. With this we have:

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{q'\phi_0}{m'c^2}}. \tag{5.7}$$

Finally, as a matter of notational convention, because (5.7) compares a neutral $q = 0$ laboratory g-clock with dt, to a charged $q' \neq 0$ g-clock with $d\tau$, the primes are no longer needed, so we re-denote q' to q and m' to m. We then use (5.7) so re-notated to define an electromagnetic time dilation factor γ_e comparing the ratio of time ticked off by the neutral g-clock of an observer to time ticked off by an observed charged g-clock, as follows:

$$\gamma_{em} \equiv \frac{dt}{d\tau} = \frac{1}{1-\frac{q\phi_0}{mc^2}} = 1 + \frac{q\phi_0}{mc^2} + \left(\frac{q\phi_0}{mc^2}\right)^2 + \left(\frac{q\phi_0}{mc^2}\right)^3 + \left(\frac{q\phi_0}{mc^2}\right)^4 + \ldots = \sum_{n=0}^{\infty}\left(\frac{q\phi_0}{mc^2}\right)^n . \tag{5.8}$$

Above, $q\phi_0 / mc^2$ is the key dimensionless ratio which determines the numerical size of γ_{em}. Because $E_e = q\phi_0$ is the energy of electromagnetic interaction between the test charge q and the source of the potential ϕ_0, we see that $q\phi_0 / mc^2 = E_e / E_0$ is the dimensionless ratio of this electromagnetic interaction energy to the rest energy $E_0 = mc^2$ of the test charge.

It is illustrative to examine (5.8) in the special case where a positive charge Q generates a Coulomb proper scalar potential $\phi_0 = k_e Q / r$, with $k_e = 1/4\pi\varepsilon_0 = \mu_0 c^2 / 4\pi = 10^{-7} c^2 \mathrm{N/A}^2$ being the Coulomb constant. For a test body with positive charge q and mass m at rest in the potential at a distance r from Q, the electromagnetic interaction energy $E_e = q\phi_0 = k_e Qq / r$ is repulsive because lower energy states are achieved by two like-charges moving farther apart. The ratio of this interaction energy to the test charge rest mass is $q\phi_0 / mc^2 = k_e Qq / mc^2 r$. Here, (5.8) becomes:

$$\gamma_{em} \equiv \frac{dt}{d\tau} = \frac{1}{1-\frac{k_e Qq}{mc^2 r}} = 1 + \frac{k_e Qq}{mc^2 r} + \left(\frac{k_e Qq}{mc^2 r}\right)^2 + \left(\frac{k_e Qq}{mc^2 r}\right)^3 + \left(\frac{k_e Qq}{mc^2 r}\right)^4 + \ldots = \sum_{n=0}^{\infty}\left(\frac{k_e Qq}{mc^2 r}\right)^n . \tag{5.9}$$

Because $dt / d\tau > 1$ when Q and q both have the same sign and are therefore repelling, the neutral laboratory g-clock will emit more "tick" signals during a given time than the observed charged g-clock being observed. So we learn that time dilates for a *repulsive* electromagnetic interactions between two like-charges, just as it dilates for the attractive gravitational interaction between what are always two like-masses. That is, time dilation occurs for interactions between *like charges*, which interactions for gravitation are attractive and for electromagnetism are repulsive, owing to the respective spin-2 gravitons and spin-1 photons that quantum-mediate these interactions. This also means that time contracts for attractive electromagnetic interactions between unlike charges.

As a numeric benchmark for classical interactions, consider that the two charges each have $Q = q = 1\mathrm{C}$, the test particle has a rest mass $m = 1\mathrm{kg}$, and the separation $r = 1\mathrm{m}$. Therefore, the dimensionless ratio of interaction to rest energy $q\phi_0 / mc^2 = k_e / c^2 = 10^{-7}$, and the time dilation is $\gamma_{em} \cong 1 + 10^{-7}$ (to parts per 10^{-14}, from the next-higher-order term in (5.9)). At the same time, this interaction energy $q\phi_0 = k_e = 10^{-7} c^2\,\mathrm{J} = 8.897 \times 10^9\,\mathrm{J}$ is exceedingly large. The release of this much energy per second would yield a power of approximately 8.897 GW, which roughly approximates seven or eight nuclear power plants, or four times the power of the Hoover Dam, or the power of about seventy five jet engines, or the power output of a single space shuttle launch, or of a single lightning bolt. So it takes tremendously large electromagnetic interactions to produce very small time dilations. For electromagnetic interactions encountered in daily experience, this dilation will be much smaller. For example, a kW-order interaction would dilate time to about one part in 10^{14}. For a cesium clock ticking every 1.09×10^{-10} seconds, the discrepancy for a kW-order interaction would be about 1 tick per ten thousand seconds – about 2.75 hours.

Knowing from (5.8) that time dilates for repulsive electromagnetic interactions, one can design an even-simpler experiment to test for these time dilations, at least qualitatively: take a first neutral g-clock, and synchronize it with a second neutral g-clock. Then charge the second g-clock and use the first g-clock as a control to measure its time oscillations. Because there will now be an internal repulsive self-interaction energy between and among the various elemental parts of the charged clock, the mere charging of the clock should cause the oscillatory period to dilate.

As we now also show, the well-known energy content of electromagnetically-interacting bodies provides direct empirical evidence time really does dilate in accordance with (5.8) and (5.9).

6. The Energy Content of Electromagnetically-Interacting, Moving and Gravitating Material Bodies

Einstein's pioneering paper [12] first used a time dilation factor γ_v in the simple calculation $E = mc^2\gamma_v = mc^2 \cdot dt/d\tau = mc^2/\sqrt{1-v^2/c^2} \cong mc^2 + \tfrac{1}{2}mv^2$ to uncover the rest energy relation now known as $E_0 = mc^2$. In this calculation, the Newtonian kinetic energy $E_v = \tfrac{1}{2}mv^2$ is shown to be a comparatively tiny addition to the huge rest energy $E_0 = mc^2$ of a mass *m*, for non-relativistic velocities $v/c \ll 1$. Moreover, the kinetic energy in general is seen to be the *nonlinear* $E_v = mc^2 \cdot (dt/d\tau - 1) = mc^2\left(1/\sqrt{1-v^2/c^2} - 1\right)$ in which the Newtonian $\tfrac{1}{2}mv^2$ is the lowest-order term in the McLaurin series $E_{\text{kin}} = \tfrac{1}{2}mv^2\Sigma_{n=0}^{\infty}\left((2n+1)!!/2^n(n+1)!\right)\left(v^2/c^2\right)^n$, with $\tfrac{1}{2}mv^2$ multiplied by higher order terms v^2/c^2, v^4/c^4, v^6/c^6, etc. times a series of numeric coefficients.

Einstein later showed in [3] that this carries over to gravitational energies, but now with a gravitational time dilation $\gamma_g = dt/d\tau = 1/\sqrt{g_{00}}$ which leads to the energy content relation $E = mc^2\gamma_g = mc^2 \cdot dt/d\tau = mc^2/\sqrt{g_{00}}$. For a Schwarzschild metric with $g_{00} = 1 - 2GM/c^2r$ this produces $E = mc^2/\sqrt{1-2GM/c^2r} \cong mc^2 + GMm/r$. Here, the negative* Newtonian gravitational interaction energy $-E_g = GMm/r$ is seen to be a comparatively tiny addition to the rest energy mc^2 for weak gravitational interactions in which the ratio of gravitational energy to rest energy $GM/c^2r = (GMm/r)/mc^2 \ll 1$. Here too, $-E_g = mc^2(dt/d\tau - 1) = mc^2/\left(1/\sqrt{1-2GM/c^2r} - 1\right)$ is a nonlinear energy, with a series $-E_g = (GMm/r)\Sigma_{n=0}^{\infty}\left((2n+1)!!/(n+1)!\right)\left(GM/c^2r\right)^n$. In this

* Even though the mass *m* gains energy in the gravitational field and thus increases its ability to do work, e.g., by falling toward *M*, the gravitational interaction energy must be negative. This is because gravitation is an attractive interaction so that lower energy states must correlate with the two masses moving closer.

situation, the Newtonian GMm/r is multiplied by a higher-order succession of terms GM/c^2r, $\left(GM/c^2r\right)^2$, $\left(GM/c^2r\right)^3$ etc. terms times a series of coefficients.

As it happens, the electromagnetic time dilation (5.8) when multiplied through by the rest energy mc^2 yields similar information about the energy content of electromagnetically-interacting bodies. Working from (5.8) in the same way as reviewed just above, it is readily calculated that:

$$E = mc^2\gamma_{em} = mc^2\frac{dt}{d\tau} = \frac{mc^2}{1-\frac{q\phi_0}{mc^2}} = mc^2 + q\phi_0\left(1+\frac{q\phi_0}{mc^2}+\left(\frac{q\phi_0}{mc^2}\right)^2+\ldots\right) = mc^2 + q\phi_0\sum_{n=0}^{\infty}\left(\frac{q\phi_0}{mc^2}\right)^n . \quad (6.1)$$

Here, the known interaction energy $E_e = q\phi_0$ is seen to be a comparatively tiny addition to the rest energy mc^2 for interactions in which the dimensionless ratio of electromagnetic interaction energy $q\phi_0$ to rest energy mc^2 is very small, $q\phi_0/mc^2 \ll 1$. Here, when $q\phi_0/mc^2$ grows measurably larger – in a new result that does not appear to have been reported in the literature at least for classical electromagnetic interactions – *the electromagnetic interaction energy becomes non-linear just like special and general relativistic energies*. Now, in general, electromagnetic interaction energy is given by the non-linear series $E_e = q\phi_0\Sigma_{n=0}^{\infty}\left(q\phi_0/mc^2\right)^n$, and the higher order multipliers of the known energy $q\phi_0$ are $q\phi_0/mc^2$, $\left(q\phi_0/mc^2\right)^2$, $\left(q\phi_0/mc^2\right)^3$ etc. So for a Coulomb potential $\phi_0 = k_eQ/r$ (6.1) above becomes:

$$\begin{aligned} E = mc^2\gamma_{em} = mc^2\frac{dt}{d\tau} &= \frac{mc^2}{1-\frac{k_eQq}{mc^2r}} = mc^2 + \frac{k_eQq}{r}\left(1+\frac{k_eQq}{mc^2r}+\left(\frac{k_eQq}{mc^2r}\right)^2+\left(\frac{k_eQq}{mc^2r}\right)^3+\ldots\right) \\ &= mc^2 + \frac{k_eQq}{r}\sum_{n=0}^{\infty}\left(\frac{k_eQq}{mc^2r}\right)^n \end{aligned} . \quad (6.2)$$

Just as with $E_v = \frac{1}{2}mv^2$ for motion and $-E_g = GMm/r$, the Coulomb interaction energy $E_e = k_eQq/r$ is likewise a tiny correction to the to the rest energy mc^2, precisely as is observed. But the complete energy $E_e = \left(k_eQq/r\right)\Sigma_{n=0}^{\infty}\left(k_eQq/mc^2r\right)^n$ is non-linear. For the classical benchmark $q\phi_0/mc^2 = k_e/c^2 = 10^{-7}$ given at the end of the last section, the interaction energy $q\phi_0 = k_e = 10^{-7}c^2\,\text{J} = 8.897\times10^9\,\text{J}$ is increased by a scant one part in 10^7 owing to the first correction term k_eQq/mc^2r in the series. Nonetheless, (6.2) gives a precise prediction of the magnitude of these newly-predicted non-linear corrections.

When there are both motion and gravitation, the special and general relativistic time dilations are compounded by multiplication, so the total time dilation $\gamma = dt/d\tau = \gamma_v\gamma_g$, with a total energy content $E = mc^2\gamma_v\gamma_g$. We may therefore expect that when there are electromagnetic interactions in addition to motion and gravitation, $\Gamma \equiv dt/d\tau = \gamma_v \gamma_g \gamma_{em}$ will be the complete time dilation, and the total energy content of the material body will be $E = mc^2\Gamma = mc^2 \gamma_v \gamma_g \gamma_{em}$. If we compute this using while also showing the linear limit, we obtain:

$$\begin{aligned} E &= mc^2\Gamma = mc^2 \frac{dt}{d\tau} = mc^2 \gamma_v \gamma_g \gamma_{em} = mc^2 \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{1}{\sqrt{1-\frac{2GM}{c^2 r}}} \frac{1}{1-\frac{k_e Qq}{mc^2 r}} \\ &\cong mc^2 \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\left(1+\frac{GM}{c^2 r}\right)\left(1+\frac{k_e Qq}{mc^2 r}\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{GMm}{r} + \frac{1}{2}\frac{GMm}{c^2 r}v^2 + \frac{k_e Qq}{r} + \frac{1}{2}\frac{k_e Qq}{c^2 r}v^2 + \frac{GM}{r}\frac{k_e Qq}{c^2 r} + \frac{1}{2}\frac{GM}{c^2 r}\frac{k_e Qq}{c^2 r}v^2 \end{aligned} \quad . \tag{6.3}$$

What we see here, in succession, are 1) the rest energy mc^2, 2) the kinetic energy of the mass *m*, 3) the gravitational interaction energy of the mass, 4) the kinetic energy of the gravitational energy, 5) the Coulomb interaction energy of the charged mass, 6) the kinetic energy of the Coulomb energy, 7) the gravitational energy of the Coulomb energy and 8) the kinetic energy of the gravitational energy of the Coulomb energy. Numbers 1 through 4 above are standard results that are obtained when one applies the Special and General theories at the same time. Numbers 1 through 4 are well-established in relativity theory. Numbers 5 through 8 incorporate the new findings (5.8) and (5.8) of an electromagnetic time dilation. All of these accords entirely with empirical observations of the linear limits of these same energies.

Of course, $E = p^0 c$ in (6.3) is the time component of the energy-momentum four-vector $cp^\mu = (E, c\mathbf{p}) = mcdx^\mu / d\tau$. By the chain rule, the relativistic four velocity $dx^\mu / d\tau = (dx^\mu / dt)(dt/d\tau)$, and because $dx^\mu = (cdt, d\mathbf{x})$ the ordinary four-velocity $dx^\mu / dt = (c, d\mathbf{x}/dt) = (c, \mathbf{v}) \equiv v^\mu$. Because the composite time dilation $\Gamma \equiv dt/d\tau = \gamma_v \gamma_g \gamma_{em}$ is validated at least at lowest order by the energy content shown in (6.3), we may combine the foregoing to deduce that $dx^\mu / d\tau = \gamma_v \gamma_g \gamma_e v^\mu = \Gamma v^\mu$. Therefore, when all of motion and gravitation and electrodynamic interactions are present, the Lorentz four-vector p^μ in (1.6), of which (6.3) sits in the time component, is deduced to be:

$$cp^\mu = \begin{pmatrix} E & c\mathbf{p} \end{pmatrix} = mcdx^\mu / d\tau = mcv^\mu \gamma_v \gamma_g \gamma_{em} = mcv^\mu \Gamma . \tag{6.4}$$

Likewise, we may deduce that in the "peculiar" quadratic metric of (3.3), the coordinate elements with all of motion and gravitation and electrodynamics are $dx^\mu = \gamma_v \gamma_g \gamma_{em} v^\mu d\tau = \Gamma v^\mu d\tau$. This is a way to reintroduce motion and gravitation and Lorentz covariance into the quadratic solution (4.4) obtained at rest and absent gravitation, and into the consequent (5.8) for a neutral laboratory g-clock used to measure time signals from an identical g-clock which is charged.

Finally, because gauge symmetry results in replacing $p^\mu \to \pi^\mu = p^\mu + qA^\mu / c$, it is necessary for A^μ to transform in the same general covariant manner as p^μ. Absent gravitation, and absent being aware of electromagnetic time dilations, the four-potential is normally defined in relation to motion by $A^\mu = \phi_0 v^\mu (dt/d\tau)/c = \phi_0 \gamma_v v^\mu / c$, where $v^\mu = (c, \mathbf{v})$. But when all time dilations are considered, energy content is changed, and general covariance requires that this now be extended to:

$$A^\mu = (\phi \quad \mathbf{A}) = \phi_0 (v^\mu / c)(dt/d\tau) = \phi_0 v^\mu \gamma_v \gamma_g \gamma_{em} / c = \phi_0 v^\mu \Gamma / c. \tag{6.5}$$

We see that both (6.4) and (6.5) contain the same time dilation and motion kernel $v^\mu \Gamma$.

7. Energy-Momentum Gradients, and Heisenberg Rules for Momentum Commutation in view of Electromagnetic Time Dilations

When the energy-momentum of a particle depends only on its rest mass and its motion, then $\gamma_g = 1$ and $\gamma_{em} = 1$, so (6.4) of course becomes the special relativistic $p^\mu = mv^\mu \gamma_v$. Because $\partial_\alpha \gamma_v = 0$ and $\partial_\alpha v^\mu = 0$, this has no spacetime dependency, which may be expressed differentially via $\partial_\alpha p^\mu = m\partial_\alpha (\gamma_v v^\mu) = 0$. When there is a gravitational field, then $p^\mu = mv\gamma_v \gamma_g$ and there is a spacetime dependency, because $\gamma_g(t, \mathbf{x}) = 1/\sqrt{g_{00}(t, \mathbf{x})}$ is a function of space and time. Thus, the four-gradient $\partial_\alpha \gamma_g = -\frac{1}{2}\partial_\alpha g_{00} / (g_{00})^{1.5} \neq 0$ and $\partial_\alpha p^\mu = m\partial_\alpha (\gamma_v \gamma_g v^\mu) \neq 0$. However, because g_{00} is a component of the metric tensor with gravitational-covariant derivative $\partial_{;\alpha} g_{\mu\nu} = 0$, the gravitational covariant derivative $\partial_{;\alpha} p^\mu = m\partial_{;\alpha} (\gamma_v \gamma_g v^\mu) = 0$ of the energy-momentum is still zero.

For electromagnetic interactions, this is no longer the case. Now, the energy-momentum $p^\mu(t, \mathbf{x}) = mv^\mu \gamma_v \gamma_g \gamma_{em}(t, \mathbf{x})$ takes on an explicit spacetime dependency, because as deduced in (5.8), the electromagnetic time dilation is a function $\gamma_{em}(t, \mathbf{x}) = 1/(1 - q\phi_0(t, \mathbf{x})/mc^2)$ of spacetime, because the proper potential $\phi_0(t, \mathbf{x})$ is (or may be) a function of space and time. Expressed differentially, $\partial_{;\alpha} \gamma_{em} = \partial_\alpha \gamma_{em} \neq 0$, so that $\partial_\alpha p^\mu \neq 0$ and even $\partial_{;\alpha} p^\mu \neq 0$. This spacetime dependency of the energy momentum stemming from $\phi_0(t, \mathbf{x})$ has a number of useful and important properties that it now behooves us to explore.

Absent gravitation, with $g_{\mu\nu} = \eta_{\mu\nu}$ thus $\gamma_g = 1$, the complete time dilation $\Gamma = dx^\mu / d\tau = \gamma_v \gamma_{em}$. From the time component of (6.5) we find that $\phi = \phi_0 \gamma_v \gamma_{em}$ which we invert to $\phi_0 = \phi / \gamma_v \gamma_{em}$. We then use this to write the electromagnetic time dilation (5.8) as:

$$\gamma_{em} = \frac{1}{1 - q\phi_0 / mc^2} = \frac{1}{1 - q\phi / \gamma_v \gamma_{em} mc^2} = \frac{\gamma_v \gamma_{em}}{\gamma_v \gamma_{em} - q\phi / mc^2} . \tag{7.1}$$

Upon dividing through by γ_{em} then taking the reciprocal of both sides we obtain:

$$1 = \frac{\gamma_v \gamma_{em} - q\phi / mc^2}{\gamma_v} = \gamma_{em} - \frac{q\phi}{mc^2 \gamma_v} , \tag{7.2}$$

which easily restructures into an alternative expression for γ_{em}, namely:

$$\gamma_{em} = 1 + \frac{q\phi}{mc^2 \gamma_v} . \tag{7.3}$$

The time component of (6.4) contains the total energy $E = mc^2 \gamma_v \gamma_{em}$ with $\gamma_g = 1$, see also (6.3), which energy, in view of (7.3), may be written as:

$$E = mc^2 \gamma_v \left(1 + \frac{q\phi}{mc^2 \gamma_v} \right) = mc^2 \gamma_v + q\phi . \tag{7.4}$$

So the total energy $E = mc^2 \gamma_v \gamma_{em}$ is alternatively written as the rest-plus motion energy $mc^2 \gamma_v$, plus the electromagnetic interaction potential energy $q\phi$. So if we take the space-gradient $-\partial^\alpha = \nabla^\alpha = \nabla$ of the above, then apply the relation $\nabla \phi = -\left(\mathbf{E} + \dot{\mathbf{A}} / c \right)$ between the potential gradient and the electric field $\mathbf{E}$ and time derivative $\dot{\mathbf{A}} = \partial \mathbf{A} / \partial t$ of the three-potential, we obtain:

$$-\partial^j E = \nabla E = -q \partial^j \phi = q \nabla \phi = -q \left(E^j + \dot{A}^j / c \right) = -q \left(\mathbf{E} + \dot{\mathbf{A}} / c \right) . \tag{7.5}$$

It is also useful to separately take the gradient of the time dilation (7.3), namely:

$$\partial^j \gamma_{em} = \frac{q}{mc^2 \gamma_v} \partial^j \phi = \frac{1}{mc^2 \gamma_v} q \left(E^j + \dot{A}^j / c \right) = -\frac{q}{mc^2 \gamma_v} \nabla \phi = -\nabla \gamma_{em} = \frac{1}{mc^2 \gamma_v} q \left(\mathbf{E} + \dot{\mathbf{A}} / c \right) . \tag{7.6}$$

We will find it useful to include γ_v inside the gradient and write this as a gradient of the total time dilation $dt / d\tau = \gamma_v \gamma_{em} = E / mc^2$, as such:

$$\partial^j \left(\frac{dt}{d\tau} \right) = \partial^j \left(\gamma_v \gamma_{em} \right) = \partial^j \left(\frac{E}{mc^2} \right) = \frac{q}{mc^2} \partial^j \phi = \frac{q}{mc^2} \left(E^j + \dot{A}^j / c \right) . \tag{7.7}$$

Having deduced the time component of (6.4) in flat spacetime, now we turn to the space components $c\mathbf{p} = cp^i = mcv^i \gamma_v \gamma_{em}$. Using (7.6), we find that:

$$\nabla c\mathbf{p} = -\partial^j cp^i = -mcv^i \gamma_v \partial^j \gamma_{em} = -\frac{1}{mc^2 \gamma_v} mcv^i \gamma_v q \partial^j \phi . \tag{7.8}$$

Multiply through by $-E = -mc^2 \gamma_v \gamma_{em}$ then applying $cp^i = mc\gamma_{em}\gamma_v v^i$ and $\partial^j \phi = E^j + \dot{A}^j / c$ yields:

$$\begin{aligned} &mc^2 \gamma_v \gamma_{em} \partial^j cp^i = mcv^i \gamma_v \gamma_{em} q \partial^j \phi \\ &= E\partial^j cp^i = q\partial^j \phi cp^i = q\left(E^j + \dot{A}^j / c\right) cp^i = -E\nabla c\mathbf{p} = -q\nabla \phi c\mathbf{p} = q\left(\mathbf{E} + \dot{\mathbf{A}} / c\right) c\mathbf{p} \end{aligned} . \tag{7.9}$$

With $cp^\mu = \begin{pmatrix} E & c\mathbf{p} \end{pmatrix}$ we then assemble equations (7.5) and (7.9) into spacetime-covariant form:

$$E\partial^j cp^\mu = q\partial^j \phi cp^\mu = q\left(E^j + \dot{A}^j / c\right) cp^\mu = -E\nabla cp^\mu = -q\nabla \phi cp^\mu = q\left(\mathbf{E} + \dot{\mathbf{A}} / c\right) cp^\mu . \tag{7.10}$$

This now reveals the heuristic rule $-E\nabla \mapsto -q\nabla \phi \mapsto q\left(\mathbf{E} + \dot{\mathbf{A}} / c\right)$ for when a gradient is applied to the flat spacetime four-momentum $cp^\mu = mc\gamma_v \gamma_{em} v^\mu$ of (6.4) with electromagnetic time dilation.

Another important and useful consequence of the relation $cp^\mu = mcv^\mu \gamma_v \gamma_{em}$ is that the three-momentum $p^i = \mathbf{p}$ no longer commutes as between momenta oriented along orthogonal spacetime axes, that is, $\left[p^i, p^j \right] = p^i p^j - p^j p^i \neq 0$ when $i \neq j$. Written in vector notation, this means that the momentum self-cross product $\mathbf{p} \times \mathbf{p} \neq 0$. To derive this with specificity, we begin with Heisenberg's canonical commutation relation $[p_x, x] = -i\hbar$ which of course underlies the uncertainty principle. It is easily calculated that $[p_x, x^n] = -i\hbar n x^{n-1}$. Moreover, because elementary calculus teaches that $\partial_x x^n = nx^{n-1}$ we may combine the foregoing into $[p_x, x^n] = -i\hbar n x^{n-1} = -i\hbar \partial_x x^n$. Therefore, for any function $b(\mathbf{x})$ expansible as a Maclaurin series in $\mathbf{x}$, noting that $\partial^i = -\nabla = -\left(\partial_x, \partial_y, \partial_z\right)$, we may generalize this in well-known fashion to the well-established relation $\left[p^i, b \right] = i\hbar \partial^i b$. This is then generalizable to any vector, tensor, etc. object $O(\mathbf{x})$ whereby $\left[p^i, O \right] = i\hbar \partial^i O$ and $\left[O, p^j \right] = -i\hbar \partial^j O$. Thus, if we generalize to a vector $b \mapsto b^j(\mathbf{x})$, with the momentum to the left of the commutator, this becomes $\left[p^i, b^j \right] = i\hbar \partial^i b^j$. With momentum to the right of $b^j(\mathbf{x})$, and with renamed indexes, the right-side relation is $\left[b^i, p^j \right] = -i\hbar \partial^j b^i$.

Now, the space components of $p^i = mv^i\gamma_v\gamma_{em}$ include both the non-relativistic momentum $p^i_{NR} = mv^i$ and the time dilation multiplier $dt/d\tau = \gamma_v\gamma_{em}$, that is, $p^i = p^i_{NR}\gamma_v\gamma_{em}$. However, the Heisenberg operator (op) matrices for momentum correspond to the *non-relativistic momentum only*, $p^i_{NR} \Leftrightarrow p^i_{op}$. With this in mind, were we to assign $b^j \mapsto p^j$ then the left-side commutator relation would become $\left[p^i, p^j\right] = i\hbar\partial^i p^j$. But, were we to instead assign $b^i \mapsto p^i$ then the right-side commutator this would become $\left[p^i, p^j\right] = -i\hbar\partial^j p^i$. So both relations together would imply that $i\hbar\partial^i p^j = -i\hbar\partial^j p^i$ is an antisymmetric tensor, which is not necessarily so. However, this overlooks the fact that *unlike any other vector*, the space-component momentum vector $p^i = mv^i\gamma_v\gamma_{em}$ is a hybrid momentum and spacetime vector. This is because it includes both $mv^i = p^i_{NR} \Leftrightarrow p^i_{op}$ which is a pure momentum against which functions of the space coordinates x^j are commuted, and because it also includes $\gamma_{em}(t,\mathbf{x}) = 1/\left(1 - q\phi_0/mc^2\right)$ which is a function of spacetime because the proper scalar potential $\phi_0(t,\mathbf{x})$ is a function of spacetime. Specifically, $p^i(t,\mathbf{x}) = p^i_{NR}\gamma_v\gamma_{em}(t,\mathbf{x})$. Consequently, there is a self-commutativity wherein when we commute $\left[p^i, p^j\right]$, we are commuting the space-dependent portion of p^j to the left past the p^i_{NR} portion of p^i, while *simultaneously* commuting the space-dependent portion of p^i to the right past the p^j_{NR} portion of p^j. It is only the pure non-relativistic momentum which is self-commuting along all orthogonal pairs of space coordinates, $\left[p^i_{NR}, p^j_{NR}\right] = 0$.

Given the foregoing, if we write the two commuting momentum vectors as $cp^i = mcv^i\gamma_v\gamma_{em} = cp^i_{NR}(\gamma_v\gamma_{em})$ and $cp^j = mcv^j\gamma_v\gamma_{em} = cp^j_{NR}(\gamma_v\gamma_{em})$, and if we write the left-momentum and right-momentum commutativity relations as $\left[cp^i_0, \gamma_v\gamma_{em}\right] = i\hbar c\partial^i(\gamma_v\gamma_{em})$ and $\left[\gamma_v\gamma_{em}, cp^j_0\right] = -i\hbar c\partial^j(\gamma_v\gamma_{em})$, then also making use of $\left[p^i_{NR}, p^j_{NR}\right] = 0$, we may calculate:

$$\begin{aligned}cp^i cp^j &= cp^i_{NR}\gamma_v\gamma_{em}cp^j_{NR}\gamma_v\gamma_{em} = \gamma_v\gamma_{em}cp^j_{NR}cp^i_{NR}\gamma_v\gamma_{em} + i\hbar c\partial^i(\gamma_v\gamma_{em})cp^j_{NR}\gamma_v\gamma_{em} \\ &= cp^j_{NR}\gamma_v\gamma_{em}cp^i_{NR}\gamma_v\gamma_{em} - i\hbar c\partial^j(\gamma_v\gamma_{em})cp^i_{NR}\gamma_v\gamma_{em} + i\hbar c\partial^i(\gamma_v\gamma_{em})cp^j_{NR}\gamma_v\gamma_{em} \\ &= cp^j cp^i - i\hbar c\partial^j(\gamma_v\gamma_{em})cp^i + i\hbar c\partial^i(\gamma_v\gamma_{em})cp^j\end{aligned} \quad . \tag{7.11}$$

Recasting this as a commutator, and then using (7.7), this becomes:

$$\begin{aligned}&\left[cp^i, cp^j\right] = -i\hbar c\partial^j(\gamma_v\gamma_{em})cp^i + i\hbar c\partial^i(\gamma_v\gamma_{em})cp^j \\ &= -i\hbar c\frac{q}{mc^2}\partial^j\phi cp^i + i\hbar c\frac{q}{mc^2}\partial^i\phi cp^j = -i\frac{\hbar cq}{mc^2}\left(E^j + \dot{A}^j/c\right)cp^i + \frac{\hbar cq}{mc^2}\left(E^i + \dot{A}^i/c\right)cp^j\end{aligned} \quad . \tag{7.12}$$

So the momentum self-commutator actually contains the gradient $\nabla\phi$ of the scalar potential, and, via $\nabla\phi = -\left(\mathbf{E} + \dot{\mathbf{A}} / c\right)$, the electric field $\mathbf{E}$ and time derivative $\dot{\mathbf{A}} = \partial \mathbf{A} / \partial t$ of the three-vector potential. Note that $\left[cp^i, cp^j \right] = 0$ when $i = j$, so that this commutation relation only becomes non-zero, $\left[cp^i, cp^j \right] \neq 0$, as to orthogonal pairs of space coordinates where $i \neq j$.

Now let us additionally consider the circumstance where p^i is the momentum of an individual charged lepton (i.e., the electron, or the mu or tau lepton), the mass m is lepton rest mass, and the charge $q = -e$ is the electric charge quantum. With the Bohr magneton $\mu_B = \hbar e / 2mc$, given the specific emergence of $\hbar cq / mc^2$ in (7.12), we may rewrite the commutator as:

$$\left[cp^i, cp^j \right] = 2i\mu_B \left(\partial^j \phi cp^i - \partial^i \phi cp^j \right) = 2i\mu_B \left(\left(E^j + \dot{A}^j / c \right) cp^i - \left(E^i + \dot{A}^i / c \right) cp^j \right). \tag{7.13}$$

Now let's return to (7.9), which contains terms very similar to those in the commutator (7.13). This gradient may be specialized to a dot (inner) product by forming

$$E \partial^i cp^i = q \partial^i \phi cp^i = q \left(E^i + \dot{A}^i / c \right) cp^i = -E \nabla \cdot c\mathbf{p} = -q \nabla \phi \cdot c\mathbf{p} = q \left(\mathbf{E} + \dot{\mathbf{A}} / c \right) \cdot c\mathbf{p} . \tag{7.14}$$

But it is especially of interest to form the cross product:

$$\begin{aligned} &\varepsilon^{kji} E \partial^j cp^i = \varepsilon^{kji} q \partial^j \phi cp^i = \varepsilon^{kji} q \left(E^j + \dot{A}^j / c \right) cp^i \\ &= -E \left(\nabla \phi \times c\mathbf{p} \right)^k = -q \left(\nabla \phi \times c\mathbf{p} \right)^k = q \left(\left(\mathbf{E} + \dot{\mathbf{A}} / c \right) \times c\mathbf{p} \right)^k \end{aligned} . \tag{7.15}$$

This is because (7.13) contains $\partial^j \phi cp^i - \partial^i \phi cp^j$ which is also a cross product. Specifically, given that $\varepsilon^{kij} \left[cp^i, cp^j \right] = \varepsilon^{kij} cp^i cp^j - \varepsilon^{kij} cp^j cp^i = 2\varepsilon^{kij} cp^i cp^j = 2 \left(c\mathbf{p} \times c\mathbf{p} \right)^k$, we may turn (7.13) into an explicit cross product by multiplying through by ε^{kij} to form (recall $\partial^j = -\nabla^j$):

$$\begin{aligned} &\varepsilon^{kij} \left[cp^i, cp^j \right] = 2i\varepsilon^{kij} \mu_B \left(\partial^j \phi cp^i - \partial^i \phi cp^j \right) = 2i\varepsilon^{kij} \mu_B \left(\left(E^j + \dot{A}^j / c \right) cp^i - \left(E^i + \dot{A}^i / c \right) cp^j \right) \\ &= 2\varepsilon^{kij} cp^i cp^j = 4i\varepsilon^{kij} \mu_B \nabla^i \phi cp^j = -4i\varepsilon^{kij} \mu_B \left(E^i + \dot{A}^i / c \right) cp^j \\ &= 2 \left(c\mathbf{p} \times c\mathbf{p} \right)^k = 2i\mu_B \left(\nabla \phi \times c\mathbf{p} \right)^k = -2i\mu_B \left(\left(\mathbf{E} + \dot{\mathbf{A}} / c \right) \times c\mathbf{p} \right)^k \end{aligned} . \tag{7.16}$$

If we now set $q = -e$ in (7.15) and so apply this to the charged leptons, then multiply through by $i\hbar c / mc^2$ and use $\mu_B = \hbar e / 2mc$, we separately obtain:

$$-\frac{E}{mc^2}\left(i\hbar c\nabla\phi\times c\mathbf{p}\right)^k = 2i\mu_B\left(\nabla\phi\times c\mathbf{p}\right)^k = -2i\mu_B\left(\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times c\mathbf{p}\right)^k . \tag{7.17}$$

Dropping the k superscript which is absorbed into $\times^k = \times$, we than combine (7.16) and (7.17) into:

$$\begin{aligned}2c\mathbf{p}\times c\mathbf{p} &= -\left(E/mc^2\right)i\hbar c\nabla\phi\times c\mathbf{p} = -\left(dt/d\tau\right)i\hbar c\nabla\phi\times c\mathbf{p} = -\gamma_v\gamma_{em}i\hbar c\nabla\phi\times c\mathbf{p} \\ &= 2i\mu_B\nabla\phi\times c\mathbf{p} = -2i\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times c\mathbf{p}\end{aligned} . \tag{7.18}$$

So as previewed in the paragraph following (7.10), we see that indeed, the momentum self-cross product $\mathbf{p}\times\mathbf{p}\neq 0$. And we have learned that this arises because the electromagnetic time dilation $\gamma_{em}\left(t,\mathbf{x}\right)=1/\left(1-q\phi_0\left(t,\mathbf{x}\right)/mc^2\right)$ is included in flat spacetime within the total energy momentum $cp^\mu = mcv^\mu\gamma_v\gamma_{em} = p^\mu_{NR}\gamma_v\gamma_{em}$ of (6.4). This is the first of numerous quantum mechanical results that we shall now begin to explore as we turn from classical to quantum electrodynamics.

PART II: COVARIANT GAUGE FIXING TO REMOVE TWO DEGREES OF FREEDOM FROM THE GAUGE POTENTIAL, YIELDING A MASSLESS PHOTON WITH TWO HELICITY STATES

8. Heisenberg / Ehrenfest Equations of Time Evolution and Space Configuration

Thus far all the development has been based on (1.6), which is the relativistic energy-momentum relation $m^2c^2 = p_\sigma p^\sigma$ turned into $m^2c^2 = \pi_\sigma\pi^\sigma$ via the prescription $p^\sigma \mapsto \pi^\sigma$ which arises from imposing local U(1) gauge symmetry, taken in the classical $\hbar = 0$ limit by regarding the commutator relation to be $\left[p_\sigma, A^\sigma\right]=0$. Now, we return to the commutator $\left[p_\sigma, A^\sigma\right]$ in (1.5) and no longer approximate this to zero, but instead treat this quantum mechanically.

It was reviewed early in section 1 how when operating on a Fourier kernel $\exp\left(-ip_\sigma x^\sigma/\hbar\right)$ with the spacetime gradient ∂_μ, we obtain $\partial_\mu \exp\left(-ip_\sigma x^\sigma/\hbar\right) = -\left(ip_\mu/\hbar\right)\exp\left(-ip_\sigma x^\sigma/\hbar\right)$, where we assume that $\partial_\mu p_\sigma = 0$ i.e. that the components of energy momentum are not functions of spacetime. So when we form a function such as $\phi = s\exp\left(-ip_\sigma x^\sigma/\hbar\right)$ with $s\left(p^\nu\right)$ a function of momentum but, importantly, not of spacetime because $\partial_\mu s\left(p^\nu\right)=0$, or such as $\psi = u\exp\left(-ip_\sigma x^\sigma/\hbar\right)$ with $u\left(p^\nu\right)$ and $\partial_\mu u\left(p^\nu\right)=0$, then we obtain $i\hbar\partial_\mu\phi = p_\mu\phi$ in the former and $i\hbar\partial_\mu\psi = p_\mu\psi$ in the latter case. Then for the Klein-Gordon and Dirac equations respectively, these operations allow for toggling between momentum and configuration space via $i\hbar\partial_\mu \leftrightarrow p_\mu$.

We now add to this, that energies $W = E - mc^2 = cp^0 - mc^2$ are eigenstates $H|s\rangle = W|s\rangle$ of a Hamiltonian operator H operating on a ket $|s\rangle$. Therefore, we may similarly form a Hamiltonian-momentum four-vector defined as $H^\mu \equiv (H + mc^2, c\mathbf{p})$ for which $H^\mu |s\rangle = cp^\mu |s\rangle$, then use this in a Fourier-type kernel $\exp(-iH_\sigma x^\sigma / \hbar c)$ with the derivative $\partial_\mu \exp(-iH_\sigma x^\sigma / \hbar c) = -(iH_\mu / \hbar c)\exp(-ip_\sigma x^\sigma / \hbar c)$, likewise assuming that $\partial_\mu H = 0$ i.e. that the Hamiltonian is spacetime-independent, whole of course $\partial_\mu (mc^2) = 0$. This is of interest because $\exp(-iH_\sigma x^\sigma / \hbar c) = \exp(-i(H + mc^2)t/\hbar + i\mathbf{p}\cdot\mathbf{x}/\hbar) = \exp(-i(H + mc^2)t/\hbar)\exp(i\mathbf{p}\cdot\mathbf{x}/\hbar)$ and because $U(t) = \exp(-iHt/\hbar)$ is the time evolution operator used in both the Heisenberg and Schrödinger pictures of quantum mechanics. The separation of this exponential into time and space operators via $\exp(A+B) = \exp A \exp B = \exp B \exp A$ is allowed because each of the four terms in $H_\sigma x^\sigma / c = (H + mc^2)t - p_x x - p_y y - p_z z = Ht - \mathbf{p}\cdot\mathbf{x}$ commutes with all other three.

Now, we generalize all of the foregoing by defining a ket $|s\rangle \equiv \exp(-iH_\sigma x^\sigma / \hbar c)|s_0\rangle$. This ket is a generalized state object including both a Fourier-type kernel $\exp(-iH_\sigma x^\sigma / \hbar c)$ which contains the Hamiltonian $H^0 = H + mc^2$, and a fixed-state ket $|s_0\rangle$ *defined* to be independent of spacetime, $\partial_\mu |s_0\rangle \equiv 0$, as designated by the subscript 0. The definition $\partial_\mu |s_0\rangle \equiv 0$ is important, and is the generalization of how we use $\partial_\mu s(p^\nu) = 0$ and $\partial_\mu u(p^\nu) = 0$ with $\partial_\mu p_\sigma = 0$ to toggle between configuration and momentum space for the Klein-Gordon and Dirac equations, respectively. As a consequence of these definitions, we may deduce that $H_\mu |s\rangle = i\hbar c\partial_\mu |s\rangle$.

Given that $H = H^\dagger$ is a Hermitian operator, we may also obtain the Hermitian conjugate of $|s\rangle$ which is the bra $\langle s| = \langle s|_0 \exp(iH_\sigma x^\sigma / \hbar c)$. As is customary we normalize the bra and ket to $\langle s|s\rangle = 1$. We then start by forming the operator relation:

$$\langle A^\nu \rangle = \langle s| A^\nu |s\rangle = \langle s_0| \exp(iH_\sigma x^\sigma / \hbar c) A^\nu \exp(-iH_\sigma x^\sigma / \hbar c)|s_0\rangle . \tag{8.1}$$

This is the expectation value for the gauge field A^ν, given that $\langle A^\nu \rangle = \langle s| A^\nu |s\rangle$. Now, our goal is to deduce the time-dependency $d\langle A^\nu \rangle / dt$, and thereafter, the space-dependency $d\langle A^\nu \rangle / d\mathbf{x}$.

The first step is to separate $\exp(-iH_\sigma x^\sigma / \hbar c) = \exp(-i(H + mc^2)t/\hbar)\exp(-ip_k x^k / \hbar)$ into time and space components with $-p_k x^k = \mathbf{p}\cdot\mathbf{x}$, via the standard $\exp(A+B) = \exp A \exp B$ because the commutator $[Ht, \mathbf{p}\cdot\mathbf{x}] = 0$. So for the ket we obtain the relation

$\exp\left(-iH_\sigma x^\sigma / \hbar c\right)\left|s_0\right\rangle = \exp\left(-i\left(H+mc^2\right)t/\hbar\right)\exp\left(i\mathbf{p}\cdot\mathbf{x}/\hbar\right)\left|s_0\right\rangle$, with a conjugate relation for the bra. Then, for convenient notation we define the bra $\left|s_{0,\mathbf{x}}\right\rangle \equiv \exp\left(i\mathbf{p}\cdot\mathbf{x}/\hbar\right)\left|s_0\right\rangle$. Because $\partial_t\left|s_0\right\rangle = 0$ by definition, it is easy to see that $\partial_t\left|s_{0,\mathbf{x}}\right\rangle = 0$, but that $\hbar\partial_k\left|s_{0,\mathbf{x}}\right\rangle = \hbar\nabla\left|s_{0,\mathbf{x}}\right\rangle = -ip_k\left|s_{0,\mathbf{x}}\right\rangle = i\mathbf{p}\left|s_{0,\mathbf{x}}\right\rangle \neq 0$, so that $\left|s_{0,\mathbf{x}}\right\rangle$ varies in space but not over time. The subscripts $0,\mathbf{x}$ thus mean that $x^\mu = \left(ct,\mathbf{x}\right) = \left(0,\mathbf{x}\right)$. If we view all of physics as describing the evolution over time of configurations of matter in space, then because $\exp\left(-iHt/\hbar\right)$ is the time evolution operator, we may regard $\exp\left(i\mathbf{p}\cdot\mathbf{x}/\hbar\right)$ as a space configuration operator. Likewise, now we may write $\left|s\right\rangle = \exp\left(-iHt/\hbar\right)\left|s_{0,\mathbf{x}}\right\rangle$. Likewise, also because $\left[Ht,\mathbf{p}\cdot\mathbf{x}\right]=0$, the bra $\left\langle s\right| = \left\langle s_{0,\mathbf{x}}\right|\exp\left(i\left(H+mc^2\right)t/\hbar\right)$. In this notation, we may then rewrite (8.1) as:

$$\left\langle A^\nu\right\rangle = \left\langle s\right|A^\nu\left|s\right\rangle = \left\langle s_{0,\mathbf{x}}\right|\exp\left(iHt/\hbar\right)A^\nu\exp\left(-iHt/\hbar\right)\left|s_{0,\mathbf{x}}\right\rangle, \tag{8.2}$$

with the rest mass term in $H+mc^2$ cancelling out because $\exp\left(imc^2t/\hbar\right)\exp\left(-imc^2t/\hbar\right)=1$. The above will be recognized as the usual starting point for deriving the Heisenberg equation of motion.

Because $\partial_t\left|s_{0,\mathbf{x}}\right\rangle = 0$, the total derivative of (8.2) with respect to time is the following:

$$\begin{aligned}\frac{d}{dt}\left\langle A^\nu\right\rangle &= \frac{d}{dt}\left\langle s\right|A^\nu\left|s\right\rangle = \frac{d}{dt}\left\langle s_{0,\mathbf{x}}\right|\left(\exp\left(iHt/\hbar\right)A^\nu\exp\left(-iHt/\hbar\right)\right)\left|s_{0,\mathbf{x}}\right\rangle \\ &= \left\langle s_{0,\mathbf{x}}\right|\exp\left(iHt/\hbar\right)\left(\frac{i}{\hbar}\left[H,A^\nu\right]+\frac{\partial A^\nu}{\partial t}\right)\exp\left(-iHt/\hbar\right)\left|s_{0,\mathbf{x}}\right\rangle \\ &= \left\langle s\right|\left(\frac{i}{\hbar}\left[H,A^\nu\right]+\frac{\partial A^\nu}{\partial t}\right)\left|s\right\rangle = \frac{i}{\hbar}\left\langle\left[H,A^\nu\right]\right\rangle + \left\langle\frac{\partial A^\nu}{\partial t}\right\rangle\end{aligned}. \tag{8.3}$$

This is recognizable as Ehrenfest's theorem, which is merely the expectation value of the Heisenberg equation of motion in the Heisenberg picture. Also applying the eigenvalue relations $H\left|s\right\rangle = E\left|s\right\rangle$ and $\left\langle s\right|H = \left\langle s\right|E$, we may rewrite this overall result, retaining bras and kets, as:

$$\left\langle s\right|\left[H,A^\nu\right]\left|s\right\rangle = \left\langle s\right|\left[E,A^\nu\right]\left|s\right\rangle = i\hbar\left\langle s\right|\frac{\partial A^\nu}{\partial t}\left|s\right\rangle - i\hbar\frac{d}{dt}\left\langle s\right|A^\nu\left|s\right\rangle. \tag{8.4}$$

Note that all of the above are also equal to $\left\langle s\right|\left[\left(H+mc^2\right),A^\nu\right]\left|s\right\rangle$, and it is really $\left(H+mc^2\right)\left|s\right\rangle = E\left|s\right\rangle$ which enables us to interchange $E\leftrightarrow H$ in this context. We then reintroduce spacetime indexes in flat spacetime, to rewrite the above using $p_0 = E/c$ as:

$$\langle s|\left[p_0, A^\nu\right]|s\rangle = i\hbar\langle s|\frac{\partial A^\nu}{\partial x^0}|s\rangle - i\hbar\frac{d}{dx^0}\langle s|A^\nu|s\rangle = i\hbar\langle s|\partial_0 A^\nu|s\rangle - i\hbar d_0\langle s|A^\nu|s\rangle . \tag{8.5}$$

Because our interest is the commutator $\left[p_\sigma, A^\sigma\right] = \left[p_0, A^0\right] + \left[p_k, A^k\right]$ in (1.5), we find that when sandwiched between a bra and a ket as defined above, the term $\langle s|\left[p_0, A^0\right]|s\rangle$ is the $\nu = 0$ component of (8.5) above.

Next, let us obtain the space-dependency $d\left\langle A^\nu\right\rangle / d\mathbf{x}$ for (8.1). We can sample, say, the *z* axis, then generalize to *x* and *y*. First, we segregate the z-axis term to the front of the kernel $\exp\left(-iH_\sigma x^\sigma / \hbar c\right) = \exp\left(-ip_3 x^3 / \hbar\right)\exp\left(-ip_{2,1}x^{2,1} / \hbar\right)\exp\left(-i\left(H_0 + mc^2\right)x^0 / \hbar c\right)$. Again, this is permitted because all four terms in $H_\sigma x^\sigma / c = \left(H + mc^2\right)t - p_x x - p_y y - p_z z$ mutually commute. Then, we define $\left|s_{t,x,y,0}\right\rangle \equiv \exp\left(-ip_{2,1}x^{2,1} / \hbar\right)\exp\left(-i\left(H_0 + mc^2\right)x^0 / \hbar c\right)\left|s_0\right\rangle$ to be another ket which varies over time and over *x* and *y* but not over *z*, thus $\partial_z\left|s_{t,x,y,0}\right\rangle = 0$. Therefore, $|s\rangle = \exp\left(-ip_3 x^3 / \hbar\right)\left|s_{t,x,y,0}\right\rangle$. Given that $p_3 = -p_z$ in flat spacetime, using this and its conjugate bra $\langle s|$ in (8.1) yields:

$$\left\langle A^\nu\right\rangle = \langle s|A^\nu|s\rangle = \left\langle s_{t,x,y,0}\right|\exp\left(-ip_z z / \hbar\right)A^\nu\exp\left(ip_z z / \hbar\right)\left|s_{t,x,y,0}\right\rangle . \tag{8.6}$$

Then, using $\partial_z\left|s_{t,x,y,0}\right\rangle = 0$, we take the *z*-axis total derivative of (8.6) to obtain:

$$\begin{aligned}
&\frac{d}{dz}\left\langle A^\nu\right\rangle = \frac{d}{dz}\langle s|A^\nu|s\rangle = \frac{d}{dz}\left\langle s_{t,x,y,0}\right|\exp\left(-ip_z z / \hbar\right)A^\nu\exp\left(ip_z z / \hbar\right)\left|s_{t,x,y,0}\right\rangle \\
&= \left\langle s_{t,x,y,0}\right|\exp\left(-ip_z z / \hbar\right)\left(-\frac{i}{\hbar}\left[p_z, A^\nu\right] + \frac{\partial A^\nu}{\partial z}\right)\exp\left(ip_z z / \hbar\right)\left|s_{t,x,y,0}\right\rangle \\
&= \langle s|\left(-\frac{i}{\hbar}\left[p_z, A^\nu\right] + \frac{\partial A^\nu}{\partial z}\right)|s\rangle = -\frac{i}{\hbar}\left\langle\left[p_z, A^\nu\right]\right\rangle + \left\langle\frac{\partial A^\nu}{\partial z}\right\rangle
\end{aligned} . \tag{8.7}$$

This is an Ehrenfest-type equation for the *z* evolution. Then generalizing to the other two space dimensions and also using $p_k = -\mathbf{p}$, we rewrite this in the form of (8.5), as:

$$\langle s|\left[p_k, A^\nu\right]|s\rangle = i\hbar\langle s|\frac{\partial A^\nu}{\partial x^k}|s\rangle - i\hbar\frac{d}{dx^k}\langle s|A^\nu|s\rangle = i\hbar\langle s|\partial_k A^\nu|s\rangle - i\hbar d_k\langle s|A^\nu|s\rangle . \tag{8.8}$$

Comparing (8.5) with (8.8), we see that these are simply the time and space parts of a Lorentz-covariant relation, and so may be combined into a single relation:

$$\langle s|\left[p_\mu, A^\nu\right]|s\rangle = i\hbar\langle s|\partial_\mu A^\nu|s\rangle - i\hbar\partial_\mu\langle s|A^\nu|s\rangle = \left\langle\left[p_\mu, A^\nu\right]\right\rangle = i\hbar\left(\left\langle\partial_\mu A^\nu\right\rangle - \partial_\mu\left\langle A^\nu\right\rangle\right). \quad (8.9)$$

Above, we have also replaced what were originally the total derivatives into partial derivatives, $d \mapsto \partial$, because we now have combined the $d_\sigma = d/dx^\sigma$ taken in all four spacetime dimensions into one relation. Now, even with the same ∂_μ in both terms on the right hand side above, we see with clarity that the expected value of the commutator, $\left\langle\left[p_\mu, A^\nu\right]\right\rangle$, measures $i\hbar$ times *the difference between the expected value of the four-gradient,* $\left\langle\partial_\mu A^\nu\right\rangle$*, and the four-gradient of the expected value,* $\partial_\mu\left\langle A^\nu\right\rangle$. Summing indexes this becomes:

$$\langle s|\left[p_\sigma, A^\sigma\right]|s\rangle = i\hbar\langle s|\partial_\sigma A^\sigma|s\rangle - i\hbar\partial_\sigma\langle s|A^\sigma|s\rangle = \left\langle\left[p_\sigma, A^\sigma\right]\right\rangle = i\hbar\left(\left\langle\partial_\sigma A^\sigma\right\rangle - \partial_\sigma\left\langle A^\sigma\right\rangle\right). \quad (8.10)$$

Now we have derived the correct quantum mechanical treatment of the commutator $\left[p_\sigma, A^\sigma\right]$ in (1.5): When this commutator is sandwiched within $\langle s|\left[p_\sigma, A^\sigma\right]|s\rangle$ using $\langle s|$ and $|s\rangle$ developed above, it is evaluated according to the Ehrenfest-type equation (8.10) above, which contains the expected value of the Heisenberg-picture equation of motion in its time term, and three space-component terms containing expectation values for Heisenberg-picture equations of configuration. Combined in the summed form of (8.10), these terms Lorentz transform as a scalar. Although derived in flat spacetime, we can generalize to curved spacetime by simply writing the commutator term as $\left\langle\left[p_\sigma, A^\sigma\right]\right\rangle = \left\langle g_{\mu\nu}\left[p^\mu, A^\nu\right]\right\rangle$.

9. Arriving at a Massless Photon by Gauge-Covariant, Lorentz-Covariant Gauge Fixing of the Klein-Gordon Equation to Remove Two Degrees of Freedom from the Gauge Field

With the result (8.10), we return to (1.5) with $A_\sigma p^\sigma = A^\sigma p_\sigma$ and $p_\sigma p^\sigma = p^\sigma p_\sigma$, but now sandwich this between the bra $\langle s|$ and the ket $|s\rangle$ developed in the previous section, to write:

$$0 = \langle s|\left(\pi_\sigma\pi^\sigma - m^2c^2\right)|s\rangle = \langle s|\left(p^\sigma p_\sigma + 2\frac{q}{c}A^\sigma p_\sigma - m^2c^2 + \frac{q}{c}\left[p_\sigma, A^\sigma\right] + \frac{q^2}{c^2}A_\sigma A^\sigma\right)|s\rangle. \quad (9.1)$$

This is just the Klein-Gordon equation $0 = \left(\hbar^2\left(\partial_\sigma - iqA_\sigma/\hbar c\right)\left(\partial^\sigma - iqA^\sigma/\hbar c\right) + m^2c^2\right)\phi$ restated in momentum space as $0 = \left(\left(p_\sigma + qA_\sigma/c\right)\left(p^\sigma + qA^\sigma/c\right) - m^2c^2\right)s$ with the earlier *s* turned into a ket $|s\rangle$ and with a front-appended bra $\langle s|$. For two random variables *A* and *B*, the expectation value is linear, $\langle A+B\rangle = \langle A\rangle + \langle B\rangle$. So the commutator term in (9.1) may be separately treated as $(q/c)\langle s|\left[p_\sigma, A^\sigma\right]|s\rangle$, enabling us to directly substitute (8.10) into (9.1). The result is:

$$0=\left\langle s\right|\left(p^{\sigma}p_{\sigma}+2\frac{q}{c}A^{\sigma}p_{\sigma}-m^{2}c^{2}+\frac{q}{c}i\hbar\partial_{\sigma}A^{\sigma}+\frac{q^{2}}{c^{2}}A_{\sigma}A^{\sigma}\right)\left|s\right\rangle-\frac{q}{c}i\hbar\partial_{\sigma}\left\langle s\right|A^{\sigma}\left|s\right\rangle. \tag{9.2}$$

Again, this is still the Klein-Gordon equation, in momentum space, with a bra in front. In (9.2), $(q/c)i\hbar\partial_{\sigma}A^{\sigma}+(q/c)^{2}A_{\sigma}A^{\sigma}=(q/c)\left(i\hbar\partial_{\sigma}+(q/c)A_{\sigma}\right)A^{\sigma}$, which contains the gauge-covariant derivative in the form $i\hbar\partial_{\sigma}+qA_{\sigma}/c=i\hbar\mathfrak{D}_{\sigma}$. Thus (9.2) becomes:

$$0=\left\langle s\right|\left(p^{\sigma}p_{\sigma}+2\frac{q}{c}A^{\sigma}p_{\sigma}-m^{2}c^{2}+i\hbar\frac{q}{c}\mathfrak{D}_{\sigma}A^{\sigma}\right)\left|s\right\rangle-i\hbar\frac{q}{c}\partial_{\sigma}\left\langle s\right|A^{\sigma}\left|s\right\rangle. \tag{9.3}$$

Now, it is very common practice in U(1) gauge theory to remove one degree of freedom by imposing the Lorenz gauge $\partial_{\sigma}A^{\sigma}=0$. However, *a priori*, the gauge field A^{σ} has four independent components, while the photon which this represents in quantum theory is massless and so only has two transverse degrees of freedom. Because $\left\langle s\right|\mathfrak{D}_{\sigma}A^{\sigma}\left|s\right\rangle=\left\langle\mathfrak{D}_{\sigma}A^{\sigma}\right\rangle$ and $\partial_{\sigma}\left\langle s\right|A^{\sigma}\left|s\right\rangle=\partial_{\sigma}\left\langle A^{\sigma}\right\rangle$, (9.3) affords us the opportunity to remove two degrees of freedom. First, we may impose the Lorentz-covariant and gauge-covariant Lorenz gauge fixing condition:

$$\boxed{\left\langle s\right|\mathfrak{D}_{\sigma}A^{\sigma}\left|s\right\rangle=\left\langle\mathfrak{D}_{\sigma}A^{\sigma}\right\rangle=\left\langle\partial_{\sigma}A^{\sigma}-i\frac{q}{\hbar c}A_{\sigma}A^{\sigma}\right\rangle=0}, \tag{9.4}$$

which sets the expected value $\left\langle\mathfrak{D}_{\sigma}A^{\sigma}\right\rangle$ of the gauge-covariant derivative $\mathfrak{D}_{\sigma}A^{\sigma}$ of the gauge field A^{σ} to zero. Second, we may impose the Lorentz-covariant gauge fixing condition:

$$\boxed{\partial_{\sigma}\left\langle s\right|A^{\sigma}\left|s\right\rangle=\partial_{\sigma}\left\langle A^{\sigma}\right\rangle=0} \tag{9.5}$$

which is the usual Lorenz gauge used to set the *expected value* $\left\langle A^{\sigma}\right\rangle$ of the gauge field A^{σ} to zero. If we impose both (9.4) and (9.5) on (9.3), then we can remove *two of the four degrees of freedom* from the gauge field, in a covariant manner, ensuring that A^{σ} will only retain two degrees of freedom which is precisely what is needed for this to represent massless photon quanta.

Therefore, we now proceed to impose both (9.4) and (9.5) on (9.3), to simplify this to:

$$0=\left\langle s\right|\left(p^{\sigma}p_{\sigma}+2\frac{q}{c}A^{\sigma}p_{\sigma}-m^{2}c^{2}\right)\left|s\right\rangle=\left\langle p^{\sigma}p_{\sigma}+2\frac{q}{c}A^{\sigma}p_{\sigma}-m^{2}c^{2}\right\rangle, \tag{9.6}$$

while the gauge field loses two of its four degrees of freedom. We may also again apply the heuristic rule $p_{\sigma}\mapsto i\hbar\partial_{\sigma}$ in the above to write this, with sign flip and the bra removed, as:

$$0=\left(\hbar^2\partial^\sigma\partial_\sigma-2i\frac{\hbar}{c}qA^\sigma\partial_\sigma+m^2c^2\right)\left|s\right\rangle. \tag{9.7}$$

We have removed the bra in the above and so *not* written this as an expected value equation, because when ∂_σ appears in the equation, it needs to operate on a ket to its right, as $\partial_\sigma\left|s\right\rangle$. This is now a gauge-fixed Klein-Gordon equation in configuration space, in which the gauge field A^σ contains two not four degrees of freedom, precisely as is required for a massless photon. By the Correspondence Principle, the classical equation obtained from (9.6) is:

$$m^2c^2=p^\sigma p_\sigma+2\frac{q}{c}A^\sigma p_\sigma. \tag{9.8}$$

This should be contrasted with (1.5) from which the final two $\left[p_\sigma,A^\sigma\right]$ and $A_\sigma A^\sigma$ terms have been removed using $\left\langle s\right|$ and $\left|s\right\rangle$ to turn (1.5) from a classical into a quantum mechanical equation, and then imposing the gauge conditions (9.5) and (9.6). What we learn from all this is that quantum mechanics, combined with two covariant gauge fixing conditions removing two degrees of freedom from the gauge fields, has brought about a wholesale change to the classical equation (1.5) by removing two of its terms.

10. Classical and Quantum Mechanical Geodesic Equations of Gravitational and Electromagnetic Motion

Now, let s work from the expectation value equation in (9.6), apply $p^\sigma=mdx^\sigma/d\tau$ throughout, and raise an index in the first term, and move the term with m^2c^2 to the left, thus:

$$\left\langle m^2c^2\right\rangle=\left\langle m^2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}+2\frac{qm}{c}A_\sigma\frac{dx^\sigma}{d\tau}\right\rangle. \tag{10.1}$$

It will be seen that this is the parallel equation to (2.1), but that two things have now changed: First, the term with $A_\sigma A^\sigma$ is gone as a consequence of the gauge conditions (9.4) and (9.5). Second the entire equation is an expectation value equation. By the Correspondence Principle and Ehrenfest's theorem, we know that the classical equation implied by (10.1) is simply (10.1) with the expectation brackets removed, which is (2.1) without the $A_\sigma A^\sigma$ term. Therefore, it is easy to see that if start with the classical equation implied by (10.1) via Correspondence, and repeat all the same steps earlier taken from (2.1) through (2.12) starting with the variational equation $0=\delta\int_A^B d\tau$ of (2.3) for *geodesic motion*, we will end up with the classical equation of motion:

$$\frac{d^2x^\beta}{c^2d\tau^2}=-\Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+\frac{q}{mc^2}F^\beta{}_\sigma\frac{dx^\sigma}{cd\tau}. \tag{10.2}$$

This has the gauge-dependent $\partial^\beta\left(A_\sigma A^\sigma\right)$ term removed as a consequence of the gauge fixing in (9.4) and (9.5), it accords precisely with the known classical physical motions for gravitation and electrodynamics, and it is entirely geodesic motion because of its derivation from a variation.

Now, however, we can also obtain the *quantum mechanical equation of motion* based on (10.1). First, we note that the mass term may be written as $\left\langle m^2c^2\right\rangle = m^2c^2$ because *m* and *c* are numbers with definite values and zero variance, not statistical values. So too for *q*. Second, dx^μ are coordinate elements and $d\tau$ is a proper time element which also represent definite, not statistical measurement numbers *against which* we measure statistical spreads. That is, even when we graph a probability distribution, we still do so against definite measurement axes. The statistical objects in (10.1) are the gravitational fields in $\left\langle g_{\mu\nu}\right\rangle$, and the gravitational fields and electromagnetic potential in $\left\langle A_\sigma\right\rangle = \left\langle g_{\sigma\tau}A^\tau\right\rangle$, though for now it will be convenient to retain the lower-indexed form A_σ to absorb the gravitational field. As a result, we may refine (10.1) into:

$$m^2c^2 = m^2\left\langle g_{\mu\nu}\right\rangle \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2\frac{qm}{c}\left\langle A_\sigma\right\rangle \frac{dx^\sigma}{d\tau}. \tag{10.3}$$

We then divide both sides through by m^2c^2 to write this as:

$$1 = \left\langle g_{\mu\nu}\right\rangle \frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + 2\frac{q}{mc^2}\left\langle A_\sigma\right\rangle \frac{dx^\sigma}{cd\tau} \tag{10.4}$$

Contrasting to (2.2), the difference is that the $A_\sigma A^\sigma$ term is now gone, and the two fields $g_{\mu\nu}$ and A_σ are now expectation values $\left\langle g_{\mu\nu}\right\rangle$ and $\left\langle A_\sigma\right\rangle$.

So if we now employ this "1" in a minimized variation as in (2.3), it turns out that all the steps taken from (2.3) through (2.11) will be exactly the same, except that $\left\langle g_{\mu\nu}\right\rangle$ will end up wherever there was a $g_{\mu\nu}$, and $\left\langle A_\sigma\right\rangle$ wherever there was a A_σ, in (2.11). Therefore, the counterpart to (2.11) based on (10.4) now turns out to be:

$$0 = \delta\int_A^B d\tau = \int_A^B \delta x^\alpha d\tau \begin{pmatrix} \frac{1}{2}\left(\partial_\alpha\left\langle g_{\mu\nu}\right\rangle - \partial_\mu\left\langle g_{\nu\alpha}\right\rangle - \partial_\nu\left\langle g_{\alpha\mu}\right\rangle\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} - \left\langle g_{\alpha\nu}\right\rangle\frac{d^2x^\nu}{c^2d\tau^2} \\ +\frac{q}{mc^2}\left(\partial_\alpha\left\langle A_\sigma\right\rangle - \partial_\sigma\left\langle A_\alpha\right\rangle\right)\frac{dx^\sigma}{cd\tau} \end{pmatrix}. \tag{10.5}$$

As before, and for the same reasons, the term inside the large parenthesis must be zero, so that:

$$\left\langle g_{\alpha\nu}\right\rangle\frac{d^2x^\nu}{c^2d\tau^2} = \frac{1}{2}\left(\partial_\alpha\left\langle g_{\mu\nu}\right\rangle - \partial_\mu\left\langle g_{\nu\alpha}\right\rangle - \partial_\nu\left\langle g_{\alpha\mu}\right\rangle\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + \frac{q}{mc^2}\left(\partial_\alpha\left\langle A_\sigma\right\rangle - \partial_\sigma\left\langle A_\alpha\right\rangle\right)\frac{dx^\sigma}{cd\tau}. \tag{10.6}$$

In contrast to its counterpart (2.12), the above needs to be treated with some care, because $-\langle \Gamma^{\beta}{}_{\mu\nu} \rangle = \frac{1}{2}\langle g^{\beta\alpha}\left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\right)\rangle$ and $\langle F_{\alpha\sigma}\rangle = \langle \partial_\alpha A_\sigma - \partial_\sigma A_\alpha \rangle$ contain *expectation values of derivatives*, while (10.6) is distinguished by having the terms $\frac{1}{2}\left(\partial_\alpha \langle g_{\mu\nu}\rangle - \partial_\mu \langle g_{\nu\alpha}\rangle - \partial_\nu \langle g_{\alpha\mu}\rangle\right)$ and $\partial_\alpha \langle A_\sigma \rangle - \partial_\sigma \langle A_\alpha \rangle$ containing *derivatives of expectation values*. This is where the Heisenberg equation of time evolution and space configuration comes back into play, because this very same distinction is measured by the commutators of the fields with energy momentum. So, from (8.9):

$$\partial_\alpha \langle A_\sigma \rangle = \langle \partial_\alpha A_\sigma \rangle + i\langle [p_\alpha, A_\sigma]\rangle / \hbar . \tag{10.7}$$

And because this applies generally to field operators, not only to A_σ, for $g_{\mu\nu}$ we may also write:

$$\partial_\alpha \langle g_{\mu\nu} \rangle = \langle \partial_\alpha g_{\mu\nu} \rangle + i\langle [p_\alpha, g_{\mu\nu}]\rangle / \hbar . \tag{10.8}$$

Then, using (10.7) and (10.8) in (10.6) and rearranging somewhat yields:

$$\begin{aligned}\langle g_{\alpha\nu}\rangle \frac{d^2 x^\nu}{c^2 d\tau^2} &= \frac{1}{2}\langle \partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\rangle \frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + \frac{q}{mc^2}\langle \partial_\alpha A_\sigma - \partial_\sigma A_\alpha \rangle \frac{dx^\sigma}{cd\tau} \\ &+ \frac{i}{2\hbar}\langle [p_\alpha, g_{\mu\nu}] - [p_\mu, g_{\nu\alpha}] - [p_\nu, g_{\alpha\mu}]\rangle \frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + \frac{iq}{\hbar mc^2}\langle [p_\alpha, A_\sigma] - [p_\sigma, A_\alpha]\rangle \frac{dx^\sigma}{cd\tau}\end{aligned} . \tag{10.9}$$

Now, we have a term $\langle \partial_\alpha A_\sigma - \partial_\sigma A_\alpha \rangle = \langle F_{\alpha\sigma}\rangle$ placed inside expectation values. Moreover, with simple re-indexing, we also find $\frac{1}{2}\langle \partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\rangle = -\langle g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu}\rangle$. So with these replacements (10.9) becomes:

$$\begin{aligned}\langle g_{\alpha\nu}\rangle \frac{d^2 x^\nu}{c^2 d\tau^2} &= -\langle g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu}\rangle \frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + \frac{q}{mc^2}\langle F_{\alpha\sigma}\rangle \frac{dx^\sigma}{cd\tau} \\ &+ \frac{i}{2\hbar}\langle [p_\alpha, g_{\mu\nu}] - [p_\mu, g_{\nu\alpha}] - [p_\nu, g_{\alpha\mu}]\rangle \frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + \frac{iq}{\hbar mc^2}\langle [p_\alpha, A_\sigma] - [p_\sigma, A_\alpha]\rangle \frac{dx^\sigma}{cd\tau}\end{aligned} . \tag{10.10}$$

To further simplify, we raise the free index α inside the expectation brackets. Although raising an index, for example, via $X^\mu = g^{\mu\sigma}X_\sigma$ for some X_σ involves multiplying by $g^{\mu\sigma}$, we may still perform this entirely within the brackets because if $\langle X \rangle = \langle Y \rangle$ then $\langle g^{\mu\sigma}X\rangle = \langle g^{\mu\sigma}Y\rangle$ for any objects *X*, *Y*. When we raise an index for $\langle g_{\alpha\nu}\rangle$ we have $\langle \delta^{\alpha}{}_{\nu}\rangle = \delta^{\alpha}{}_{\nu}$ which removes the expectation value because the Kronecker delta $\delta^{\alpha}{}_{\nu}$ is just a 4x4 identity matrix; likewise for $\langle g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu}\rangle$ we have $\langle \delta^{\alpha}{}_{\beta}\Gamma^{\beta}{}_{\mu\nu}\rangle = \delta^{\alpha}{}_{\beta}\langle \Gamma^{\beta}{}_{\mu\nu}\rangle$. And when we do this for e.g. $[p_\nu, g_{\alpha\mu}]$ we obtain $[p_\nu, \delta^{\alpha}{}_{\mu}] = 0$. So this removes the two commutators $[p_\mu, g_{\nu\alpha}]$ and $[p_\nu, g_{\alpha\mu}]$ which have an index α in the metric tensor. With all of this, also raising the remaining indexes to explicitly show all appearances of the gravitational field, we arrive at our final result:

$$\boxed{\begin{aligned}\frac{d^2x^\alpha}{c^2d\tau^2}&=-\left\langle\Gamma^\alpha{}_{\mu\nu}\right\rangle\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+\frac{q}{mc^2}\left\langle g_{\sigma\beta}F^{\alpha\beta}\right\rangle\frac{dx^\sigma}{cd\tau}\\&+\frac{i}{2\hbar}\left\langle\left[p^\alpha,g_{\mu\nu}\right]\right\rangle\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+\frac{iq}{\hbar mc^2}\left\langle g_{\sigma\beta}\left[p^\alpha,A^\beta\right]-g_{\sigma\beta}\left[p^\beta,A^\alpha\right]\right\rangle\frac{dx^\sigma}{cd\tau}\end{aligned}}. \tag{10.11}$$

For classical theory, where all the commutators become zero and the expectation values are removed via the Correspondence Principle, (10.11) becomes the well-settled classical equation (10.2). So – very importantly – using the gauge conditions (9.4) and (9.5) to remove two terms from (9.3) which descended from (1.5), has caused the gauge-dependent term $\partial^\beta\left(A_\sigma A^\sigma\right)$ to vanish from (2.12) in favor of (10.2), which accords entirely with the robustly confirmed motions of particles in gravitational and electromagnetic fields, and which motions are now seen to *both* be geodesic motions. When a classical system approaches a scale where quantum commutation cannot be neglected, (10.11) applies. And in a fully-quantum setting, where the commutators are large enough so the classical terms with $\left\langle\Gamma^\alpha{}_{\mu\nu}\right\rangle$ and $\left\langle g_{\sigma\beta}F^{\alpha\beta}\right\rangle$ become negligible due to the very tiny $\hbar$ in the *denominator* of the commutator terms, (10.11) becomes a quantum motion equation:

$$\frac{d^2x^\alpha}{c^2d\tau^2}=\frac{i}{2\hbar}\left\langle\left[p^\alpha,g_{\mu\nu}\right]\right\rangle\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+\frac{iq}{\hbar mc^2}\left\langle g_{\sigma\beta}\left[p^\alpha,A^\beta\right]-g_{\sigma\beta}\left[p^\beta,A^\alpha\right]\right\rangle\frac{dx^\sigma}{cd\tau}. \tag{10.12}$$

Finally, in (10.12) we can make good use of the generalized uncertainty relation $\sigma(A)\sigma(B)\ge\frac{1}{2}\left|i\left\langle[A,B]\right\rangle\right|$ for any two objects which are non-commuting, where σ represents statistical standard deviation. By this relation, $\sigma\left(p^\alpha\right)\sigma\left(g_{\mu\nu}\right)/\hbar\ge(i/2\hbar)\left|\left\langle\left[p^\alpha,g_{\mu\nu}\right]\right\rangle\right|$. Therefore, when we consider gravitation alone by setting $q=0$ or $A^\alpha=0$, (10.12) becomes:

$$\sigma\left(p^\alpha\right)\sigma\left(g_{\mu\nu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}\ge\hbar\frac{d^2x^\alpha}{c^2d\tau^2}=\frac{i}{2}\left\langle\left[p^\alpha,g_{\mu\nu}\right]\right\rangle\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}, \tag{10.13}$$

which is an uncertainty relation for quantum gravitational interactions. Conversely, when we consider electromagnetic interactions alone in flat spacetime, (10.12) becomes:

$$\frac{q}{mc^2}\eta_{\sigma\beta}\left(\sigma\left(p^\alpha\right)\sigma\left(A^\beta\right)-\sigma\left(p^\beta\right)\sigma\left(A^\alpha\right)\right)\frac{dx^\sigma}{cd\tau}\ge\frac{\hbar}{2}\frac{d^2x^\alpha}{c^2d\tau^2}=\frac{iq}{2mc^2}\eta_{\sigma\beta}\left\langle\left[p^\alpha,A^\beta\right]-\left[p^\beta,A^\alpha\right]\right\rangle\frac{dx^\sigma}{cd\tau}. \tag{10.14}$$

This is an uncertainty relation for quantum electromagnetic interactions. Both (10.13) and (10.14) are actually four independent equations, with the free index α. In both of these relations, the lower bound on the uncertainty spread is established by the four-acceleration $d^2x^\alpha/c^2d\tau^2$. For gravitation, the coefficient of the acceleration is $\hbar$. And for electromagnetism, it is noteworthy that the coefficient is $\hbar/2$ which is also the magnitude of fermion spins. And it is again worth noting that because of the gauge conditions (9.4) and (9.5) all unphysical gauge freedom has been removed from A^μ, so that there is no gauge ambiguity in $\sigma\left(p^\alpha\right)\sigma\left(A^\mu\right)-\sigma\left(p^\mu\right)\sigma\left(A^\alpha\right)$.

Finally, it is helpful to directly contrast the classical equations of motion with the quantum equations of *motion uncertainty*. For gravitation absent electromagnetism this contrast is:

$$-\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}=\frac{d^2x^\alpha}{c^2d\tau^2} \quad \text{versus} \quad \sigma\left(p^\alpha\right)\sigma\left(g_{\mu\nu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}\geq\hbar\frac{d^2x^\alpha}{c^2d\tau^2}. \tag{10.15}$$

For electromagnetism absent gravitation, mindful that $F^{\alpha\mu}=\partial^\alpha A^\mu-\partial^\mu A^\alpha$, this is:

$$\frac{q}{mc^2}\eta_{\mu\nu}F^{\alpha\mu}\frac{dx^\nu}{cd\tau}=\frac{d^2x^\alpha}{c^2d\tau^2} \quad \text{versus} \quad \frac{q}{mc^2}\eta_{\mu\nu}\left(\sigma\left(p^\alpha\right)\sigma\left(A^\mu\right)-\sigma\left(p^\mu\right)\sigma\left(A^\alpha\right)\right)\frac{dx^\nu}{cd\tau}\geq\frac{\hbar}{2}\frac{d^2x^\alpha}{c^2d\tau^2}. \tag{10.16}$$

In (10.15) we see that for a given acceleration, as the momentum uncertainty $\sigma\left(p^\alpha\right)$ for the mass in the gravitational field becomes smaller the field uncertainty $\sigma\left(g_{\mu\nu}\right)$ grows larger, and vice versa. In (10.16) we see a similar incompatibility between momentum uncertainty and electromagnetic potential uncertainty.

11. The Simplified Quadratic Line Element following Gauge Fixing

If we again start with (10.4) and multiply each side through by $c^2d\tau^2$ we obtain the metric:

$$c^2d\tau^2=\left\langle g_{\mu\nu}\right\rangle dx^\mu dx^\nu+2\frac{q}{mc^2}\left\langle A_\sigma\right\rangle dx^\sigma cd\tau \tag{11.1}$$

It will be seen that this is the "unusual" quadratic metric (3.3) from earlier, but with the same two changes reviewed after (10.4): the $A_\sigma A^\sigma$ is gone, and we now have expectation values $\left\langle g_{\mu\nu}\right\rangle$ and $\left\langle A_\sigma\right\rangle$. This remains quadratic in $c^2d\tau^2$, as is seen if we write this as (contrast (3.4)):

$$0=c^2d\tau^2-2\frac{q}{mc^2}\left\langle A_\sigma\right\rangle dx^\sigma cd\tau-\left\langle g_{\mu\nu}\right\rangle dx^\mu dx^\nu \tag{11.2}$$

But now the quadratic solution takes on a much simpler form that its counterpart (3.5), namely:

$$cd\tau=\frac{q}{mc^2}\left\langle A_\sigma\right\rangle dx^\sigma\pm\sqrt{\left(\left\langle g_{\mu\nu}\right\rangle+\frac{q^2}{m^2c^4}\left\langle A_\mu\right\rangle\left\langle A_\nu\right\rangle\right)dx^\mu dx^\nu}. \tag{11.3}$$

In particular, this no longer contains the ratio form of (3.5), and the term inside the square root is significantly simplified. In fact, if we make the two definitions:

$$G_{\mu\nu} \equiv \langle g_{\mu\nu} \rangle + \frac{q^2}{m^2c^4} \langle A_\mu \rangle \langle A_\nu \rangle; \qquad c^2\mathrm{T}^2 \equiv G_{\mu\nu} dx^\mu dx^\nu , \tag{11.4}$$

then also employing $\langle A \rangle = \langle A_\sigma \rangle dx^\sigma$ which is the expected value of the differential one-form $A = A_\sigma dx^\sigma$ for the gauge field, we see that (11.3) can be written in the very simple form:

$$cd\tau = \frac{q}{mc^2} \langle A_\sigma \rangle dx^\sigma \pm \sqrt{G_{\mu\nu} dx^\mu dx^\nu} = \frac{q}{mc^2} \langle A \rangle \pm cd\mathrm{T} . \tag{11.5}$$

The above have several very interesting properties. First, the object $c^2 d\mathrm{T}^2 \equiv G_{\mu\nu} dx^\mu dx^\nu$ has a form very similar to the metric scalar $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$. Of course, $G_{\mu\nu}$ defined above cannot be formally regarded as a metric tensor because it does not have the metricity properties of $g_{\mu\nu}$ whereby $g_{\mu\nu;\sigma} = 0$ and $g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu{}_\nu$. Nor is $d\mathrm{T}$ (necessarily) invariant; rather, the invariant is $cd\tau = q\langle A \rangle / mc^2 \pm cd\mathrm{T}$ with the possibility of some sub-relation between $q\langle A \rangle / mc^2$ and $cd\mathrm{T}$ which leaves $cd\tau$ unchanged. But what makes this of keen interest is that we may still think of of $G_{\mu\nu}$ as being a "quasi-geometric" object in the manner of $g_{\mu\nu}$ merely because $\pm\sqrt{G_{\mu\nu} dx^\mu dx^\nu}$ standing alone still does define a line element $cd\mathrm{T}$ (which differs from $cd\tau$ precisely by $q\langle A \rangle / mc^2$). Further, the $\pm\sqrt{G_{\mu\nu} dx^\mu dx^\nu}$ square root is very reminiscent of how Dirac's equation $\left(i\hbar\Gamma^\mu \partial_\mu - mc\right)\psi = 0$ is developed in flat spacetime from $\pm\sqrt{\eta_{\mu\nu} p^\mu p^\nu}$ and then generalized into curved spacetime using a tetrad $e^\mu_a \gamma^a \equiv \Gamma^\mu$, as earlier reviewed in section 1.

This point will be of keen interest here, because while Dirac's equation teaches about how individual electrons behave in an *electromagnetic field*, (11.5) will lead us to a variant of Dirac's equation which can be used to understand *how individual photons interact with individual electrons*. And in fact, (11.5) only has the form that it does (versus the earlier (3.5)), because at (9.4) and (9.5) we removed two of the four degrees of freedom from A^σ giving it precisely the properties expected of a massless photon. Indeed, the foregoing is why, following Dirac, part of the title of this paper is "Quantum Theory of the Electron *and the Photon*."

12. The Electromagnetic Time Dilation and Energy Content Relations, following Gauge Fixing

Before we proceed to this new variant of Dirac's equation, we first wish to determine the impact of the foregoing quantum development and gauge fixing on the electromagnetic time dilations (5.8) and (5.9). To do so, we develop the quadratic solution for the metric (11.1) when taken at rest in flat spacetime, just as we earlier did for the metric (3.3). To place (11.1) into flat spacetime, we need to set $\langle g_{\mu\nu} \rangle = \langle \eta_{\mu\nu} \rangle = \eta_{\mu\nu}$. So following the same steps that led to (4.1), it is easy to see that (11.1) will become:

$$d\tau^2 = dt^2 + 2\frac{q\langle\phi_0\rangle}{mc^2}dt d\tau . \tag{12.1}$$

Following the development from (4.1) to (4.4), again choosing to solve for dt, we see that in place of (4.4) we now have:

$$\frac{dt}{d\tau} = 1 - \frac{q\langle\phi_0\rangle}{mc^2} . \tag{12.2}$$

So the only difference is that now the scalar potential appears as an expectation value. Otherwise there is no change to the overall form of the equation. This is because when we had the earlier terms with ${\phi_0}^2$ that have now been eliminated because of the gauge fixing at (9.4) and (9.5), these terms nonetheless ended up cancelling inside the square root term in (4.3).

So if we repeat the development from (4.4) to (5.8), nothing else changes, and the earlier (5.8) and (6.1) for the time electromagnetic time dilation at rest in flat spacetime and its energy content via the relation $E = \gamma_{em} mc^2$, now becomes:

$$\gamma_{em} \equiv \frac{dt}{d\tau} = \frac{E}{mc^2} = \frac{1}{1-\dfrac{q\langle\phi_0\rangle}{mc^2}} = 1 + \frac{q\langle\phi_0\rangle}{mc^2} + \left(\frac{q\langle\phi_0\rangle}{mc^2}\right)^2 + \left(\frac{q\langle\phi_0\rangle}{mc^2}\right)^3 + \left(\frac{q\langle\phi_0\rangle}{mc^2}\right)^4 + \ldots = \sum\nolimits_{n=0}^{\infty}\left(\frac{q\langle\phi_0\rangle}{mc^2}\right)^n . \tag{12.3}$$

Now, the time dilation is based on the expected value of the scalar potential. When we employ a Coulomb potential, this will enter as $\langle\phi_0\rangle = k_e Q\langle 1/r\rangle$ where $\langle 1/r\rangle$ is the expectation value of the inverse separation between the two charges. Note, we have not used $1/\langle r\rangle$ because statistically, $\langle 1/r\rangle \neq 1/\langle r\rangle$. Rather, as is well known, $\langle 1/r\rangle \geq 1/\langle r\rangle$ for positive random variable *r*. The only distribution with $\langle 1/r\rangle = 1/\langle r\rangle$ is a Dirac delta $\delta(r)$. So the (5.9), (6.2) counterpart is:

$$\gamma_{em} \equiv \frac{dt}{d\tau} = \frac{E}{mc^2} = \frac{1}{1-\dfrac{k_e Qq}{mc^2}\left\langle\dfrac{1}{r}\right\rangle} = 1 + \frac{k_e Qq}{mc^2}\left\langle\frac{1}{r}\right\rangle + \left(\frac{k_e Qq}{mc^2}\left\langle\frac{1}{r}\right\rangle\right)^2 + \ldots = \sum\nolimits_{n=0}^{\infty}\left(\frac{k_e Qq}{mc^2}\left\langle\frac{1}{r}\right\rangle\right)^n . \tag{12.4}$$

So we naturally find ourselves in a situation where must use an *expected separation* between *Q* and *q*, which is precisely where we do end up once we talk about interactions between electrons, protons, etc. which do not have positions with classical certainty. Thus, (12.4) naturally embeds the existence of Heisenberg position uncertainty via the appearance of $\langle 1/r\rangle$. In general, cf. (6.3), the energy content relation $E = \Gamma mc^2 = \gamma_v \gamma_g \gamma_{em} mc^2$ holds for both classical and quantum systems. The expectation values of quantum systems are embedded in the individual γ_v, γ_g, γ_{em}. The energy in excess of mc^2, is then $W = E - mc^2 = mc^2(\Gamma - 1)$. This means as well that the relation $p^\mu = m\Gamma v^\mu = m\gamma_v \gamma_g \gamma_{em} v^\mu$ obtained at (6.4) also continues to hold for a quantum system.

PART III: THE HYPER-CANONICAL DIRAC EQUATION FOR INDIVIDUAL ELECTRON AND PHOTON INTERACTIONS

13. Dirac's Equation with Electromagnetic Tetrads

Now we turn to Dirac's equation. As reviewed in section 1, to obtain Dirac's equation, we start with the entirely-classical relation $m^2c^2 = \eta^{\mu\nu} p_\mu p_\nu$ in flat spacetime, define a set of 4x4 γ^μ operator matrices $\frac{1}{2}\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\} \equiv \eta^{\mu\nu}$, then use $(\gamma^\mu p_\mu)^2 = \frac{1}{2}\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\} p_\mu p_\nu = \eta^{\mu\nu} p_\mu p_\nu$ to take the square root equation $mc = \pm\sqrt{\eta^{\mu\nu} p_\mu p_\nu} = \gamma^\mu p_\mu$ with the $\pm$ sign absorbed in the γ^μ definitions. Finally, because this result only makes sense if it operates on a spinor $u(p^\nu)$ which following the development in section 8 we represent as the ket $|u_0\rangle$ with $\partial_\mu |u_0\rangle = 0$, we are able to form $(\gamma^\mu p_\mu - mc)|u_0\rangle = 0$. If we then use the ket $|\psi\rangle \equiv \exp(-ip_\sigma x^\sigma)|u_0\rangle$ this readily becomes $(i\hbar\gamma^\mu\partial_\mu - mc)|\psi\rangle = 0$. We then introduce electromagnetic interactions by requiring local U(1) electromagnetic interactions which provides us with the gauge-covariant derivative $\partial_\mu \mapsto \mathfrak{D}_\mu \equiv \partial_\mu - iqA_\mu / \hbar c$. Finally, in curved spacetime, where the underlying equation is $mc = \pm\sqrt{g^{\mu\nu} p_\mu p_\nu}$, we also employ tetrads defined such that $g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu$. In this way, we turn $mc = \pm\sqrt{\eta^{\mu\nu} p_\mu p_\nu}$ or $mc = \pm\sqrt{g^{\mu\nu} p_\mu p_\nu}$ which is a classical equation, into the quintessentially quantum mechanical operator equation of Dirac.

As also reviewed in section 1, a similar process occurs with the Klein Gordon equation. Here we start with the same classical $m^2c^2 = \eta_{\mu\nu} p^\mu p^\nu$, have this operate on what we now write as the ket $|s_0\rangle$ with $\partial_\mu |s_0\rangle = 0$ in the form $(p_\sigma p^\sigma - m^2c^2)|s_0\rangle = 0$, then use $|s\rangle \equiv \exp(-ip_\sigma x^\sigma)|s_0\rangle$ to advance this to $0 = (\hbar^2\partial_\sigma\partial^\sigma + m^2c^2)|s\rangle$, then use $\partial_\mu \mapsto \mathfrak{D}_\mu$ to add interactions. Here too, we turn a purely classical equation $m^2c^2 = \eta_{\mu\nu} p^\mu p^\nu$ into a quantum mechanical equation. The key point of both these examples for the discussion to follow is this: the tried and true recipe of both Klein-Gordon and Dirac teaches us that we can start with a classical equation such as $mc = \pm\sqrt{\eta^{\mu\nu} p_\mu p_\nu}$ or $m^2c^2 = \eta_{\mu\nu} p^\mu p^\nu$, use it to operate on a ket such as $|\psi\rangle$ or $|s\rangle$, and thereby produce a valid quantum mechanical equation.

With this in mind, we return to (11.3) which is the quadratic solution for the metric (11.1), which in turn descends from (9.6) which in turn is the Klein-Gordon equation in the form (1.5) sandwiched between a bra and a ket after applying the gauge conditions (9.4) and (9.5). By the

Ehrenfest/Correspondence Principle, the classical equation we may extract from (11.3) by turning all expectation values into ordinary classical objects is:

$$cd\tau = \frac{q}{mc^2} A_\sigma dx^\sigma \pm \sqrt{\left(g_{\mu\nu} + \frac{q^2}{m^2c^4} A_\mu A_\nu \right) dx^\mu dx^\nu} = \frac{q}{mc^2} A_\sigma dx^\sigma \pm \sqrt{G_{\mu\nu} dx^\mu dx^\nu} \,, \tag{13.1}$$

Above, we also insert the classical value $G_{\mu\nu} = g_{\mu\nu} + \left(q^2 / m^2c^4\right) A_\mu A_\nu$ from (11.4), so this is (11.5) in its classical limit. This is also the "peculiar" quadratic solution (3.5), once its Klein-Gordon counterpart is converted to a quantum operator equation and its gauge fixed using (9.4) and (9.5).

Because our present interest is in Dirac's equation, we multiply this classical result (13.1) through by $m / d\tau$ and swap upper and lower indexes, to obtain:

$$mc = \frac{q}{mc^2} A^\sigma p_\sigma \pm \sqrt{\left(g^{\mu\nu} + \frac{q^2}{m^2c^4} A^\mu A^\nu \right) p_\mu p_\mu} = \frac{q}{mc^2} A^\sigma p_\sigma \pm \sqrt{G^{\mu\nu} p_\mu p_\mu} \,, \tag{13.2}$$

so we have the square root in the exact same form as the classical curved spacetime equation $mc = \pm\sqrt{g^{\mu\nu} p_\mu p_\nu}$. Just as we do for Dirac's equation in curved spacetime, we now turn (13.2) above into an alternative form of Dirac's equation which applies specifically to the quantum interactions between individual electrons and individual photons, because the covariant removal of two degrees of freedom to produce a massless photon is *structurally embedded* in (13.2). Specifically, in the same way we generalize Dirac's equation into flat spacetime by defining a set of Γ^μ in terms of $g^{\mu\nu}$ by $\frac{1}{2}\left\{\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu\right\} \equiv g^{\mu\nu}$ and in terms of the tetrads $e^\mu_{\ a}$ by $e^\mu_{\ a}\gamma^a \equiv \Gamma^\mu$ so that $g^{\mu\nu} = \frac{1}{2}\left\{\gamma^a\gamma^b + \gamma^b\gamma^a\right\} e^\mu_{\ a} e^\nu_{\ b} = \eta^{ab} e^\mu_{\ a} e^\nu_{\ b}$, let us now use exactly the same approach to (13.2). From (11.4), we may extract classical equation:

$$G_{\mu\nu} = g_{\mu\nu} + \left(q^2 / m^2c^4\right) A_\mu A_\nu \tag{13.3}$$

from the expectation value. To start we will work in flat spacetime so that $g_{\mu\nu} = \eta_{\mu\nu}$ and $G_{\mu\nu} = \eta_{\mu\nu} + \left(q^2 / m^2c^4\right) A_\mu A_\nu$. Later, we will generalize back to curved spacetime.

Just as the gravitational tetrads $e^\mu_{\ a}$ contain both an upper Greek spacetime index and a lower early-in-the-alphabet Latin Lorentz index, we begin by defining a similar *electromagnetic tetrad* $\varepsilon^\mu_{\ y}$ (ε denoting electromagnetism) with an upper Greek spacetime index and a lower late-in-the alphabet Latin electromagnetic index. We also use these in flat spacetime to define a set of electromagnetic gamma matrices by the relation $\Gamma^\mu_{(\varepsilon)} \equiv \varepsilon^\mu_{\ y}\gamma^y$. Finally, we further define these $\Gamma^\mu_{(\varepsilon)}$ in terms of $G^{\mu\nu}$ by $\frac{1}{2}\left\{\Gamma^\mu_{(\varepsilon)}\Gamma^\nu_{(\varepsilon)} + \Gamma^\nu_{(\varepsilon)}\Gamma^\mu_{(\varepsilon)}\right\} \equiv G^{\mu\nu}$, then combine all these definitions by writing:

$$G^{\mu\nu} = \eta^{\mu\nu} + \frac{q^2}{m^2c^4} A^\mu A^\nu \equiv \tfrac{1}{2}\left\{\Gamma^\mu_{(\varepsilon)}\Gamma^\nu_{(\varepsilon)} + \Gamma^\nu_{(\varepsilon)}\Gamma^\mu_{(\varepsilon)}\right\} = \tfrac{1}{2}\left\{\gamma^y\gamma^z + \gamma^y\gamma^z\right\}\varepsilon^\mu_y\varepsilon^\nu_z = \eta^{yz}\varepsilon^\mu_y\varepsilon^\nu_z , \tag{13.4}$$

Just as $\left(\gamma^\mu p_\mu\right)^2 = \eta^{\mu\nu} p_\mu p_\nu$ in flat spacetime and $\left(\Gamma^\mu p_\mu\right)^2 = g^{\mu\nu} p_\mu p_\nu$ in curved spacetime, it is simple to deduce from the above definitions that $\left(\Gamma^\mu_{(\varepsilon)} p_\mu\right)^2 = G^{\mu\nu} p_\mu p_\nu$. Then, the square root in (13.2) may be written as $\pm\sqrt{G^{\mu\nu} p_\mu p_\mu} = \Gamma^\mu_{(\varepsilon)} p_\mu$ which, as with $\Gamma^\mu p_\mu$ in Dirac's equation, is a 4x4 matrix. So this will now have to operate on a 4-component column vector.

For Dirac's momentum space flat spacetime equation $\left(\gamma^\mu\left(p_\mu + qA_\mu / c\right) - mc\right)\left|u_0\right\rangle = 0$ we employ a Dirac spinor $u\left(p^\mu\right)$ that is independent of space and time which, in accord with the conventions developed in section 8, we denote as $\left|u_0\right\rangle$. Here, we use a similar four-component fixed-state ket $\left|U_0\right\rangle$ defined to be independent of spacetime, $\partial_\mu\left|U_0\right\rangle \equiv 0$. Then, appending $\left|U_0\right\rangle$ to the right of (13.2), using $\pm\sqrt{G^{\mu\nu} p_\mu p_\mu} = \Gamma^\sigma_{(\varepsilon)} p_\sigma$ and setting everything to a zero (13.2) becomes:

$$\left(\left(\Gamma^\sigma_{(\varepsilon)} + \frac{q}{mc^2} A^\sigma\right) c p_\sigma - mc^2\right)\left|U_0\right\rangle = 0 . \tag{13.5}$$

This is to be contrasted (13.6) with Dirac's $\left(\gamma^\sigma \pi_\sigma - mc\right)\left|u_0\right\rangle = \left(\gamma^\sigma\left(p_\sigma + qA_\sigma / c\right) - mc\right)\left|u_0\right\rangle = 0$. In the absence of electromagnetic fields, where either $q = 0$ or $A^\sigma = 0$, the tetrad ε^μ_y becomes a 4x4 unit matrix, and $\Gamma^\sigma_{(\varepsilon)} \equiv \varepsilon^\sigma_y \gamma^y = \gamma^\mu$, so that (13.5) this reduces to $\left(\gamma^\sigma p_\sigma - mc\right)\left|U_0\right\rangle = 0$. Likewise, Dirac's momentum space equation reduces to $\left(\gamma^\sigma p_\sigma - mc\right)\left|u_0\right\rangle = 0$. Because these two equations now have exactly the same operator $\gamma^\sigma p_\sigma - mc$, this also means that $\left|U_0\right\rangle \to \left|u_0\right\rangle$ when electromagnetic interactions vanish. Thus (13.5) becomes synonymous with Dirac's momentum space equation for free fermions. However, when there are electromagnetic interactions, (13.5) is a somewhat different equation from $\left(\gamma^\sigma\left(p_\sigma + qA_\sigma / c\right) - mc\right)\left|u_0\right\rangle = 0$. Shortly, we shall study these differences. As a result of appending $\left|U_0\right\rangle$, the classical (13.2) is now a quantum mechanical equation (13.5).

If the further define a ket $\left|\Psi\right\rangle \equiv \exp\left(-iH_\sigma x^\sigma / \hbar c\right)\left|U_0\right\rangle$ which is a function of space and time due to the kernel $\exp\left(-iH_\sigma x^\sigma / \hbar c\right)$, then we may deduce $H_\sigma\left|\Psi\right\rangle = cp_\sigma\left|\Psi\right\rangle = i\hbar c\partial_\sigma\left|\Psi\right\rangle$ just as we did previously prior to (8.1) for $\left|s\right\rangle$. With this (13.5) can be turned into:

$$\left(i\hbar c\left(\Gamma^\sigma_{(\varepsilon)} + \frac{q}{mc^2} A^\sigma\right)\partial_\sigma - mc^2\right)\left|\Psi\right\rangle = 0 . \tag{13.6}$$

This is the new variant of Dirac's equation in configuration space in flat spacetime, which should be contrasted to the usual $0=\left(i\hbar\gamma^{\mu}\mathfrak{D}_{\mu}-mc\right)\left|\psi\right\rangle=\left(\gamma^{\mu}\left(i\hbar\partial_{\mu}+qA_{\mu}/c\right)-mc\right)\left|\psi\right\rangle$ for Dirac's configuration space equation in flat spacetime, as reviewed in section 1. As with (13.5), the two operators become identical when $q=0$ or $A^{\sigma}=0$ so that $\left|\Psi\right\rangle\rightarrow\left|\psi\right\rangle$, in which circumstance, (13.6) becomes synonymous with Dirac's configuration apace equation for free fermions.

Importantly, (13.5) and (13.6) also answer the question how to make sense of the "peculiar" line element in (3.3) and its equally perplexing solution (3.5): The quadratic solution (3.5) is in fact a new variant (13.5) of Dirac's equation in thick disguise, which is unmasked once we use the Heisenberg/Ehrenfest equations of motion and configuration, then remove two degrees of freedom from the gauge field A^{σ} via (9.4) and (9.5), thereby turning A^{σ} into a true massless photon. So as we shall also shortly see, (13.5) allows us to study interactions between *individual electrons and individual photons*. For pedagogic reference, given that the Dirac equation $0=\left(i\hbar\gamma^{\mu}\mathfrak{D}_{\mu}-mc\right)\left|\psi\right\rangle$ is the *canonical* result of applying local U(1) gauge symmetry to the ordinary $0=\left(i\hbar\gamma^{\mu}\partial_{\mu}-mc\right)\left|\psi\right\rangle$, we shall refer to equations (13.5) and (13.6) as Dirac's equation with electromagnetic tetrads embedded in $\Gamma^{\mu}_{(\varepsilon)}\equiv\varepsilon^{\mu}_{\ y}\gamma^{y}$ as the "hyper-canonical" Dirac equation.

14. The Electromagnetic Interaction Tetrad

Now we wish to derive the electromagnetic tetrad $\varepsilon^{\mu}_{\ y}$, in explicit component representation. The key relation for doing so is $\eta^{yz}\varepsilon^{\mu}_{\ y}\varepsilon^{\nu}_{\ z}\equiv\eta^{\mu\nu}+\left(q^{2}/m^{2}c^{4}\right)A^{\mu}A^{\nu}$ in (13.4). For compact notation we define the substitute variable $\rho\equiv q/mc^{2}$. Given that $\varepsilon^{\mu}_{\ y}=\delta^{\mu}_{\ y}$ is a 4x4 identify matrix when $q=0$ or A^{μ}, it also helps to define $\varepsilon'^{\mu}_{\ y}$ via $\varepsilon^{\mu}_{\ y}\equiv\delta^{\mu}_{\ y}+\varepsilon'^{\mu}_{\ y}$, to represent how $\varepsilon^{\mu}_{\ y}$ differs from the unit $\delta^{\mu}_{\ y}$. With these definitions we write the salient portion of (13.4) as:

$$\eta^{yz}\varepsilon^{\mu}_{\ y}\varepsilon^{\nu}_{\ z}=\eta^{yz}\left(\delta^{\mu}_{\ y}+\varepsilon'^{\mu}_{\ y}\right)\left(\delta^{\nu}_{\ z}+\varepsilon'^{\nu}_{\ z}\right)=\eta^{yz}\left(\delta^{\mu}_{\ y}\delta^{\nu}_{\ z}+\varepsilon'^{\mu}_{\ y}\delta^{\nu}_{\ z}+\delta^{\mu}_{\ y}\varepsilon'^{\nu}_{\ z}+\varepsilon'^{\mu}_{\ y}\varepsilon'^{\nu}_{\ z}\right)=\eta^{\mu\nu}+\rho A^{\mu}\rho A^{\nu}. \quad (14.1)$$

With $\eta^{yz}\delta^{\mu}_{\ y}\delta^{\nu}_{\ z}=\eta^{\mu\nu}$ and $\eta^{yz}\delta^{\nu}_{\ z}=\eta^{y\nu}$ and $\eta^{yz}\delta^{\mu}_{\ y}=\eta^{\mu z}$, and also subtracting $\eta^{\mu\nu}$ from each side, this easily simplifies to:

$$\eta^{y\nu}\varepsilon'^{\mu}_{\ y}+\eta^{\mu z}\varepsilon'^{\nu}_{\ z}+\eta^{yz}\varepsilon'^{\mu}_{\ y}\varepsilon'^{\nu}_{\ z}=\rho A^{\mu}\rho A^{\nu}. \quad (14.2)$$

The above contains sixteen (16) equations for each of $\mu=0,1,2,3$ and $\nu=0,1,2,3$. But, this is symmetric in μ and ν so in fact there are only ten (10) independent equations. Moreover, because A^{μ} has only four independent components, and also because we have already removed two degrees of freedom from A^{μ} via the gauge conditions (9.4) and (9.5), we anticipate that (14.2) will highlight this limited freedom by imposing definitive constraints on A^{μ}. Given that $\mathrm{diag}\left(\eta^{yz}\right)=\left(1,-1,-1,-1\right)$, the four $\mu=\nu$ "diagonal" equations in (14.2) produce the relations:

$$\begin{aligned}
&2\varepsilon_0^{\prime 0}+\varepsilon_0^{\prime 0}\varepsilon_0^{\prime 0}-\varepsilon_1^{\prime 0}\varepsilon_1^{\prime 0}-\varepsilon_2^{\prime 0}\varepsilon_2^{\prime 0}-\varepsilon_3^{\prime 0}\varepsilon_3^{\prime 0}=\rho A^0\rho A^0\\
&-2\varepsilon_1^{\prime 1}+\varepsilon_0^{\prime 1}\varepsilon_0^{\prime 1}-\varepsilon_1^{\prime 1}\varepsilon_1^{\prime 1}-\varepsilon_2^{\prime 1}\varepsilon_2^{\prime 1}-\varepsilon_3^{\prime 1}\varepsilon_3^{\prime 1}=\rho A^1\rho A^1\\
&-2\varepsilon_2^{\prime 2}+\varepsilon_0^{\prime 2}\varepsilon_0^{\prime 2}-\varepsilon_1^{\prime 2}\varepsilon_1^{\prime 2}-\varepsilon_2^{\prime 2}\varepsilon_2^{\prime 2}-\varepsilon_3^{\prime 2}\varepsilon_3^{\prime 2}=\rho A^2\rho A^2\\
&-2\varepsilon_3^{\prime 3}+\varepsilon_0^{\prime 3}\varepsilon_0^{\prime 3}-\varepsilon_1^{\prime 3}\varepsilon_1^{\prime 3}-\varepsilon_2^{\prime 3}\varepsilon_2^{\prime 3}-\varepsilon_3^{\prime 3}\varepsilon_3^{\prime 3}=\rho A^3\rho A^3
\end{aligned}\;. \tag{14.3a}$$

Likewise the three $\mu=0$, $\nu=1,2,3$ mixed time and space relations in (14.2) are:

$$\begin{aligned}
&-\varepsilon_1^{\prime 0}+\varepsilon_0^{\prime 1}+\varepsilon_0^{\prime 0}\varepsilon_0^{\prime 1}-\varepsilon_1^{\prime 0}\varepsilon_1^{\prime 1}-\varepsilon_2^{\prime 0}\varepsilon_2^{\prime 1}-\varepsilon_3^{\prime 0}\varepsilon_3^{\prime 1}=\rho A^0\rho A^1\\
&-\varepsilon_2^{\prime 0}+\varepsilon_0^{\prime 2}+\varepsilon_0^{\prime 0}\varepsilon_0^{\prime 2}-\varepsilon_1^{\prime 0}\varepsilon_1^{\prime 2}-\varepsilon_2^{\prime 0}\varepsilon_2^{\prime 2}-\varepsilon_3^{\prime 0}\varepsilon_3^{\prime 2}=\rho A^0\rho A^2\\
&-\varepsilon_3^{\prime 0}+\varepsilon_0^{\prime 3}+\varepsilon_0^{\prime 0}\varepsilon_0^{\prime 3}-\varepsilon_1^{\prime 0}\varepsilon_1^{\prime 3}-\varepsilon_2^{\prime 0}\varepsilon_2^{\prime 3}-\varepsilon_3^{\prime 0}\varepsilon_3^{\prime 3}=\rho A^0\rho A^3
\end{aligned}\;. \tag{14.3b}$$

Finally, the pure-space relations with $\mu,\nu=1,2$, $\mu,\nu=2,3$ and $\mu,\nu=3,1$ are:

$$\begin{aligned}
&-\varepsilon_2^{\prime 1}-\varepsilon_1^{\prime 2}+\varepsilon_0^{\prime 1}\varepsilon_0^{\prime 2}-\varepsilon_1^{\prime 1}\varepsilon_1^{\prime 2}-\varepsilon_2^{\prime 1}\varepsilon_2^{\prime 2}-\varepsilon_3^{\prime 1}\varepsilon_3^{\prime 2}=\rho A^1\rho A^2\\
&-\varepsilon_3^{\prime 2}-\varepsilon_2^{\prime 3}+\varepsilon_0^{\prime 2}\varepsilon_0^{\prime 3}-\varepsilon_1^{\prime 2}\varepsilon_1^{\prime 3}-\varepsilon_2^{\prime 2}\varepsilon_2^{\prime 3}-\varepsilon_3^{\prime 2}\varepsilon_3^{\prime 3}=\rho A^2\rho A^3\\
&-\varepsilon_1^{\prime 3}-\varepsilon_3^{\prime 1}+\varepsilon_0^{\prime 3}\varepsilon_0^{\prime 1}-\varepsilon_1^{\prime 3}\varepsilon_1^{\prime 1}-\varepsilon_2^{\prime 3}\varepsilon_2^{\prime 1}-\varepsilon_3^{\prime 3}\varepsilon_3^{\prime 1}=\rho A^3\rho A^1
\end{aligned}\;. \tag{14.3c}$$

Now, the right hand side of all ten of (14.3) have nonlinear products $\rho A^{\mu}\rho A^{\nu}$ of two field terms. On the left of each there is a mix of linear and nonlinear expressions containing the ε_y^{μ}. In (14.3a) the linear appearances are of $\varepsilon_0^{\prime 0}$, $\varepsilon_1^{\prime 1}$, $\varepsilon_2^{\prime 2}$ and $\varepsilon_3^{\prime 3}$ respectively. Given that the complete tetrad $\varepsilon_y^{\mu}\equiv\delta_y^{\mu}+\varepsilon_y^{\prime\mu}$, let us require that $\varepsilon_y^{\mu}=\delta_y^{\mu}$ for the four $\mu=y$ components, therefore, $\varepsilon_0^{\prime 0}=\varepsilon_1^{\prime 1}=\varepsilon_2^{\prime 2}=\varepsilon_3^{\prime 3}$ for $\mu=y$. This is consistent with $\varepsilon_y^{\mu}=\delta_y^{\mu}$ generally when $q=0$ or A^{μ}, and it means that the field components ρA^{μ} will all appear in off-diagonal components of ε_y^{μ}. In (14.3b), let us eliminate the linear terms by requiring $\varepsilon_1^{\prime 0}=\varepsilon_0^{\prime 1}$, $\varepsilon_2^{\prime 0}=\varepsilon_0^{\prime 2}$, and $\varepsilon_3^{\prime 0}=\varepsilon_0^{\prime 3}$, which is symmetric in μ and *y*. In (14.3c) we likewise remove the linear terms by requiring $\varepsilon_2^{\prime 1}=-\varepsilon_1^{\prime 2}$, $\varepsilon_3^{\prime 2}=-\varepsilon_2^{\prime 3}$ and $\varepsilon_1^{\prime 3}=-\varepsilon_3^{\prime 1}$ which is antisymmetric in μ and *y*. With all of this (14.3) reduce to:

$$\begin{aligned}
&-\varepsilon_1^{\prime 0}\varepsilon_1^{\prime 0}-\varepsilon_2^{\prime 0}\varepsilon_2^{\prime 0}-\varepsilon_3^{\prime 0}\varepsilon_3^{\prime 0}=\rho A^0\rho A^0\\
&+\varepsilon_0^{\prime 1}\varepsilon_0^{\prime 1}-\varepsilon_2^{\prime 1}\varepsilon_2^{\prime 1}-\varepsilon_3^{\prime 1}\varepsilon_3^{\prime 1}=\rho A^1\rho A^1\\
&+\varepsilon_0^{\prime 2}\varepsilon_0^{\prime 2}-\varepsilon_1^{\prime 2}\varepsilon_1^{\prime 2}-\varepsilon_3^{\prime 2}\varepsilon_3^{\prime 2}=\rho A^2\rho A^2\\
&+\varepsilon_0^{\prime 3}\varepsilon_0^{\prime 3}-\varepsilon_1^{\prime 3}\varepsilon_1^{\prime 3}-\varepsilon_2^{\prime 3}\varepsilon_2^{\prime 3}=\rho A^3\rho A^3
\end{aligned}\;, \tag{14.4a}$$

$$\begin{aligned} &-\varepsilon'^{0}_{2}\varepsilon'^{1}_{2}-\varepsilon'^{0}_{3}\varepsilon'^{1}_{3}=\rho A^{0}\rho A^{1} \\ &-\varepsilon'^{0}_{1}\varepsilon'^{2}_{1}-\varepsilon'^{0}_{3}\varepsilon'^{2}_{3}=\rho A^{0}\rho A^{2}\ , \\ &-\varepsilon'^{0}_{1}\varepsilon'^{3}_{1}-\varepsilon'^{0}_{2}\varepsilon'^{3}_{2}=\rho A^{0}\rho A^{3} \end{aligned} \tag{14.4b}$$

$$\begin{aligned} &\varepsilon'^{1}_{0}\varepsilon'^{2}_{0}-\varepsilon'^{1}_{3}\varepsilon'^{2}_{3}=\rho A^{1}\rho A^{2} \\ &\varepsilon'^{2}_{0}\varepsilon'^{3}_{0}-\varepsilon'^{2}_{1}\varepsilon'^{3}_{1}=\rho A^{2}\rho A^{3}\ . \\ &\varepsilon'^{3}_{0}\varepsilon'^{1}_{0}-\varepsilon'^{3}_{2}\varepsilon'^{1}_{2}=\rho A^{3}\rho A^{1} \end{aligned} \tag{14.4c}$$

Next, for the space components of A^{μ}, we assign $\varepsilon'^{1}_{0}=-\rho A^{1}$, $\varepsilon'^{2}_{0}=-\rho A^{2}$ and $\varepsilon'^{3}_{0}=-\rho A^{3}$ for the components of the tetrad which have a space world index and a time Lorentz index. By the earlier symmetric relations $\varepsilon'^{0}_{1}=\varepsilon'^{1}_{0}$, $\varepsilon'^{0}_{2}=\varepsilon'^{2}_{0}$, and $\varepsilon'^{0}_{3}=\varepsilon'^{3}_{0}$ this means $\varepsilon'^{0}_{1}=-\rho A^{1}$, $\varepsilon'^{0}_{2}=-\rho A^{2}$ and $\varepsilon'^{0}_{3}=-\rho A^{3}$ as well. Substituting this in (14.4) and reducing then brings us to:

$$\begin{aligned} &-\rho A^{1}\rho A^{1}-\rho A^{2}\rho A^{2}-\rho A^{3}\rho A^{3}=\rho A^{0}\rho A^{0} \\ &+\rho A^{1}\rho A^{1}-\varepsilon'^{1}_{2}\varepsilon'^{1}_{2}-\varepsilon'^{1}_{3}\varepsilon'^{1}_{3}=\rho A^{1}\rho A^{1} \\ &+\rho A^{2}\rho A^{2}-\varepsilon'^{2}_{1}\varepsilon'^{2}_{1}-\varepsilon'^{2}_{3}\varepsilon'^{2}_{3}=\rho A^{2}\rho A^{2} \\ &+\rho A^{3}\rho A^{3}-\varepsilon'^{3}_{1}\varepsilon'^{3}_{1}-\varepsilon'^{3}_{2}\varepsilon'^{3}_{2}=\rho A^{3}\rho A^{3} \end{aligned}\ , \tag{14.5a}$$

$$\begin{aligned} &-\rho A^{2}\varepsilon'^{1}_{2}-\rho A^{3}\varepsilon'^{1}_{3}=\rho A^{0}\rho A^{1} \\ &-\rho A^{1}\varepsilon'^{2}_{1}-\rho A^{3}\varepsilon'^{2}_{3}=\rho A^{0}\rho A^{2}\ , \\ &-\rho A^{1}\varepsilon'^{3}_{1}-\rho A^{2}\varepsilon'^{3}_{2}=\rho A^{0}\rho A^{3} \end{aligned} \tag{14.5b}$$

$$\begin{aligned} &-\varepsilon'^{1}_{3}\varepsilon'^{2}_{3}=0 \\ &-\varepsilon'^{2}_{1}\varepsilon'^{3}_{1}=0\ . \\ &-\varepsilon'^{3}_{2}\varepsilon'^{1}_{2}=0 \end{aligned} \tag{14.5c}$$

Because (14.4) all contain products of two tetrads it would be possible to make the oppositely-signed assignments $\varepsilon'^{1}_{0}=+\rho A^{1}$, $\varepsilon'^{2}_{0}=+\rho A^{2}$ and $\varepsilon'^{3}_{0}=+\rho A^{3}$ without changing the results (14.5) at all, because as to this sign ambiguity, $(\pm 1)^{2}=+1$. As we shall later see at (19.13) supra, we choose the minus sign because this is required to ensure that (13.5) produces solutions identical to Dirac's usual $\left(\gamma^{\sigma}\left(p_{\sigma}+qA_{\sigma}/c\right)-mc\right)\left|u_{0}\right\rangle=0$ in the weak field linear limit.

Next, one way to satisfy the earlier relation $\varepsilon'^{1}_{2}=-\varepsilon'^{2}_{1}$, $\varepsilon'^{2}_{3}=-\varepsilon'^{3}_{2}$ and $\varepsilon'^{3}_{1}=-\varepsilon'^{1}_{3}$ following (14.3) is to set all six of these to zero. This will satisfy all of (14.5c) identically, and will also satisfy the last three relations (14.5a) identically. We may also divide out ρ^2 from the first relation (14.5a), and all of (14.5b) may be combined into one, so now all we have left to satisfy are:

$$A^0 A^0 + A^1 A^1 + A^2 A^2 + A^3 A^3 = 0 \,, \tag{14.6a}$$

$$0 = \rho A^0 \rho A^1 = \rho A^0 \rho A^2 = \rho A^0 \rho A^3 . \tag{14.6b}$$

If we posit that at least one of the three $A^1 \neq 0$, $A^2 \neq 0$ and $A^3 \neq 0$, then we are required by (14.6b) to set $A^0 = 0$. The only relation we now have left to satisfy is (14.6a), which is the $\mu\nu = 00$ pure-time component of (14.2). Because of (14.6b), (14.6a) becomes:

$$A^1 A^1 + A^2 A^2 + A^3 A^3 = \mathbf{A} \cdot \mathbf{A} = \mathbf{A}^2 = 0 \,. \tag{14.7}$$

Consolidating (14.6) and (14.7) into generally covariant form, we obtain:

$$A^0 = A_0 = 0; \quad -A^k A_k = \mathbf{A} \cdot \mathbf{A} = \mathbf{A}^2 = 0; \quad A^\sigma A_\sigma = 0 \,. \tag{14.8}$$

Now, subject to (14.8) which we shall review in depth momentarily, we obtained each component of the tetrad $\varepsilon^\mu_{\ y}$. Collecting all of the results from (14.3) through (14.8), reassembling the complete tetrad $\varepsilon^\mu_{\ y} \equiv \delta^\mu_{\ y} + \varepsilon'^\mu_{\ y}$, and restoring $\rho = q / mc^2$, what we have deduced is that the simultaneous equations in (14.1) are solved by:

$$\varepsilon^\mu_{\ y} = \begin{pmatrix} 1 & -\rho A^1 & -\rho A^2 & -\rho A^3 \\ -\rho A^1 & 1 & 0 & 0 \\ -\rho A^2 & 0 & 1 & 0 \\ -\rho A^3 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -qA^1/mc^2 & -qA^2/mc^2 & -qA^3/mc^2 \\ -qA^1/mc^2 & 1 & 0 & 0 \\ -qA^2/mc^2 & 0 & 1 & 0 \\ -qA^3/mc^2 & 0 & 0 & 1 \end{pmatrix} . \tag{14.9}$$

The A^1, A^2 and A^3 above subject to the further constraint (14.8) which means that only two of the three A^k in (14.9) are truly independent. Thus, there are indeed only two degrees of freedom in the original A^μ which again is a downstream result of the gauge conditions (11.4) and (11.5).

15. Massless Photons with Two Helicity States and Coulomb Gauge

Equation (14.7) also part of (14.8), which is the $\mu\nu = 00$ pure-time component of (14.2) shown expressly in the top line of (14.3a), is consequential. First, because the Pythagorean sum in (14.7) is equal to zero, it is impossible for all three of A^1, A^2 and A^3 to simultaneously be non-zero and real. In fact, if any of these is real, then at least one other must be imaginary. This means that A^μ under conditions (14.8) no longer represents a classical field $A^\mu = (\phi, \mathbf{A})$ with four real components and inherent gauge ambiguity, but rather a *massless* photon quantum with two degrees of freedom and no gauge ambiguity. This all is confirmed by the fact that $A^0 = 0$, making it impossible for a massive gauge field travelling along the z axis (denoted $\hat{z}$) to keep a longitudinal polarization $\varepsilon_\mu(\hat{z}) = (c|\mathbf{p}|, 0, 0, E)/Mc^2$, see, e.g., section 6.12 of [13] at [6.92]. In the discussion to follow, we shall use "γ," a customary photon notation, as a subscript to designate when particular fields are those of an individual photon. This will distinguish from classical fields external to the photon, for which shall use the subscript "c".

Second, if this photon propagates along the z axis and has energy $cq^0 = E = h\nu = \hbar\omega$ also using $h = 2\pi\hbar$ and the radian frequency $\omega = 2\pi\nu$, then its energy-momentum four-vector is

$$cq^\mu(\hat{z}) = \begin{pmatrix} E & 0 & 0 & cq_z \end{pmatrix} = \begin{pmatrix} h\nu & 0 & 0 & cq_z \end{pmatrix} = \begin{pmatrix} \hbar\omega & 0 & 0 & cq_z \end{pmatrix}. \tag{15.1}$$

Also, because the photon is massless, we must have $0 = m_\gamma^2 c^4 = c^2 q_\sigma q^\sigma$ (we shall now use the subscript "γ" to denote photon). Together with the above, this implies that $c^2\mathbf{q}\cdot\mathbf{q} = (h\nu)^2$. With (15.1), the longitudinal orthogonal polarization component must be $\varepsilon^3 = 0$, thus $A^3 = 0$. Now (14.7) reduces given (15.1), to $A^1A^1 + A^2A^2 = 0$ which in turn means $A^1 = \pm iA^2$. Then, the relation $A^1 = \pm iA^2$ is solved by the right- and left-polarization vectors:

$$\varepsilon^\mu_{R,L}(\hat{z}) \equiv \begin{pmatrix} 0 & \mp 1 & -i & 0 \end{pmatrix}/\sqrt{2}, \tag{15.2}$$

again, [13] at [6.92]. In general, this means that:

$$A^\mu_\gamma = A\varepsilon^\mu \exp(-iq_\sigma x^\sigma/\hbar), \tag{15.3}$$

with a dimensionless polarization vector ε^μ and an amplitude A having dimensions of energy-per-charge to keep balance because those are the dimensions of A^μ_γ. So not only have the two covariant gauge conditions (9.4) and (9.5) forced A^μ_γ to be massless photons, but they have also forced A^μ_γ to assume the known *right- and left-handed photon helicities*. This is what has become of our remaining two degrees of freedom, precisely in accord with known theory and observation. The above (15.2) and (15.3) represent a *real photon*, not a "virtual" photon, because the longitudinal and scalar polarizations have been entirely eliminated and all that is left are the two transverse polarizations. And, because of (15.2), the gauge potential now introduces imaginary terms into the Riemannian geometry above and beyond the Fourier kernels often used to transform between configuration and momentum space, thus producing a type of Kähler Geometry. Because the only part of (15.3) which is a function of spacetime is the Fourier kernel $\exp(-iq_\sigma x^\sigma/\hbar)$, this means that in general:

$$i\hbar c\partial_\sigma A^\mu_\gamma = cq_\sigma A^\mu_\gamma = \begin{pmatrix} i\hbar\partial_t & i\hbar c\nabla \end{pmatrix} A^\mu_\gamma = \begin{pmatrix} \hbar\omega & -c\mathbf{q} \end{pmatrix} A^\mu_\gamma. \tag{15.4}$$

Third, it is clear from the above that $q_\sigma\varepsilon^\sigma = 0$, which is a form of the Lorentz gauge that emerges from the classical rendition of (9.5). But because $A^0_\gamma = 0$ thus $\varepsilon^0 = 0$ we may also deduce that $\mathbf{q}\cdot\boldsymbol{\varepsilon}=0$ which is the Coulomb gauge. Ordinarily this is a non-covariant gauge choice, see section 6.9 of [13] at [6.67]. Yet here, this is a *covariant* gauge, because it is a consequence of the covariant gauge conditions (9.4) and (9.5). Indeed, (9.4) and (9.5) are responsible for the very structure of (13.5), having caused the peculiar quadratic solution (3.5) to eventually turn into (13.5) via the definitions (13.4) that led among other results, to $A^0 = A^0_\gamma = 0$ in (14.6). Assembling this with other immediate corollaries and (15.2), we find that:

$$q_\sigma \varepsilon^\sigma = 0; \quad q^k \varepsilon^k = \mathbf{q}\cdot\boldsymbol{\varepsilon} = 0; \quad q_\sigma A^\sigma_\gamma = 0; \quad q^k A^k_\gamma = \mathbf{q}\cdot\mathbf{A}_\gamma = 0; \quad \partial_\sigma A^\sigma_\gamma = 0; \quad \partial_k A^k_\gamma = \nabla\cdot\mathbf{A}_\gamma = \mathbf{0}\,. \tag{15.5}$$

The above, together with (14.8), will be used extensively to zero out many contracted terms in subsequent calculations.

Finally, let us consider the relation $\mathbf{B}=\nabla\times\mathbf{A}$ between the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$, which via (15.4) may be written as $\hbar\mathbf{B}_\gamma = i\left(\mathbf{q}\times\mathbf{A}_\gamma\right)$ for an individual photon. Referring to just prior to (15.3), it is helpful to note that $\mathbf{q}(\hat{z})=(0,0,q_z)$ and $\boldsymbol{\varepsilon}_{R,L}(\hat{z})=(\mp 1,-i,0)/\sqrt{2}$. Thus, $\mathbf{q}\times\boldsymbol{\varepsilon}=(-iq_z \quad \mp q_z \quad 0)/\sqrt{2}$ and $(\mathbf{q}\times\boldsymbol{\varepsilon})^2=0$. Because $A^\mu_\gamma = A\varepsilon^\mu \exp\left(-iq_\sigma x^\sigma/\hbar\right)$, this also means that $\left(\mathbf{q}\times\mathbf{A}_\gamma\right)^2 = -\hbar^2\mathbf{B}_\gamma{}^2 = 0$. Recognizing that we can rotate to propagate any other direction without changing the invariant features of this result, this leads to several important observations:

First, this emphasizes how $\mathbf{A}=\mathbf{A}_\gamma$ has now been fully converted to the quantum potential for a photon, and *is not and can no longer be regarded* as a classical potential $\mathbf{A}=\mathbf{A}_\mathrm{c}$. Second, as the mediator of electromagnetic interactions, the *z*-traversing photon must have a magnetic field which we now know has the components

$$\hbar\mathbf{B}_\gamma = i\left(\mathbf{q}\times\mathbf{A}_\gamma(\hat{z})\right) = A\left(q_z \quad \mp iq_z \quad 0\right)\exp\left(-iq_\sigma x^\sigma/\hbar\right)/\sqrt{2}\,. \tag{15.6}$$

Just as $\mathbf{A}_\gamma$ is orthogonal to $\mathbf{q}$, so too $\mathbf{B}_\gamma$ is orthogonal to $\mathbf{q}$. Third, although the photon magnetic field is nonzero, this $\mathbf{B}_\gamma$ has imaginary components just like $\mathbf{A}_\gamma$, over and above the complex kernel $\exp\left(-iq_\sigma x^\sigma/\hbar\right)$. Fourth, from (15.6) we may calculate that $\mathbf{B}^2_\gamma = 0$, which stems directly from $\mathbf{A}^2 = \mathbf{A}^2_\gamma = 0$ found in (14.8). Thus, just as a photon carries energy even though as a luminous boson it is massless, so too a photon has a non-zero magnetic field even though the *magnitude* of that magnetic field is zero, $\left|\mathbf{B}_\gamma\right| = 0$. Fifth, the fact that a classical magnetic field can have a non-zero magnitude $\left|\mathbf{B}_\mathrm{c}\right| \neq 0$ is one clear indicator why $\mathbf{A}=\mathbf{A}_\gamma$ *must* be regarded as a photon rather than a classical potential. Sixth, the magnetic field still carries the kernel $\exp\left(-iq_\sigma x^\sigma/\hbar\right)$. As a result, also using (15.1), the four-gradient has the identical form to (15.4) for the photon:

$$i\hbar c\partial_\sigma \mathbf{B}_\gamma = cq_\sigma \mathbf{B}_\gamma = \left(i\hbar\partial_t \quad i\hbar c\nabla\right)\mathbf{B}_\gamma = \left(\hbar\omega \quad -c\mathbf{q}\right)\mathbf{B}_\gamma \tag{15.7}$$

Seventh, although the classical $A_c^{\ \mu}=(\phi,\mathbf{A}_c)$ which is an amalgamation of countless individual photons can always be Lorentz transformed into a rest frame $A_c^{\ \mu}=(\phi_0,\mathbf{0})$, the photon with $A_\gamma^{\ \mu}=A\varepsilon^\mu \exp(-iq_\sigma x^\sigma/\hbar)$ as a luminous particle can never be so-transformed. No matter what direction the photon travels, its time component $A_\gamma^0=0$ as we deduced at (14.8).

Eighth, although there is no *Lorentz* transformation $A_\gamma^{\ \mu}\to A_c^{\ \mu}$ that can place the photon into a rest frame, it is *formally* possible to use a *U(1) gauge* transformation $qA^\mu \to qA'^\mu \equiv qA^\mu+\hbar c\partial^\mu\Lambda$ to do transform a photon potential into a classical potential. . Specifically, we assign $A^\mu=A_\gamma^\mu$ and $A'^\mu=A_c^\mu$ and write the transformation as $qA_\gamma^\mu\to qA_c^\mu\equiv qA_\gamma^\mu+\hbar c\partial^\mu\Lambda$, whereby the arbitrary gauge parameter $\Lambda(t,\mathbf{x})$ is defined by $\hbar c\partial^\mu\Lambda\equiv qA_c^\mu-qA_\gamma^\mu=e\left(A_\gamma^\mu-A_c^\mu\right)$ for $q=-e$. For the time component, because $A_\gamma^{\ 0}=0$ and $A_c^{\ 0}=\phi$, rather simply, we obtain $\hbar c\partial^0\Lambda\equiv q\phi$. For the space components, we find an interesting wrinkle, owing to the fact that (15.2) and (15.3) are complex, not real, because $\sqrt{2}A_\gamma^\mu=A(0,\mp1,-i,0)\exp(-iq_\sigma x^\sigma/\hbar)$ for a z-traversing photon is a complex vector. Therefore, together with $\hbar c\partial^0\Lambda\equiv q\phi$ above, and mindful that $\partial^k=-(\partial_x,\partial_y,\partial_z)=-\nabla$ whereby $-\hbar c\nabla\Lambda\equiv q\mathbf{A}_c-q\mathbf{A}_\gamma$, we find:

$$
\begin{aligned}
&\hbar c\partial^0\Lambda=+\hbar c\partial\Lambda/\partial t\equiv q\phi\\
&\hbar c\partial^1\Lambda=-\hbar c\partial\Lambda/\partial x\equiv qA_c^1\pm qA\exp(-iq_\sigma x^\sigma/\hbar)/\sqrt{2}\\
&\hbar c\partial^2\Lambda=-\hbar c\partial\Lambda/\partial y\equiv qA_c^2+iqA\exp(-iq_\sigma x^\sigma/\hbar)/\sqrt{2}\\
&\hbar c\partial^3\Lambda=-\hbar c\partial\Lambda/\partial x\equiv qA_c^3
\end{aligned}
\qquad (15.8)
$$

Because the above sets $\partial^1\Lambda$ and $\partial^2\Lambda$ to complex numbers stemming from A_γ^μ being a complex vector, the gauge parameter $\Lambda(t,\mathbf{x})$ used in this transformation must also be a *complex* number, $\Lambda=a+ib$, once again signifying a type of Kähler Geometry. This is also of interest because historically, Weyl spent over a decade [6], [7], [8] pursuing the ultimately incorrect view that equations of nature should be invariant under a true "gauge" transformation $\varphi\to\varphi'\equiv\exp(\Lambda)\varphi$ rather than what we know is the correct $\varphi\to\varphi'\equiv\exp(i\Lambda)\varphi$. Writing Weyl's original misconception as $\varphi\to\varphi'\equiv\exp(i(-i\Lambda))\varphi$, we see that what we understand today to be a real gauge angle was in Weyl's original view equivalent to an imaginary gauge angle. What (15.8) shows is that when we wish to transform between $A_\gamma^{\ \mu}\leftrightarrow A_c^{\ \mu}$, we actually require a hybrid of both Weyl's original view and his eventual result: a complex gauge parameter $\Lambda=a+ib$, with an underlying transformation $\varphi\to\varphi'\equiv\exp(i\Lambda)\varphi=\exp(i(a+ib))\varphi$. And it is very luminosity of non-material, massless photons with complex components, which is the cause of this. The imaginary part of $\Lambda = a + ib$ only becomes non-zero for transformations between a material $A_c^{\ \mu}$ and a luminous $A_\gamma^{\ \mu}$.

Ninth, although it is *formally* possible to transform between $A_\gamma^{\ \mu} \leftrightarrow A_c^{\ \mu}$, *physically* we cannot do so: At (9.4) and (9.5) we removed two degrees of freedom from A^μ, thereby removing any remaining freedom to transform $A_\gamma^{\ \mu} \leftrightarrow A_c^{\ \mu}$. Equivalently, once two degrees of freedom were covariantly removed from the unrestricted A^μ turning it into the A_γ^μ of (15.3) with the properties (14.8) and (15.5) of a photon which is restricted to the Lorenz and Coulomb gauges, we "broke" the gauge symmetry, and can no longer transform $A_\gamma^{\ \mu}$ back to $A_c^{\ \mu}$. In other words, we cannot "unbreak" a broken gauge symmetry. But we can always trace back the breaking.

Tenth, and finally, although we cannot transform between $A_\gamma^{\ \mu} \leftrightarrow A_c^{\ \mu}$, the electric and magnetic fields contained in the field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ are invariant under a gauge transformation, as has long been well known, because the antisymmetry of $F^{\mu\nu}$ washes out any gauge transformation. Although gauge theory was not known in the late-19th century, in retrospect one reason that Heaviside reformulated Maxwell's equations to eliminate the potential, and went so far as to erroneously argue that physics ought not even bother with a potential and should only use electric and magnetic fields, was because of what we now understand to be the gauge symmetry of **E** and **B**. Therefore, the electric and magnetic fields are invariant under a gauge transformation between $A_\gamma^{\ \mu} \rightarrow A_c^{\ \mu}$, and so the transformation from $|\mathbf{B}_\gamma| = 0$ to $|\mathbf{B}_c| \neq 0$ is gauge invariant. However, knowing this, whenever we start with $\mathbf{A}_\gamma$ and end up with a quantity such as $\mathbf{B}_\gamma^2$ in an equation, it is best to leave this as is rather than set $\mathbf{B}_\gamma^2 = 0$, in order to preserve the ability to let conduce a gauge transformation $A_\gamma^{\ \mu} \rightarrow A_c^{\ \mu}$ from which $\mathbf{B}_\gamma^2 = 0 \rightarrow \mathbf{B}_c^2 \neq 0$.

This is important to keep in mind, because in the next several sections we will be developing a Dirac Hamiltonian using (14.8) and (15.1) through (15.7) to reduce terms containing **A**, and will set $A^0 = \phi = 0$ throughout using (14.8), (15.2) and (15.3), effectively removing the two degrees of gauge freedom from A^μ as a downstream consequence of the gauge fixing conditions (9.4) and (9.5). Again, this may be thought of as "breaking" the gauge symmetry. Once this is done, however, those terms in the Hamiltonian which contain **A** will *not* be invariant under the quantum-to-classical potential gauge transformation $A_\gamma^{\ \mu} \rightarrow A_c^{\ \mu}$. So **A** must be interpreted as the gauge potential for a single individual photon. Conversely, **A** cannot be regarded as part of a classical external potential $A_c^{\ \mu} = (\phi, \mathbf{A}_c)$, because the symmetry breaking conditions we will have imposed using (14.8) and (15.1) through (15.7) are conditions that are not followed by a classical potential which has a rest frame, but only by a luminous photon which can never be at rest. Again, this is why, contrasting Dirac's original theory, this paper is titled a "Quantum Theory of *Individual* Electron and Photon Interactions."

This is also important to keep in mind because although **A** cannot be interpreted as an external potential owing to how the gauge symmetry has been broken, in terms which will also arise containing the electric and magnetic fields **E** and **B**, these fields *can* in interpreted as either classical or quantum fields, precisely because **E** and **B** are invariant under gauge transformations. Thus, **E** and **B** will enter the Hamiltonian in exactly the same form whether the gauge potentials are classical A_c^{μ} or quantum mechanical A_{γ}^{μ}, cf. earlier reference to Heaviside. Put differently, the Hamiltonian terms containing **E** and **B** sans **A** are invariant under gauge transformations and so are invariant under transformations between classical and quantum potentials. Thus, the **E** and **B** which appear, regardless of how we interpret the A^{μ} from which they arise via $-\nabla\phi = \mathbf{E} + \partial\mathbf{A}/c\partial t$ or by $\mathbf{B} = \nabla\times\mathbf{A}$ (which of course have the generally-covariant formulation $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$), can be interpreted and used either as the **E**, **B** fields of an individual photon, or as classical external **E**, **B** fields arising from the stationary linear amalgamation of a countless multitude of individual luminously-propagating photons.

Additionally, it will be of use to examine the electric and magnetic fields associated with a single photon quantum. As with a classical field, the photon field strength is $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ which is of course gauge symmetric and thus invariant under a gauge transformation $A_{\gamma}^{\mu} \leftrightarrow A_c^{\mu}$. Using (15.4) we write this as:

$$i\hbar c F_{\gamma}^{\ \mu\nu} = i\hbar c\partial^{\mu}A_{\gamma}^{\nu} - i\hbar c\partial^{\nu}A_{\gamma}^{\mu} = cq^{\mu}A_{\gamma}^{\nu} - cq^{\nu}A_{\gamma}^{\mu}. \tag{15.9}$$

Using (15.5) and $q_{\sigma}q^{\sigma} = 0$ for a luminous photon, the photon current density four-vector is then:

$$\begin{aligned} -4\pi\hbar^2 J_{\gamma}^{\nu} &= -4\pi\hbar^2\begin{pmatrix} c\rho_{em\,\gamma} & \mathbf{J}_{\gamma}\end{pmatrix} = -\hbar^2 c\partial_{\mu}F_{\gamma}^{\ \mu\nu} = i\hbar c\partial_{\mu}\left(i\hbar\partial^{\mu}A_{\gamma}^{\nu} - i\hbar\partial^{\nu}A_{\gamma}^{\mu}\right) \\ &= i\hbar c\partial_{\mu}\left(q^{\mu}A_{\gamma}^{\nu} - q^{\nu}A_{\gamma}^{\mu}\right) = cq_{\mu}q^{\mu}A_{\gamma}^{\nu} - cq_{\mu}q^{\nu}A_{\gamma}^{\mu} = -cq^{\nu}\left(q_0A_{\gamma}^{0} + q_kA_{\gamma}^{k}\right) = 0 \end{aligned}, \tag{15.10}$$

that is, $\begin{pmatrix}\rho_{em} & \mathbf{J}\end{pmatrix} = 0$, using the notation ρ_{em} to distinguish this charge density from the substitute variable $\rho = q/mc^2$. Note that this zero arises from $q_{\mu}q^{\mu} = 0$ because of the massless photon, from $A_{\gamma}^{0} = 0$ in (14.8), and from $\mathbf{q}\cdot\mathbf{A}_{\gamma} = 0$ in (15.5). So as expected, the photon is not an electromagnetic *source*, but rather is the electromagnetic mediator. But the electric and magnetic fields are still not zero. Specifically, using (14.8) and (15.1) in (15.9) we find:

$$i\hbar c\mathbf{E}_{\gamma} = i\hbar cE_{\gamma}^{\ j} = i\hbar cF_{\gamma}^{\ j0} = i\hbar cF_{\gamma}^{\ j0} = i\hbar cF_{\gamma}^{\ j0} = cq^{j}A_{\gamma}^{\ 0} - cq^{0}A_{\gamma}^{\ j} = -\hbar\omega A_{\gamma}^{\ j} = -\hbar\omega\mathbf{A}_{\gamma}, \tag{15.11}$$

i.e., $\hbar\omega\mathbf{A}_{\gamma} = -i\hbar c\mathbf{E}_{\gamma}$ which simplifies to $\omega\mathbf{A}_{\gamma} = -ic\mathbf{E}_{\gamma}$. Using the over-dot notation $\dot{\mathbf{A}}_{\gamma} = \partial_t\mathbf{A}_{\gamma}$ and combining (15.11) with $i\dot{\mathbf{A}}_{\gamma} = \omega\mathbf{A}_{\gamma}$ from (15.4), we also deduce that:

$$\dot{\mathbf{A}}_{\gamma} = -i\omega\mathbf{A}_{\gamma} = -c\mathbf{E}_{\gamma}. \tag{15.12}$$

Next, from (15.11) and (14.8), the magnitude $\mathbf{E}_{\gamma}^{\ 2} = 0$. For the magnetic field:

$$-i\hbar\mathbf{B}_\gamma = -i\hbar B^i_\gamma = \tfrac{1}{2}\varepsilon^{ijk} i\hbar F_\gamma{}^{jk} = \tfrac{1}{2}\left(\varepsilon^{ijk} q^j A^k_\gamma - \varepsilon^{ijk} q^k A^j_\gamma\right) = \varepsilon^{ijk} q^j A^k_\gamma = \mathbf{q}\times\mathbf{A}_\gamma = -i\hbar\nabla\times\mathbf{A}_\gamma \,, \qquad (15.13)$$

which is the usual relation $\mathbf{B} = \nabla\times\mathbf{A}$. However, when we take the magnitude using (14.8) and (15.5) we likewise obtain what we already saw at (15.6) namely:

$$-\hbar^2 \mathbf{B}_\gamma{}^2 = -\hbar^2 B^i_\gamma B^i_\gamma = \varepsilon^{ijk}\varepsilon^{ilm} q^j A^k_\gamma q^l A^m_\gamma = q^j A^k_\gamma q^j A^k_\gamma - q^j A^k_\gamma q^k A^j_\gamma = 0 \,. \qquad (15.14)$$

So, an individual photon has energy but no mass, and has electric and magnetic fields which are non-zero but have zero magnitude. Additionally, writing (15.11) as $\hbar c\mathbf{E}_\gamma = ih\nu\mathbf{A}_\gamma = 2\pi i\hbar\nu\mathbf{A}_\gamma$, then taking the spacetime gradient of each side and using (15.4), we obtain (contrast (15.7) for **B**):

$$i\hbar c\partial_\sigma \mathbf{E}_\gamma = -\hbar\omega\partial_\sigma \mathbf{A}_\gamma = i\omega q_\sigma \mathbf{A}_\gamma = cq_\sigma \mathbf{E}_\gamma = \left(i\hbar\partial_t \quad i\hbar c\nabla\right)\mathbf{E}_\gamma = \left(\hbar\omega \quad -c\mathbf{q}\right)\mathbf{E}_\gamma \,. \qquad (15.15)$$

Again, what will be of particular interest is that while $\mathbf{A}_\gamma$ for the photon is not invariant under the quantum-to-classical gauge transformation $A_\gamma{}^\mu \to A_c{}^\mu$, the electric and magnetic fields in $F^{\mu\nu}$ *are* invariant. Therefore, when we encounter composite terms such as $i\hbar\omega\mathbf{A}_\gamma = \hbar c\mathbf{E}_\gamma$ where the gauge-dependent $\mathbf{A}_\gamma$ is multiplied by the photon energy $\hbar\omega$, or such as $i\mathbf{q}\times\mathbf{A}_\gamma = \hbar\mathbf{B}$ where the gauge-dependent $\mathbf{A}_\gamma$ is crossed with the photon momentum **q**, these *composite* terms are invariant under gauge transformations. This means that these composite terms, and the field strength $F^{\mu\nu}$ generally, are gauge-invariant whether they represent the electric and magnetic fields of a single photon, or classical electric and magnetic fields externally-applied to a single photon. Thus, wherever **E** and **B** appear, whether obtained from a classical potential $A_c{}^\mu$ or a single-photon $A_\gamma{}^\mu$, these **E** and **B** fields (but not the **A** alone) may be regarded at will (under some carefully-proscribed restraints) as either classical fields or as individual photon fields. We shall see all of this in detail over the next several sections.

Finally, on occasion we shall encounter the commutator of a *luminous* photon momentum **q** with functions of spacetime $b(t,\mathbf{x})$. Specifically, unlike **p** which does not commute with functions of **x** as a result of the Heisenberg commutation relation reviewed following (7.10), the photon momentum **q** *does* commute with functions of **x**. To establish this, first recall from (15.1) that $q_\sigma q^\sigma = 0$ a.k.a. $c^2\mathbf{q}\cdot\mathbf{q} = (h\nu)^2$ because a photon is massless, which is well-established. Thus, let us rotate our choice of space coordinates so that the photon propagates in the +*z* direction, whereby (15.1) becomes $cq^\mu(\hat{z}) = (\hbar\omega, 0, 0, \hbar\omega)$ because $cq_z = \hbar\omega = h\nu$. Next, we posit a $b(z)$ which is a function of *z*, expansible as a Maclaurin series in *z*, and which this appears adjacent q_z in the form $b(z)cq_z = b(z)\hbar\omega$. Clearly, because the photon energy $\hbar\omega$ commutes with $b(z)$, we find that $b(z)cq_z = b(z)\hbar\omega = \hbar\omega b(z) = cq_z b(z)$, which we may write as the commutator relation $\left[b(z), q_z\right] = 0$. Then, deconstructing the series, we learn that $\left[z, q_z\right] = 0$ for a *luminous particle momentum commutator*, in contrast to Heisenberg's $\left[z, p_z\right] = i\hbar$ for a *material particle* *momentum commutator*. By rotational symmetry, we then obtain the general relation $\left[O(\mathbf{x}),\mathbf{q}\right]=0$ for any object $O(\mathbf{x})$ which is a function of the space coordinates, and the deconstructed relation $\left[\mathbf{x},\mathbf{q}\right]=\left[x^i,q^j\right]=0$ in contrast to Heisenberg's $\left[\mathbf{x},\mathbf{p}\right]=\left[x^i,p^j\right]=i\delta^{ij}\hbar$.

16. Maxwell's Equations for Individual Photons

To illustrate the foregoing considerations about the relation between a classical potential $A_c{}^\mu$ and the potential $A_\gamma^\mu = A\varepsilon^\mu \exp\left(-iq_\sigma x^\sigma / \hbar\right)$ for an individual photon, in this section we shall apply Maxwell's equations to individual photons. Not only is this study of independent value in its own right, but it will illustrate how to properly navigate between classical fields and those of a single photon quantum. This will be indispensable when we return to developing the hyper-canonical Dirac equation, starting in the next section.

The relation between a four-potential **A** and the electric and magnetic fields **E** and **B** is covariantly-formulated by $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$, which separates into the component equations $\nabla\phi=-\mathbf{E}-\partial\mathbf{A}/c\partial t=-\mathbf{E}-\dot{\mathbf{A}}/c$ and $\nabla\times\mathbf{A}=\mathbf{B}$. In turn, Maxwell's equations are covariantly-formulated by $4\pi J^\mu=\partial_\alpha F^{\alpha\mu}$ and $\partial^\alpha F^{\mu\nu}+\partial^\mu F^{\nu\alpha}+\partial^\nu F^{\alpha\mu}=0$, a.k.a. $\partial_\alpha {}^*F^{\alpha\mu}=0$ using the dual fields ${}^*F_{\mu\nu}=\frac{1}{2!}\varepsilon_{\sigma\tau\mu\nu}F^{\sigma\tau}$. The former separates into the component equations $\nabla\cdot\mathbf{E}=4\pi\rho_{em}$ and $\nabla\times\mathbf{B}=\left(4\pi\mathbf{J}+\partial\mathbf{E}/\partial t\right)/c=\left(4\pi\mathbf{J}+\dot{\mathbf{E}}\right)/c$ which are Gauss' and Ampere's Laws for electricity. The latter separates into $\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mathbf{E}=-\partial\mathbf{B}/c\partial t=-\dot{\mathbf{B}}/c$ which are Gauss' and Faraday's Laws for magnetism. The absence of a magnetic field divergence in $\nabla\cdot\mathbf{B}=0$ – colloquially expressed as the non-existence and non-observation of magnetic monopoles – is a mathematical identity that results from inserting the antisymmetric $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ into the cyclic field combination $\partial^\alpha F^{\mu\nu}+\partial^\mu F^{\nu\alpha}+\partial^\nu F^{\alpha\mu}=0$. Using vectors, this is expressed by the identity $\nabla\cdot\mathbf{B}=\nabla\cdot\left(\nabla\times\mathbf{A}\right)=0$, namely, the divergence of the curl is zero. In the language of exterior calculus, this has the simplified form $ddA=0$, and exemplifies the geometric rule $dd=0$ that the exterior derivative of an exterior derivative, or the boundary of a boundary, is zero.

Most importantly for the present development, these relations apply invariantly, whether $A^\mu = A_c^\mu$ is a classical material potential or $A^\mu = A_\gamma^\mu$ is the quantum potential for a single photon. Thus, $F_c{}^{\mu\nu}=\partial^\mu A_c^\nu-\partial^\nu A_c^\mu$, $4\pi J^\mu=\partial_\alpha F_c{}^{\alpha\mu}$ and $\partial^\alpha F_c{}^{\mu\nu}+\partial^\mu F_c{}^{\nu\alpha}+\partial^\nu F_c{}^{\alpha\mu}=0$ for a classical potential, and $F_\gamma{}^{\mu\nu}=\partial^\mu A_\gamma^\nu-\partial^\nu A_\gamma^\mu$, $4\pi J^\mu=\partial_\alpha F_\gamma{}^{\alpha\mu}$ and $\partial^\alpha F_\gamma{}^{\mu\nu}+\partial^\mu F_\gamma{}^{\nu\alpha}+\partial^\nu F_\gamma{}^{\alpha\mu}=0$ for an individual quantum photon potential. First, let us study these equations as applied to an individual photon, with the energy-momentum four-vector of (15.1) and the photon potential (15.3), absent any external potentials or fields or charge densities.

For an individual photon, for which the scalar potential $A^0=\phi=0$ because of (14.8) as represented in (15.2) and (15.3), the curl equation $\mathbf{B}=\nabla\times\mathbf{A}$ remains the same, but the gradient equation reduces to $\nabla\phi = 0 = -\mathbf{E} - \dot{\mathbf{A}} / c$ or, more directly, $\mathbf{E} = -\dot{\mathbf{A}} / c$. Employing (15.4) which contains both $i\hbar\dot{\mathbf{A}} = h\nu\mathbf{A} = \hbar\omega\mathbf{A}$ and $i\hbar\nabla\mathbf{A} = -\mathbf{q}\mathbf{A}$, we multiply through by $i\hbar c$, then convert into momentum space, to obtain:

$$\begin{aligned} &i\hbar c\mathbf{E}_\gamma = -i\hbar\dot{\mathbf{A}}_\gamma = -\hbar\omega\mathbf{A}_\gamma \\ &i\hbar c\mathbf{B}_\gamma = i\hbar c\nabla\times\mathbf{A}_\gamma = -c\mathbf{q}\times\mathbf{A}_\gamma \end{aligned} . \tag{16.1}$$

This relates the fields $\mathbf{E}_\gamma$ and $\mathbf{B}_\gamma$ *of an individual photon*, to the photon three- potential $\mathbf{A}_\gamma$. We showed via $\mathbf{E}_\gamma{}^2 = 0$ at (15.11) and $\mathbf{B}_\gamma^2 = 0$ at (15.6) how these two fields, although nonzero, do have zero magnitudes, just as a photon has non-zero energy $h\nu = \hbar\omega$ but zero rest mass. Now let's turn to Maxwell's equations.

Still for an individual photon, (15.10) teaches that $\begin{pmatrix} c\rho_{em} & \mathbf{J} \end{pmatrix} = 0$, i.e., that the luminous photon does not act as an electromagnetic source but only as an interaction mediator. Therefore, in covariant form Maxwell's equations reduce to the source-free, duality-symmetric $\partial_\alpha F^{\alpha\mu} = 0$, and $\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0$ a.k.a. $\partial_\alpha {}^* F^{\alpha\mu} = 0$. In component form, this produces $\nabla\cdot\mathbf{E}_\gamma = 0$ and $\nabla\cdot\mathbf{B}_\gamma = 0$ for the divergence equations, and $c\nabla\times\mathbf{E}_\gamma = -\dot{\mathbf{B}}_\gamma$ and $c\nabla\times\mathbf{B}_\gamma = \dot{\mathbf{E}}_\gamma$ for the curl equations. To convert these into momentum space, we turn to (15.7) and (15.15), which again using ω, contain $i\hbar\dot{\mathbf{B}}_\gamma = \hbar\omega\mathbf{B}_\gamma$ and $i\hbar\nabla\mathbf{B}_\gamma = -\mathbf{q}\mathbf{B}_\gamma$, $i\hbar\dot{\mathbf{E}}_\gamma = \hbar\omega\mathbf{E}_\gamma$ and $i\hbar\nabla\mathbf{E}_\gamma = -\mathbf{q}\mathbf{E}_\gamma$. As a result, Maxwell's equations for an individual photon convert to momentum space as follows:

$$\begin{aligned} &i\hbar c\nabla\cdot\mathbf{E}_\gamma = -c\mathbf{q}\cdot\mathbf{E}_\gamma = 0 \\ &i\hbar c\nabla\cdot\mathbf{B}_\gamma = -c\mathbf{q}\cdot\mathbf{B}_\gamma = 0 \\ &i\hbar c\nabla\times\mathbf{E}_\gamma = -c\mathbf{q}\times\mathbf{E}_\gamma = -i\hbar\dot{\mathbf{B}}_\gamma = -\hbar\omega\mathbf{B}_\gamma \\ &i\hbar c\nabla\times\mathbf{B}_\gamma = -c\mathbf{q}\times\mathbf{B}_\gamma = i\hbar\dot{\mathbf{E}}_\gamma = \hbar\omega\mathbf{E}_\gamma \end{aligned} . \tag{16.2}$$

Equations (16.2) establish paired relations $c\mathbf{q}\times\mathbf{E}_\gamma = \hbar\omega\mathbf{B}_\gamma$ and $c\mathbf{q}\times\mathbf{B}_\gamma = -\hbar\omega\mathbf{E}_\gamma$ between the electric and magnetic fields $\mathbf{E}_\gamma$ and $\mathbf{B}_\gamma$ of a photon. Again, these are non-zero, but have zero magnitude.

It is further possible to use (16.1) to write (16.2) multiplied through by another $i\hbar c$ in terms of the photon three-potential $\mathbf{A}_\gamma$ as:

$$
\begin{aligned}
&-\hbar^2c^2\nabla\cdot\mathbf{E}_\gamma=-i\hbar c c\mathbf{q}\cdot\mathbf{E}_\gamma=c\mathbf{q}\cdot\left(\hbar\omega\mathbf{A}_\gamma\right)=0\\
&-\hbar^2c^2\nabla\cdot\mathbf{B}_\gamma=-i\hbar c c\mathbf{q}\cdot\mathbf{B}_\gamma=c\mathbf{q}\cdot\left(c\mathbf{q}\times\mathbf{A}_\gamma\right)=0\\
&-\hbar^2c^2\nabla\times\mathbf{E}_\gamma=-i\hbar c c\mathbf{q}\times\mathbf{E}_\gamma=-i\hbar c\left(\hbar\omega\mathbf{B}_\gamma\right)=\hbar\omega\left(c\mathbf{q}\times\mathbf{A}_\gamma\right)\\
&-\hbar^2c^2\nabla\times\mathbf{B}_\gamma=-i\hbar c c\mathbf{q}\times\mathbf{B}_\gamma=i\hbar c\left(\hbar\omega\mathbf{E}_\gamma\right)=c\mathbf{q}\times\left(c\mathbf{q}\times\mathbf{A}_\gamma\right)=-\hbar\omega\left(\hbar\omega\mathbf{A}_\gamma\right)
\end{aligned}
\quad . \tag{16.3}
$$

The first equation contains $\mathbf{q}\cdot\mathbf{A}_\gamma=0$, which is merely (15.5) for the Coulomb gauge which characterizes a photon. Again, it is of consequence that is was *covariantly* derived from (9.4) and (9.5). The second equation contains $\mathbf{q}\cdot\left(\mathbf{q}\times\mathbf{A}_\gamma\right)=0$, which is the momentum space formulation of the identity that the divergence of the curl is zero. From the latter two equations in (16.3) we may extract the momentum space relations:

$$
\begin{aligned}
&c\mathbf{q}\times\mathbf{E}_\gamma=i\omega\left(\mathbf{q}\times\mathbf{A}_\gamma\right)\\
&c\mathbf{q}\times\mathbf{B}_\gamma=-i\omega\left(\hbar\omega\mathbf{A}_\gamma/c\right)
\end{aligned}
\tag{16.4}
$$

between the fields $\mathbf{E}_\gamma$ and $\mathbf{B}_\gamma$ of a photon, and the photon three-potential $\mathbf{A}_\gamma$. Taken together, these are the Maxwell's equations in momentum space, for an individual photon, absent any external potentials or fields or sources.

Second, starting with the individual photon studied in (16.1) through (16.4), we next introduce a classical external potential ϕ, which of course is the time component of the four-vector $A_c^\mu=\begin{pmatrix}\phi & \mathbf{A}_c\end{pmatrix}$. We also place an observer at rest in the potential so that $A_c^\mu=\begin{pmatrix}\phi & \mathbf{A}_c\end{pmatrix}=\begin{pmatrix}\phi_0 & \mathbf{0}\end{pmatrix}$ or that observer. We further introduce a classical external charge density having the four-vector $J^\mu=\begin{pmatrix}c\rho_{em} & \mathbf{J}\end{pmatrix}$. And finally, we shall have the have the photon traverse a region of spacetime in which these A_c^μ and J^μ are non-zero. Under these new circumstances, let us now repeat the calculations of equations (16.1) through (16.4).

As to (16.1), the photon magnetic field continues $\mathbf{B}_\gamma$ to bear the relation $\mathbf{B}_\gamma=\nabla\times\mathbf{A}_\gamma$ to the photon three-potential $\mathbf{A}_\gamma$. However, the photon electric field $\mathbf{E}_\gamma$ will now bear the relation $\mathbf{E}_\gamma=-\nabla\phi-\dot{\mathbf{A}}_\gamma/c$ to $\mathbf{A}_\gamma$, and specifically, this relation will now be modified by the new term $\nabla\phi$ which was zero in (16.1). As such, given the non-zero $\phi=\phi_0$, (16.1) now becomes:

$$
\begin{aligned}
&i\hbar c\mathbf{E}_\gamma=-i\hbar c\nabla\phi_0-i\hbar\dot{\mathbf{A}}_\gamma=c\mathbf{q}\phi_0-\hbar\omega\mathbf{A}_\gamma\\
&i\hbar c\mathbf{B}_\gamma=i\hbar c\nabla\times\mathbf{A}_\gamma=-c\mathbf{q}\times\mathbf{A}_\gamma
\end{aligned}
\tag{16.5}
$$

Clearly, this will revert to (16.1) when $\phi_0=0$, as it must.

As to (16.2), because we are now allowing a non-zero source J^μ, we must use the complete Maxwell equations with sources, so that (16.2) now becomes:

$$\begin{aligned} &i\hbar c\nabla\cdot\mathbf{E}_\gamma = -c\mathbf{q}\cdot\mathbf{E}_\gamma = 4\pi i\hbar c\rho_{em} \\ &i\hbar c\nabla\cdot\mathbf{B}_\gamma = -c\mathbf{q}\cdot\mathbf{B}_\gamma = 0 \\ &i\hbar c\nabla\times\mathbf{E}_\gamma = -c\mathbf{q}\times\mathbf{E}_\gamma = -i\hbar\dot{\mathbf{B}}_\gamma = -\hbar\omega\mathbf{B}_\gamma \\ &i\hbar c\nabla\times\mathbf{B}_\gamma = -c\mathbf{q}\times\mathbf{B}_\gamma = i\hbar\left(4\pi\mathbf{J}+\dot{\mathbf{E}}_\gamma\right) = 4\pi i\hbar\mathbf{J}+\hbar\omega\mathbf{E}_\gamma \end{aligned} \quad . \tag{16.6}$$

It is easily seen that when $J^\mu = 0$, the above will revert to (16.2), as it must.

Now, as we did at (16.3), let us again multiply the above through by another $i\hbar c$ and then combine with (16.5). This produces:

$$\begin{aligned} &-\hbar^2c^2\nabla\cdot\mathbf{E}_\gamma = -i\hbar c c\mathbf{q}\cdot\mathbf{E}_\gamma = -c\mathbf{q}\cdot c\mathbf{q}\phi_0 + c\mathbf{q}\cdot\left(\hbar\omega\mathbf{A}_\gamma\right) = -\left(\hbar\omega\right)^2\phi_0 = -4\pi\hbar^2c^2\rho_{em} \\ &-\hbar^2c^2\nabla\cdot\mathbf{B}_\gamma = -i\hbar c c\mathbf{q}\cdot\mathbf{B}_\gamma = c\mathbf{q}\cdot\left(c\mathbf{q}\times\mathbf{A}_\gamma\right) = 0 \\ &-\hbar^2c^2\nabla\times\mathbf{E}_\gamma = -i\hbar c c\mathbf{q}\times\mathbf{E}_\gamma = c\mathbf{q}\times\left(c\mathbf{q}\phi_0+\hbar\omega\mathbf{A}_\gamma\right) = -i\hbar c\left(\hbar\omega\mathbf{B}_\gamma\right) = \hbar\omega\left(c\mathbf{q}\times\mathbf{A}_\gamma\right) \\ &-\hbar^2c^2\nabla\times\mathbf{B}_\gamma = -i\hbar c c\mathbf{q}\times\mathbf{B}_\gamma = -4\pi\hbar^2c\mathbf{J}+\hbar\omega i\hbar c\mathbf{E}_\gamma = c\mathbf{q}\times\left(c\mathbf{q}\times\mathbf{A}_\gamma\right) \\ &\qquad = c\mathbf{q}\left(c\mathbf{q}\cdot\mathbf{A}_\gamma\right)-\mathbf{A}_\gamma\left(c\mathbf{q}\cdot c\mathbf{q}\right) = -\left(\hbar\omega\right)^2\mathbf{A}_\gamma = -\left(\hbar\omega\right)^2\mathbf{A}_\gamma + c\mathbf{q}\hbar\omega\phi_0 - 4\pi\hbar^2c\mathbf{J} \end{aligned} \quad . \tag{16.7}$$

To reduce, we use $c\mathbf{q}\cdot c\mathbf{q} = \left(h\nu\right)^2 = \left(\hbar\omega\right)^2$ from (15.1) and $\mathbf{q}\cdot\mathbf{A}_\gamma = 0$ from (15.5) in the first line. We see the identity $c\mathbf{q}\times c\mathbf{q}\phi_0 = 0$ in the third line. In the final equation we use the triple cross identity $c\mathbf{q}\times\left(c\mathbf{q}\times\mathbf{A}_\gamma\right) = c\mathbf{q}\left(c\mathbf{q}\cdot\mathbf{A}_\gamma\right)-\mathbf{A}_\gamma\left(c\mathbf{q}\cdot c\mathbf{q}\right)$, then $c\mathbf{q}\cdot c\mathbf{q} = \left(\hbar\omega\right)^2$, again together with $\mathbf{q}\cdot\mathbf{A}_\gamma = 0$ from (15.5). It is easily seen that (16.7) reverts to (16.3) when $J^\mu = 0$ and $\phi_0 = 0$, as it must.

Finally, as we did at (16.4), we isolate the momentum space relations in the above. But first, we reorder the final equation in (16.7) above into the second position in (16.8) below, so as to group together the Maxwell's equations pairs which are generally covariant, also showing underlying the covariant equation. The result is:

$$\begin{aligned} &\left.\begin{aligned} &c\mathbf{q}\cdot\mathbf{E}_\gamma = -i\omega\hbar\omega\phi_0/c = -4\pi i\hbar c\rho_{em} \\ &c\mathbf{q}\times\mathbf{B}_\gamma = -i\omega\hbar\omega\mathbf{A}_\gamma/c = -i\omega\hbar\omega\mathbf{A}_\gamma/c + i\omega\mathbf{q}\phi_0 - 4\pi i\hbar\mathbf{J} \end{aligned}\right\} \quad \partial_\alpha F^{\alpha\mu} = 4\pi J^\mu \\ &\left.\begin{aligned} &c\mathbf{q}\cdot\mathbf{B}_\gamma = ic\mathbf{q}\cdot\left(\mathbf{q}\times\mathbf{A}_\gamma\right)/\hbar = 0 \\ &c\mathbf{q}\times\mathbf{E}_\gamma = i\omega\left(\mathbf{q}\times\mathbf{A}_\gamma\right) \end{aligned}\right\} \quad \partial^\alpha F^{\mu\nu}+\partial^\mu F^{\nu\alpha}+\partial^\nu F^{\alpha\mu} = 0 \end{aligned} \quad . \tag{16.8}$$

Following this reordering, we see that the second and fourth equations above are respectively identical to the lower and upper equations in (16.4).

Further, from the first and second $4\pi J^{\mu} = \partial_{\alpha} F^{\alpha\mu}$ equations, we may deduce:

$$\begin{aligned} &4\pi\hbar^2 c^2 \rho_{em} = \hbar\omega\left(\hbar\omega\phi_0\right) \\ &4\pi\hbar^2 c\mathbf{J} = c\mathbf{q}\left(\hbar\omega\phi_0\right) \end{aligned}, \tag{16.9}$$

which combines with clarity into the covariant relation:

$$4\pi\hbar^2 cJ^{\mu} = \hbar\omega\phi_0 q^{\mu} = 4\pi\hbar^2 c\begin{pmatrix} c\rho_{em} & \mathbf{J} \end{pmatrix} = \hbar\omega\phi_0\begin{pmatrix} \hbar\omega & c\mathbf{q} \end{pmatrix}. \tag{16.10}$$

This, as it must be, is manifestly the same as (15.10) written as $4\pi\hbar^2 cJ^{\nu} = cq^{\nu} cq_0 A^0$ after the replacement $A_{\gamma}^{0} = 0 \mapsto A_{\mathrm{c}}^{0} = \phi = \phi_0$ of the photon potential with an external potential at rest. This occurred prior to (16.5) when we placed the photon in the external potential $A_{\mathrm{c}}^{\mu} = \begin{pmatrix} \phi_0 & \mathbf{0} \end{pmatrix}$ and also introduced a non-zero $J^{\mu} = \begin{pmatrix} c\rho_{em} & \mathbf{J} \end{pmatrix}$. The offsetting terms $-i\omega\hbar\omega\mathbf{A}_{\gamma} / c$ in the second line of (16.8) which cancel in (16.9), stem from the relation $q_{\mu} q^{\mu} = 0$ for a luminous photon, which was applied in (15.10). We see from (16.10) that as soon as $\phi_0 = 0$, so too does $J^{\mu} = 0$. So, introducing the external scalar potential $\phi = \phi_0$ is synonymous with introducing J^{μ}. On reflection, this is obvious: Equations (16.1) through (16.4) describe source-free electromagnetic fields. But a scalar potential must have a material electrical source. Equation (16.10) says exactly that.

Third, let us next Lorentz transform the external potential out of the rest frame and into some relative motion, so that $A_{\mathrm{c}}^{\mu} = \begin{pmatrix} \phi_0 & \mathbf{0} \end{pmatrix} \to \begin{pmatrix} \phi & \mathbf{A}_{\mathrm{c}} \end{pmatrix}$. What happens to equations (16.7)? Because $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$ and Maxwell's $4\pi J^{\mu} = \partial_{\alpha} F^{\alpha\mu}$ and $\partial^{\alpha} F^{\mu\nu} + \partial^{\mu} F^{\nu\alpha} + \partial^{\nu} F^{\alpha\mu} = 0$ have identical form whether applied to a classical material A_{c}^{μ} potential or a quantum luminous potential A_{γ}^{μ} related thereto by the gauge transformation reviewed at (15.8), *nothing at all changes in the form of equations (16.7).* Because electromagnetism is a linear, abelian interaction, potentials are additive, so that the photon potential add to the external potential, yielding an overall potential $A^{\mu} = A_{\mathrm{c}}^{\mu} + A_{\gamma}^{\mu}$. (For practical purposes, $A^{\mu} = A_{\mathrm{c}}^{\mu} + A_{\gamma}^{\mu} \cong A_{\mathrm{c}}^{\mu}$ because the individual photon potential is swamped by the external potential.) So in (16.7), all we need to do *as to form*, is replace $\phi_0 \mapsto \phi$ (to take this out of rest) and replace $\mathbf{A}_{\gamma} \mapsto \mathbf{A}$ (to add the motion components $\mathbf{A}_{\mathrm{c}}$ to the photon components $\mathbf{A}_{\gamma}$). As to substance, however, there is an important change: Whereas $\mathbf{q} \cdot \mathbf{A}_{\gamma} = 0$ for an individual photon from (15.5), this is generally not true for an external classical three-potential. Rather, we must regard $\mathbf{q} \cdot \mathbf{A}_{\mathrm{c}} \neq 0$ to be nonzero. So, we no longer remove this term. As a result, (16.7), rearranged to have the covariant orderings of (16.8), now becomes:

$$\begin{aligned}
&-\hbar^2c^2\nabla\cdot\mathbf{E}_\gamma=-i\hbar c c\mathbf{q}\cdot\mathbf{E}_\gamma=-(\hbar\omega)^2\phi+\hbar c\omega\mathbf{q}\cdot\mathbf{A}=-4\pi\hbar^2c^2\rho_{em}\\
&-\hbar^2c^2\nabla\times\mathbf{B}_\gamma=-i\hbar c c\mathbf{q}\times\mathbf{B}_\gamma=-(\hbar\omega)^2\mathbf{A}+\hbar c\omega\mathbf{q}\phi-4\pi\hbar^2c\mathbf{J}=-(\hbar\omega)^2\mathbf{A}+c\mathbf{q}(c\mathbf{q}\cdot\mathbf{A})\\
&-\hbar^2c^2\nabla\cdot\mathbf{B}_\gamma=-i\hbar c c\mathbf{q}\cdot\mathbf{B}_\gamma=c\mathbf{q}\cdot(c\mathbf{q}\times\mathbf{A})=0\\
&-\hbar^2c^2\nabla\times\mathbf{E}_\gamma=-i\hbar c c\mathbf{q}\times\mathbf{E}_\gamma=\hbar c\omega(c\mathbf{q}\times\mathbf{A})
\end{aligned}\quad. \tag{16.11}$$

The source equations contained in the first two lines above are now:

$$\begin{aligned}
&4\pi\hbar^2c^2\rho_{em}=\hbar\omega(\hbar\omega\phi-c\mathbf{q}\cdot\mathbf{A})\\
&4\pi\hbar^2c\mathbf{J}=c\mathbf{q}(\hbar\omega\phi-c\mathbf{q}\cdot\mathbf{A})
\end{aligned}\quad, \tag{16.12}$$

which, as they must, reduce to (16.9) when $\mathbf{q}\cdot\mathbf{A}=0$ and the external is Lorentz transformed to a rest frame, $\phi\to\phi_0$. This are, and also must be, the same as $4\pi\hbar^2cJ^\nu=cq^\nu\left(cq_0A^0+cq_kA^k\right)$ from (15.10), with $A^\mu_\gamma\mapsto A^\mu=A^\mu_c+A^\mu_\gamma$.

Fourth and finally, the electric and magnetic fields $\mathbf{E}_\gamma$ and $\mathbf{B}_\gamma$ in (16.11) are still those of an individual photon. Now, let us introduce classical, external electric and magnetic fields $\mathbf{E}_c$ and $\mathbf{B}_c$, and ask: what now happens to (16.11)? Again, because electromagnetism is abelian, these fields are additive to those of the photon, so that the total $F^{\mu\nu}=F_c^{\ \mu\nu}+F_\gamma^{\ \mu\nu}\cong F_c^{\ \mu\nu}$, again with the photon's $F_\gamma^{\ \mu\nu}$ swamped by the external classical $F_c^{\ \mu\nu}$. Moreover, although the photon potential $A^\mu_\gamma=A\varepsilon^\mu\exp\left(-iq_\sigma x^\sigma/\hbar\right)$ of (15.3) has broken gauge symmetry and is not invariant under the transformation $A_\gamma^{\ \mu}\to A_c^{\ \mu}$, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ *are gauge-invariant fields.* Again, although unknown in the late 19th century, this was central though unbeknownst to Heaviside's reformulation of Maxwell's Treatise to contain only electric and magnetic fields without potentials. So, the introduction of external $\mathbf{E}_c$ and $\mathbf{B}_c$ does not in any way change the *form* of equations (16.11). All we need do is replace $\mathbf{E}_\gamma\mapsto\mathbf{E}=\mathbf{E}_c+\mathbf{E}_\gamma$ and $\mathbf{B}_\gamma\mapsto\mathbf{B}=\mathbf{B}_c+\mathbf{B}_\gamma$ throughout. Therefore, with these external electromagnetic fields, (16.11) finally becomes:

$$\begin{aligned}
&-\hbar^2c^2\nabla\cdot\mathbf{E}=-i\hbar c c\mathbf{q}\cdot\mathbf{E}=-(\hbar\omega)^2\phi+\hbar c\omega\mathbf{q}\cdot\mathbf{A}=-4\pi\hbar^2c^2\rho_{em}\\
&-\hbar^2c^2\nabla\times\mathbf{B}=-i\hbar c c\mathbf{q}\times\mathbf{B}=-(\hbar\omega)^2\mathbf{A}+\hbar c\omega\mathbf{q}\phi-4\pi\hbar^2c\mathbf{J}=-(\hbar\omega)^2\mathbf{A}+c\mathbf{q}(c\mathbf{q}\cdot\mathbf{A})\\
&-\hbar^2c^2\nabla\cdot\mathbf{B}=-i\hbar c c\mathbf{q}\cdot\mathbf{B}=c\mathbf{q}\cdot(c\mathbf{q}\times\mathbf{A})=0\\
&-\hbar^2c^2\nabla\times\mathbf{E}=-i\hbar c c\mathbf{q}\times\mathbf{E}=\hbar c\omega(c\mathbf{q}\times\mathbf{A})
\end{aligned}\quad. \tag{16.13}$$

In (16.13), *all* of the fields and potentials are now classical and external, added to and swamping those of the individual photon. All that remains to represent the individual photon is its energy-momentum vector cq^μ which, via (15.10), is $cq^\mu(\hat{z}) = (h\nu, 0, 0, cq_z) = (\hbar\omega, 0, 0, \hbar\omega)$ for propagation along the positive *z* axis. So (16.13) now characterizes the behavior of the luminous photon energy-momentum cq^μ propagating through external potentials **A** and fields **E** and **B**, and even through spacetime regions with non-zero charge densities ρ_{em} and currents **J**.

Before we conclude, there are a few other lessons we may learn from the foregoing development which will be important as we momentarily return to the development of the hyper-canonical Dirac equation. First, and of great usefulness, the relations $i\hbar\partial_\sigma A^\mu_\gamma = q_\sigma A^\mu_\gamma$ in (15.4), $i\hbar\partial_\sigma \mathbf{B}_\gamma = q_\sigma \mathbf{B}_\gamma$ in (15.7) and $i\hbar\partial_\sigma \mathbf{E}_\gamma = q_\sigma \mathbf{E}_\gamma$ in (15.15) all allow the heuristic replacement $i\hbar\partial_\sigma \mapsto q_\sigma$ whenever the spacetime gradient ∂_σ operates on *any* of ϕ, $\mathbf{A}_\gamma$, $\mathbf{B}_\gamma$ or $\mathbf{E}_\gamma$. But this is not a general replacement that can be used indiscriminately; its use depends integrally on the operand of ∂_σ. For counterexamples, consider $i\hbar\partial_\sigma|\Psi\rangle = p_\sigma|\Psi\rangle$ thus $i\hbar\partial_\sigma \mapsto p_\sigma$ used at (13.6) when the operand is a fermion wavefunction, and $\nabla p^\mu = -\left(q\left(\mathbf{E} + \dot{\mathbf{A}}/c\right)/E\right)p^\mu$ from (7.10) when the operand is a material energy-momentum p^μ. This leads to the question: can we still apply $i\hbar\partial_\sigma \mapsto q_\sigma$ when the operand is an external classical field $\mathbf{A}_c$, $\mathbf{E}_c$ or $\mathbf{B}_c$?

We need look no further than (16.13) above to directly see that $i\hbar\nabla\mathbf{E} = -\mathbf{qE}$ and $i\hbar\nabla\mathbf{B} = -\mathbf{qB}$ from (15.15) and (15.7) remain fully intact. Likewise, because $\mathbf{B} = \nabla\times\mathbf{A}$, we may discern from the $c\mathbf{q}\times\mathbf{A}$ terms that so too does $i\hbar\nabla\mathbf{A} = -\mathbf{qA}$ from (15.4). Thus, although the classical fields do not contain a Fourier kernel $\exp\left(-iq_\sigma x^\sigma/\hbar\right)$, the symmetry relations applied above to go from (16.2) to (16.13) lead us to conclude that that (15.4), (15.7) and (15.15) do generalize to external classical fields, without the γ designation. Therefore, generally:

$$i\hbar c\partial_\sigma A^\mu = cq_\sigma A^\mu; \quad i\hbar c\partial_\sigma \mathbf{B} = cq_\sigma \mathbf{B}; \quad i\hbar c\partial_\sigma \mathbf{E} = cq_\sigma \mathbf{E}. \tag{16.14}$$

In sum, from the development that led from (16.2) to (16.13), we may conclude that $i\hbar\partial_\sigma \mapsto q_\sigma$ can still be used as a heuristic rule whenever the operand is a classical $\mathbf{A}_c$, $\mathbf{E}_c$ or $\mathbf{B}_c$.

Second, while the relations $A^0_\gamma = \phi = 0$ and $\mathbf{A}^2_\gamma = 0$ from (14.8), and $\mathbf{q}\cdot\mathbf{A}_\gamma = 0$ and $\nabla\cdot\mathbf{A}_\gamma = 0$ from (15.5) apply to individual photons and will be very helpful to reduce many terms from the Dirac equation when we are considering individual photon behavior, these relations all do *not* apply for a classical potential. Specifically, $A^0_c = \phi \neq 0$ and $\mathbf{A}^2_c \neq 0$ and $\nabla\cdot\mathbf{A}_c \neq 0$ for a classical potential, and for a photon in a classical potential, $\mathbf{q}\cdot\mathbf{A}_c \neq 0$. Therefore, although the symmetry relations reviewed and used to go from (16.2) to (16.13) do enable the heuristic replacement $i\hbar\partial_\sigma \mapsto q_\sigma$ to be inherited by the classical $\mathbf{A}_c$, $\mathbf{E}_c$ or $\mathbf{B}_c$ whenever they are operands of ∂_σ as generalized in (16.14), we will wish to leave $\mathbf{q}\cdot\mathbf{A}_\gamma = 0$ as is without zeroing it out, in those situations where we anticipate later wishing to generalize to a classical potential for which $\mathbf{q}\cdot\mathbf{A}_c \neq 0$. For example, consider $-\hbar^2c^2\nabla\cdot\mathbf{E} = -i\hbar c c\mathbf{q}\cdot\mathbf{E} = -(\hbar\omega)^2\phi + \hbar c\omega\mathbf{q}\cdot\mathbf{A} = -4\pi\hbar^2c^2\rho_{em}$ from the top line of (16.13), which includes $i\hbar\nabla\cdot\mathbf{E} = -\mathbf{q}\cdot\mathbf{E}$. For an individual photon, $\phi = 0$ and $\mathbf{q}\cdot\mathbf{A} = 0$, which would imply that $\rho_{em} = 0$, which is the time component of (15.10). But if we encounter a $\nabla\cdot\mathbf{E}_\gamma$ such as in (16.2) but anticipate wanting to examine $\nabla\cdot\mathbf{E}$ generally, we will refrain from setting $\phi = 0$ and $\mathbf{q}\cdot\mathbf{A} = 0$ even when these are zero. Simply put, it is easier to set $\phi = 0$ and $\mathbf{q}\cdot\mathbf{A} = 0$ in (16.13) and revert to (16.2), than to start with (16.2) and generalize to (16.13) (as we have done here to illustrate this very point). We shall keep this in mind as we now return to the hyper-canonical Dirac equation (13.6) and seek to develop this in the most general form so we can study the interactions of individual fermions and photons in external classical fields and with external sources.

17. The Hyper-Canonical Dirac Equation Generalized to Curved Spacetime

As it stands, while hyper-canonical Dirac equation (13.6) is modeled after Dirac's equation in curved spacetime because of its use of the tetrad $\varepsilon^\mu_{\ y}$ with components deduced in (14.9), it does not yet apply to gravitation. To advance (13.6) to gravitation, let us consider the electromagnetic $\varepsilon^\mu_{\ y}$ alongside the ordinary gravitational tetrad $e^\mu_{\ a}$, as well as the electromagnetic gamma matrices $\Gamma^\mu_{(\varepsilon)} \equiv \varepsilon^\mu_{\ y}\gamma^y$ alongside the ordinary gravitational $\Gamma_{(g)}{}^\mu \equiv e^\mu_{\ a}\gamma^a$. Now, when we have both electromagnetism *and* gravitation, we are required to define a set of complete Γ^μ containing both electromagnetism and gravitation which generalize (13.4) from $\eta^{\mu\nu} \mapsto g^{\mu\nu}$ and also satisfy (13.3), and so are defined *such that* $G^{\mu\nu} \equiv g^{\mu\nu} + \left(q^2/m^2c^4\right)A^\mu A^\nu \equiv \tfrac{1}{2}\left\{\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu\right\}$.

Given the two separate tetrad definitions $\Gamma^\mu_{(\varepsilon)} \equiv \varepsilon^\mu_{\ y}\gamma^y$ and $\Gamma_{(g)}{}^\mu \equiv e^\mu_{\ a}\gamma^a$, there are two possible choices for constructing the complete Γ^μ. The first is to start with the electromagnetic $\Gamma^a_{(\varepsilon)} \equiv \varepsilon^a_{\ y}\gamma^y$ developed above (note index switch from μ to *a*), then compound this with gravitation by defining $\Gamma^\mu \equiv e^\mu_{\ a}\Gamma^a_{(\varepsilon)} = e^\mu_{\ a}\varepsilon^a_{\ y}\gamma^y$. The second is to start with the usual gravitational $\Gamma_{(g)}{}^y \equiv e^y_{\ a}\gamma^a$ (note index switch from μ to *y*), then compound this with electromagnetism by defining $\Gamma^\mu \equiv \varepsilon^\mu_{\ y}\Gamma_{(g)}{}^y = \varepsilon^\mu_{\ y}e^y_{\ a}\gamma^a$. If we place no restrictions on the ordinary metric tensor $g^{\mu\nu}$ (other than its usual $\mu \leftrightarrow \nu$ symmetry), then these two choices are *not* the same, because (again with some index renaming) $e^\mu_{\ a}\varepsilon^a_{\ y}\gamma^y \neq \varepsilon^\mu_{\ a}e^a_{\ y}\gamma^y$ i.e. $\left[e^\mu_{\ a}\varepsilon^a_{\ y} - \varepsilon^\mu_{\ a}e^a_{\ y}\right]\gamma^y \neq 0$. Formally stated: the electromagnetic and gravitational tetrads operating on the Dirac gamma do not commute, $[e,\varepsilon]\gamma \neq 0$. Generally, two objects not commuting means they are not independent; presently, $[e,\varepsilon]\gamma \neq 0$ tells us that the electromagnetic interaction energies contained in ε gravitate thus changing the gravitational *e*, as they should. Now, let us examine these two possible choices.

Choosing $\Gamma^{\mu} \equiv \varepsilon^{\mu}_{\ y} e^{y}_{\ a} \gamma^{a}$ would yield $G^{\mu\nu} = \varepsilon^{\mu}_{\ y} \varepsilon^{\nu}_{\ z} e^{y}_{\ a} e^{z}_{\ b} \eta^{ab} = \varepsilon^{\mu}_{\ y} \varepsilon^{\nu}_{\ z} g^{yz}$. A simple calculation shows that this is the *incorrect* choice: Sample $G^{00} = \varepsilon^{0}_{\ y} \varepsilon^{0}_{\ z} g^{yz}$ and insert the tetrad (14.9) for *z* axis photon propagation, thus $A^3 = 0$. Then set $g^{\mu\nu} = \eta^{\mu\nu}$. Because $G^{\mu\nu} = g^{\mu\nu} + \left(q^2 / m^2 c^4\right) A^{\mu} A^{\nu}$ and using $A^0 = 0$ from (14.8), we must have $G^{\mu\nu} = \eta^{\mu\nu}$. But in fact this ordering of the tetrads produces the contradictory $G^{00} = \eta^{00} - \rho A^1 \rho A^1 - \rho A^2 \rho A^2$. So this is wrong.

The correct choice is rather to define a complete tetrad

$$\mathrm{E}^{\mu}_{\ y} \equiv e^{\mu}_{\ a} \varepsilon^{a}_{\ y}, \tag{17.1}$$

and likewise to define the complete Γ^{μ} for electromagnetism and gravitation are by:

$$\Gamma^{\mu} \equiv e^{\mu}_{\ a} \Gamma^{a}_{(\varepsilon)} = e^{\mu}_{\ a} \varepsilon^{a}_{\ y} \gamma^{y} = \mathrm{E}^{\mu}_{\ y} \gamma^{y}. \tag{17.2}$$

Importantly, because $\eta^{\mu\nu} = \frac{1}{2}\left\{\gamma^{\mu}, \gamma^{\nu}\right\}$, it is the electromagnetic tetrad which directly couples to flat Minkowski spacetime via $\Gamma^{a}_{(\varepsilon)} = \varepsilon^{a}_{\ y} \gamma^{y}$. This is in turn coupled to curved spacetime thus gravitation by the subsequent $\Gamma^{\mu} \equiv e^{\mu}_{\ a} \Gamma^{a}_{(\varepsilon)}$.

Combining these definitions and the $G^{\mu\nu} \equiv g^{\mu\nu} + \left(q^2 / m^2 c^4\right) A^{\mu} A^{\nu} \equiv \frac{1}{2}\left\{\Gamma^{\mu}\Gamma^{\nu} + \Gamma^{\nu}\Gamma^{\mu}\right\}$ requirement, we obtain:

$$\begin{aligned} G^{\mu\nu} &\equiv \tfrac{1}{2}\left\{\Gamma^{\mu}\Gamma^{\nu} + \Gamma^{\nu}\Gamma^{\mu}\right\} = e^{\mu}_{\ a} e^{\nu}_{\ b} \varepsilon^{a}_{\ y} \varepsilon^{b}_{\ z} \tfrac{1}{2}\left\{\gamma^{y}\gamma^{z} + \gamma^{z}\gamma^{y}\right\} = e^{\mu}_{\ a} e^{\nu}_{\ b} \varepsilon^{a}_{\ y} \varepsilon^{b}_{\ z} \eta^{yz} = \mathrm{E}^{\mu}_{\ y} \mathrm{E}^{\nu}_{\ z} \eta^{yz} \\ &= e^{\mu}_{\ a} e^{\nu}_{\ b} \left(\eta^{ab} + \rho A^{a} \rho A^{b}\right) = e^{\mu}_{\ a} e^{\nu}_{\ b} \eta^{ab} + e^{\mu}_{\ a} e^{\nu}_{\ b} \rho A^{a} \rho A^{b} = g^{\mu\nu} + e^{\mu}_{\ a} e^{\nu}_{\ b} \rho A^{a} \rho A^{b} \equiv g^{\mu\nu} + \rho A^{\mu} \rho A^{\nu} \end{aligned}. \tag{17.3}$$

For this to all be correct, it is necessary via the final definition that:

$$A^{\mu} A^{\nu} = e^{\mu}_{\ a} e^{\nu}_{\ b} A^{a} A^{b}. \tag{17.4}$$

be true. It will be seen making use of (14.9) for each of the sixteen pairwise μ, ν combinations, as well as $A^0 = 0$ from (14.8), that (17.4) is indeed true; thus so is (17.3).

We may then use Γ^{μ} from (17.2) in place of $\Gamma^{\sigma}_{(\varepsilon)}$ to advance (13.6) to:

$$\left(i\hbar c\left(\Gamma^{\sigma} + \frac{q}{mc^2} A^{\sigma}\right)\partial_{\sigma} - mc^2\right)\left|\Psi\right\rangle = 0. \tag{17.5}$$

Clearly, when $g_{\mu\nu}=\eta_{\mu\nu}$ thus $e^{\mu}_{a}=\delta^{\mu}_{a}$ the above will revert to (13.6). The above is the hyper-canonical Dirac equation encompassing electromagnetism and gravitation, via the successive couplings of the electromagnetic and gravitational tetrads in $\Gamma^{\mu}=e^{\mu}_{a}\varepsilon^{a}_{y}\gamma^{y}$. But it does not yet have a proper spin connection. We now review why this is needed and how it is introduced.

18. The Hyper-Canonical Spin Connection

As reviewed in section 1, in curved spacetime, in order to couple the spinor fields ψ to gravitation, we must advance ∂_{μ} in the Dirac equation $\left(i\hbar\Gamma^{\mu}\partial_{\mu}-mc\right)\psi=0$ to a spin-covariant derivative $\partial_{\mu}\mapsto\nabla_{\mu}\equiv\partial_{\mu}-\frac{i}{4}\omega^{ab}_{\mu}\sigma_{ab}$ using a spin connection $\omega^{ab}_{\mu}\equiv e_{\nu}{}^{a}\partial_{;\mu}e^{\nu b}$, whereby Dirac's equation becomes $\left(i\hbar\Gamma^{\mu}\nabla_{\mu}-mc\right)\psi=0$, and where $\partial_{;\mu}e^{\nu b}=\partial_{\mu}e^{\nu b}+\Gamma^{\nu}_{\sigma\mu}e^{\sigma b}$ is the gravitational covariant derivative of $e^{\nu b}$. More formally, in the usual Dirac equation in curved spacetime, ∇_{μ} in $\nabla_{\mu}\psi$ does correctly operate as a covariant vector because it contains the covariant $\partial_{;\mu}$, while ∂_{σ} in $\partial_{\sigma}\psi$ does not. This is why the spin connection is required. The same considerations must now be applied to ∂_{μ} in the hyper-canonical Dirac equation with gravitation, (17.5).

To guide us on how to construct the required spin connection for (17.5), let us briefly review in more detail how this is ordinarily done for Dirac's equation. First, we note from the product rule that $\partial_{;\mu}\left(e_{\nu}{}^{a}e^{\nu b}\right)=\partial_{;\mu}e_{\nu}{}^{a}e^{\nu b}+e_{\nu}{}^{a}\partial_{;\mu}e^{\nu b}=\partial_{;\mu}e_{\nu}{}^{a}e^{\nu b}+\omega^{ab}_{\mu}$. So ω^{ab}_{μ} is actually one of the two terms in the covariant derivative $\partial_{;\mu}$ of $e_{\nu}{}^{a}e^{\nu b}=g_{\mu\nu}e^{\mu a}e^{\nu b}$. With this in mind, we start with $g^{\mu\nu}=e^{\mu}_{a}e^{\nu}_{b}\eta^{ab}$ and calculate that $g^{\mu\nu}=e^{\mu}_{a}e^{\nu}_{b}\eta^{ab}=e^{\mu}_{a}e^{\nu b}\delta^{a}{}_{b}=e^{\mu}_{a}e^{\nu a}$. This true because η^{ab} is used to raise and lower the flat spacetime Lorentz indexes. We may now lower a world index to obtain $\delta^{\mu}{}_{\nu}=e^{\mu}_{a}e^{a}_{\nu}$. Then, we again start with $g^{\mu\nu}=e^{\mu}_{a}e^{\nu}_{b}\eta^{ab}$ but this time we form the identity $g^{\mu\nu}=\delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\tau}g^{\sigma\tau}=e^{\mu}_{a}e^{\nu}_{b}\eta^{ab}$. We then use $\delta^{\mu}{}_{\nu}=e^{\mu}_{a}e^{a}_{\nu}$ with renamed indexes to write this identity as $g^{\mu\nu}=e^{\mu}_{a}e^{a}_{\sigma}e^{\nu}_{b}e^{b}_{\tau}g^{\sigma\tau}=e^{\mu}_{a}e^{\nu}_{b}\eta^{ab}$. We then divide out the two tetrads common to each side to deduce $\eta^{ab}=e^{a}_{\sigma}e^{b}_{\tau}g^{\sigma\tau}$, which is the inverse of $g^{\mu\nu}=e^{\mu}_{a}e^{\nu}_{b}\eta^{ab}$. By simple rearrangement of indexes in this inverse we obtain $\eta^{ab}=e^{a}_{\nu}e^{\nu b}$ which is the term for which the derivative is taken at the start of this paragraph using the product.

Therefore, because η^{ab} is a constant, $0=\partial_{;\mu}\eta^{ab}=\partial_{;\mu}e^{a}_{\nu}e^{\nu b}+e^{a}_{\nu}\partial_{;\mu}e^{\nu b}$. This in turn leads us with some further rearrangement to deduce that $\omega^{ab}_{\mu}=e^{a}_{\nu}\partial_{;\mu}e^{\nu b}=-e^{b}_{\nu}\partial_{;\mu}e^{\nu a}=-\omega^{ba}_{\mu}$, from which we learn that ω^{ab}_{μ} is antisymmetric in its *a*, *b* indexes. This is important, because antisymmetric tensors $A^{[\mu\nu]}$ can always be constructed by $A^{[\mu\nu]}\equiv\frac{1}{2}\left[A^{\mu\nu}-A^{\nu\mu}\right]$ from arbitrary tensors $A^{\mu\nu}$, which in the present context would have us defining $\omega^{ab}_{\mu}\equiv\frac{1}{2}\left[e_{\nu}{}^{a}\partial_{;\mu}e^{\nu b}-e^{b}_{\nu}\partial_{;\mu}e^{\nu a}\right]$. However, this construction is unnecessary here, because $\omega^{ab}_{\mu}=e^{a}_{\nu}\partial_{;\mu}e^{\nu b}=-e^{b}_{\nu}\partial_{;\mu}e^{\nu a}=-\omega^{ba}_{\mu}$ shows how

$\omega_\mu^{ab} \equiv e_\nu{}^a \partial_{;\mu} e^{\nu b}$ is naturally antisymmetric without special construction. Therefore, we also deduce that $\partial_{;\mu}\left(e_\nu{}^a e^{\nu b}\right) = \partial_{;\mu} e_\nu{}^a e^{\nu b} + e_\nu{}^a \partial_{;\mu} e^{\nu b} = \omega_\mu^{ba} + \omega_\mu^{ab} = 0$.

Now we turn to the spin-covariant derivative $\nabla_\mu = \partial_\mu - \frac{i}{4}\omega_\mu^{ab}\sigma_{ab}$. Starting with $g^{\mu\nu} = \frac{1}{2}\left\{\Gamma^\mu_{(g)}\Gamma^\nu_{(g)} + \Gamma^\nu_{(g)}\Gamma^\mu_{(g)}\right\}$ with $\Gamma^\mu_{(g)} = e^\mu_a \gamma^\alpha$ we see that $\Gamma_{(g)\nu}\Gamma^\nu_{(g)} = \delta_\nu{}^\nu = 4$ and via the product rule that $\partial_{;\mu}\left(\Gamma_{(g)\nu}\Gamma^\nu_{(g)}\right) = \partial_{;\mu}\Gamma_{(g)\nu}\Gamma^\nu_{(g)} + \Gamma_{(g)\nu}\partial_{;\mu}\Gamma^\nu_{(g)} = 0$ thus $-\partial_{;\mu}\Gamma_{(g)\nu}\Gamma^\nu_{(g)} = \Gamma_{(g)\nu}\partial_{;\mu}\Gamma^\nu_{(g)}$. So writing the derivative out fully and using $\Gamma_{(g)\nu} = e_\nu{}^a\gamma_a$ and $\partial_{;\mu}\Gamma^\nu_{(g)} = \partial_{;\mu}e^{\nu b}\gamma_b$, and with the Dirac covariants $\sigma_{ab} = \frac{i}{2}\left[\gamma_a\gamma_b - \gamma_b\gamma_a\right]$, we obtain:

$$\begin{aligned}\nabla_\mu &= \partial_\mu - \tfrac{i}{4}\omega_\mu^{ab}\sigma_{ab} = \partial_\mu + \tfrac{1}{8}\left(e_\nu{}^a\partial_{;\mu}e^{\nu b}\right)\left[\gamma_a\gamma_b - \gamma_b\gamma_a\right] = \partial_\mu + \tfrac{1}{8}\left[\Gamma_{(g)\nu}\partial_{;\mu}\Gamma^\nu_{(g)} - \partial_{;\mu}\Gamma^\nu_{(g)}\Gamma_{(g)\nu}\right] \\ &= \partial_\mu + \tfrac{1}{4}\Gamma_{(g)\nu}\partial_{;\mu}\Gamma^\nu_{(g)} = \partial_\mu + \tfrac{1}{4}\Gamma_{(g)\nu}\partial_\mu\Gamma^\nu_{(g)} + \tfrac{1}{4}\Gamma^\nu_{\sigma\mu}\Gamma_{(g)\nu}\Gamma^\sigma_{(g)}\end{aligned}, \tag{18.1}$$

where $\Gamma^\nu{}_{\sigma\mu} = \frac{1}{2}g^{\nu\beta}\left(\partial_\sigma g_{\mu\beta} + \partial_\mu g_{\beta\sigma} - \partial_\beta g_{\sigma\mu}\right)$ are the Christoffel connections. With the derivative written in this way, Dirac's free-fermion equation in curved spacetime now becomes:

$$\left(i\hbar c\Gamma^\sigma_{(g)}\nabla_\sigma - mc^2\right)\psi = \left(i\hbar c\Gamma^\sigma_{(g)}\partial_\sigma + \tfrac{1}{4}i\hbar c\Gamma^\sigma_{(g)}\Gamma_{(g)\nu}\partial_{;\sigma}\Gamma^\nu_{(g)} - mc^2\right)\psi = 0. \tag{18.2}$$

It is easier to calculate using this form of ∇_μ because the Lorentz indexes *a*, *b* are entirely hidden. Furthermore, we see that the factor of ¼ in (18.1) simply normalizes the $\frac{1}{4}\Gamma_{(g)\nu}\partial_{;\mu}\Gamma^\nu_{(g)}$ term to $\frac{1}{4}\Gamma_{(g)\nu}\Gamma^\nu_{(g)} = 1$ deduced just above. And, we see clearly why the spin connection containing $\partial_{;\sigma}\Gamma^\nu_{(g)} = \partial_\sigma\Gamma^\nu_{(g)} + \Gamma^\nu_{\sigma\tau}\Gamma^\tau_{(g)}$ rather than merely $\partial_\sigma\Gamma^\nu_{(g)}$ is needed to ensure that the derivative ∇_μ operates covariantly on ψ. Then, when we reduce to flat spacetime, $\partial_{;\sigma}\Gamma^\nu_{(g)}$ of course becomes $\partial_\sigma\Gamma^\nu_{(g)}$. But also, $\partial_\sigma\Gamma^\nu_{(g)} \rightarrow \partial_\sigma\gamma^\nu = 0$ because $e^\nu_b \rightarrow \delta^\nu_b$. As a consequence, the entire $\nabla_\sigma \rightarrow \partial_\sigma$ and Dirac's free-fermion equation reduces to the usual familiar $\left(i\hbar c\gamma^\sigma\partial_\sigma - mc^2\right)\psi = 0$.

With the foregoing in mind, we return to (17.3) and specifically the relation $G^{\mu\nu} = \mathrm{E}^\mu_y\mathrm{E}^\nu_z\eta^{yz}$ which is analogous to $g^{\mu\nu} = e^\mu_a e^\nu_b\eta^{ab}$ which was central to the review just concluded. Now, our objective is to obtain a spin connection analogous to that in $\partial_{;\mu}\left(e_\nu{}^a e^{\nu b}\right) = \partial_{;\mu}e_\nu{}^a e^{\nu b} + e_\nu{}^a\partial_{;\mu}e^{\nu b} = \partial_{;\mu}e_\nu{}^a e^{\nu b} + \omega_\mu^{ab}$. To focus for the moment simply on the electromagnetic tetrad, we set $g_{\mu\nu} = \eta_{\mu\nu}$ thus $e^\mu_a = \delta^\mu_a$ thus $G^{\mu\nu} = \eta^{\mu\nu} + \rho A^\mu_\gamma\rho A^\nu_\gamma = \varepsilon^\mu_y\varepsilon^\nu_z\eta^{yz}$, including the subscript γ to make clear that A^μ_γ is for individual photons. Similarly to the above, the Lorentz indexes are raised and lowered with η^{yz}, so we may write $G^{\mu\nu} = \eta^{\mu\nu} + \rho A^\mu_\gamma\rho A^\nu_\gamma = \varepsilon^\mu_y\varepsilon^{\nu y}$, i.e., $\varepsilon^\mu_y\varepsilon^{\nu y} = \eta^{\mu\nu} + \rho A^\mu_\gamma\rho A^\nu_\gamma$. This differs from the form of the earlier $e_{a}^{\mu}e^{\nu a}=g^{\mu\nu}$ because of the extra $\rho A_{\gamma}^{\mu}\rho A_{\gamma}^{\nu}$ term, so we cannot calculate the inverse in the same way. Instead, because $e_{\nu}^{\ a}e^{\nu b}$ is the starting point for the relation $\partial_{;\mu}\left(e_{\nu}^{\ a}e^{\nu b}\right)=\partial_{;\mu}e_{\nu}^{\ a}e^{\nu b}+e_{\nu}^{\ a}\partial_{;\mu}e^{\nu b}=\omega_{\mu}^{ba}+\omega_{\mu}^{ab}=0$, let us simply start by using (14.9) to explicitly construct $\varepsilon_{\nu}^{y}\varepsilon^{\nu z}$. As in the above we may use η_{yz} to operate on the Lorentz indexes and $g_{\mu\nu}=\eta_{\mu\nu}$ to operate on the spacetime indexes, so that $\varepsilon_{\nu}^{y}=\eta_{\mu\nu}\eta^{wy}\varepsilon_{w}^{\mu}$ and $\varepsilon^{\nu z}=\eta^{xz}\varepsilon_{x}^{\nu}$, thus, $\varepsilon_{\nu}^{y}\varepsilon^{\nu z}=\eta_{\mu\nu}\eta^{wy}\eta^{xz}\varepsilon_{w}^{\mu}\varepsilon_{x}^{\nu}$. After we do this explicit construction, we obtain $\varepsilon_{\nu}^{y}\varepsilon^{\nu z}=\eta^{yz}+\rho^{2}A_{\gamma}^{y}A_{\gamma}^{z}$, which is the inverse of $\varepsilon_{y}^{\mu}\varepsilon^{\nu y}=\eta^{\mu\nu}+\rho A_{\gamma}^{\mu}\rho A_{\gamma}^{\nu}$. So the inverse again swaps Lorentz and spacetime indexes, but there is now an extra term with $\rho^{2}A_{\gamma}^{y}A_{\gamma}^{z}$.

As a consequence, using the *ordinary* derivative because we are presently considering $g_{\mu\nu}=\eta_{\mu\nu}$ thus $\partial_{;\sigma}=\partial_{\sigma}$, we obtain:

$$\partial_{\sigma}\left(\varepsilon_{\nu}^{y}\varepsilon^{\nu z}\right)=\partial_{\sigma}\varepsilon_{\nu}^{y}\varepsilon^{\nu z}+\varepsilon_{\nu}^{y}\partial_{\sigma}\varepsilon^{\nu z}=\rho^{2}\partial_{\sigma}A_{\gamma}^{y}A_{\gamma}^{z}+\rho^{2}A_{\gamma}^{y}\partial_{\sigma}A_{\gamma}^{z}. \tag{18.3}$$

We may rewrite this to define an electromagnetic tetrad spin connection by:

$$\Omega_{\sigma}^{yz}\equiv\varepsilon_{\nu}^{y}\partial_{\sigma}\varepsilon^{\nu z}-\rho^{2}A_{\gamma}^{y}\partial_{\sigma}A_{\gamma}^{z}=\varepsilon_{\nu}^{z}\partial_{\sigma}\varepsilon^{\nu y}+\rho^{2}A_{\gamma}^{z}\partial_{\sigma}A_{\gamma}^{y}=-\Omega_{\sigma}^{zy}. \tag{18.4}$$

This is naturally antisymmetric in the Lorentz indexes, $\Omega_{\sigma}^{yz}=-\Omega_{\sigma}^{zy}$, but only with the extra term $\rho^{2}A_{\gamma}^{y}\partial_{\sigma}A_{\gamma}^{z}$ included.

Contrasting (18.1), we then define a spin-covariant derivative for the electromagnetic tetrads by $\nabla_{\sigma}\equiv\partial_{\sigma}-\frac{i}{4}\Omega_{\sigma}^{yz}\sigma_{yz}$. Using $\Gamma_{(e)\nu}=\varepsilon_{\nu}^{y}\gamma_{y}$ and $\sigma_{ab}=\frac{i}{2}\left[\gamma_{a},\gamma_{b}\right]$, this is rewritten as:

$$\begin{aligned}\nabla_{\sigma}&\equiv\partial_{\sigma}-\tfrac{i}{4}\Omega_{\sigma}^{yz}\sigma_{yz}=\partial_{\sigma}+\tfrac{1}{8}\left(\varepsilon_{\nu}^{y}\partial_{\sigma}\varepsilon^{\nu z}-\rho^{2}A_{\gamma}^{y}\partial_{\sigma}A_{\gamma}^{z}\right)\left[\gamma_{y}\gamma_{z}-\gamma_{z}\gamma_{y}\right]\\&=\partial_{\sigma}+\tfrac{1}{8}\left[\varepsilon_{\nu}^{y}\gamma_{y}\partial_{\sigma}\varepsilon^{\nu z}\gamma_{z}-\partial_{\sigma}\varepsilon^{\nu z}\gamma_{z}\varepsilon_{\nu}^{y}\gamma_{y}\right]-\tfrac{1}{8}\rho^{2}\gamma_{y}\gamma_{z}\left[A_{\gamma}^{y}\partial_{\sigma}A_{\gamma}^{z}-A_{\gamma}^{z}\partial_{\sigma}A_{\gamma}^{y}\right]\\&=\partial_{\sigma}+\tfrac{1}{8}\left[\Gamma_{(e)\nu}\partial_{\sigma}\Gamma_{(e)}^{\nu}-\partial_{\sigma}\Gamma_{(e)}^{\nu}\Gamma_{(e)\nu}\right]-\tfrac{1}{8}\rho^{2}\gamma_{y}\gamma_{z}\left[-\partial_{\sigma}A_{\gamma}^{y}A_{\gamma}^{z}+\partial_{\sigma}A_{\gamma}^{z}A_{\gamma}^{y}\right]\\&=\partial_{\sigma}+\tfrac{1}{4}\Gamma_{(e)\nu}\partial_{\sigma}\Gamma_{(e)}^{\nu}-\tfrac{1}{8}i\rho^{2}\gamma_{y}\gamma_{z}q_{\sigma}\left[A_{\gamma}^{y},A_{\gamma}^{z}\right]/\hbar=\partial_{\sigma}+\tfrac{1}{4}\Gamma_{(e)\nu}\partial_{\sigma}\Gamma_{(e)}^{\nu}\end{aligned}. \tag{18.5}$$

The final result, $\nabla_{\sigma}=\partial_{\sigma}+\frac{1}{4}\Gamma_{(e)\nu}\partial_{\sigma}\Gamma_{(e)}^{\nu}$, has identical form to (18.1), except that $\frac{1}{4}\Gamma_{(e)\nu}\partial_{\sigma}\Gamma_{(e)}^{\nu}$ contains an ordinary derivative because at the moment we are using $g_{\mu\nu}=\eta_{\mu\nu}$. To reduce in the above, just as before (18.1) we may deduce that $G_{\nu}^{\ \nu}=\Gamma_{(e)\nu}\Gamma_{(e)}^{\nu}=\delta_{\nu}^{\ \nu}+\rho A_{\gamma\nu}\rho A_{\gamma}^{\nu}=4$ using (17.3) with $g_{\mu\nu}=\eta_{\mu\nu}$ and applying $A_{\gamma}^{\sigma}A_{\gamma\sigma}=0$ from (14.8). Therefore $-\partial_{\sigma}\Gamma_{(e)}^{\nu}\Gamma_{(e)\nu}=\Gamma_{(e)\nu}\partial_{\sigma}\Gamma_{(e)}^{\nu}$. Additionally we apply $\partial_{\sigma}\left(A^{y}A^{z}\right)=\partial_{\sigma}A_{\gamma}^{y}A_{\gamma}^{z}+A_{\gamma}^{y}\partial_{\sigma}A_{\gamma}^{z}$ and the fact that photon vectors are commuting $\left[A_\gamma^y, A_\gamma^z\right]=0$ because electrodynamics is an abelian gauge theory. So although Ω_σ^{yz} in (18.4) has an extra term $-\rho^2 A_\gamma^y \partial_\sigma A_\gamma^z$ not contained in the gravitational $\omega_\mu^{ab}=e_\nu^a \partial_{;\mu} e^{\nu b}$, this term washes out from ∇_σ in (18.5) owing – not to any of the zero relations in (15.5) – but to electrodynamics being an abelian gauge theory.

With this we return first to (13.6) because that applies to electromagnetism absent gravitation, and we advance $\partial_\sigma \mapsto \nabla_\sigma$ using (18.5), thus obtaining:

$$\left(\left(\Gamma_{(\varepsilon)}^\sigma+\frac{q}{mc^2}A_\gamma^\sigma\right)i\hbar c\nabla_\sigma-mc^2\right)|\Psi\rangle=\left(\left(\Gamma_{(\varepsilon)}^\sigma+\frac{q}{mc^2}A_\gamma^\sigma\right)i\hbar c\left(\partial_\sigma+\frac{1}{4}\Gamma_{(e)\nu}\partial_\sigma\Gamma_{(e)}^\nu\right)-mc^2\right)|\Psi\rangle=0\,. \quad (18.6)$$

To then broaden this to apply to gravitation, we merely generalize $\eta_{\mu\nu}\mapsto g_{\mu\nu}$. Via the minimal coupling principle we simultaneously generalize $\Gamma_{(\varepsilon)}^a=\varepsilon_y^a\gamma^y$ back to $\Gamma^\mu=\mathrm{E}_y^\mu\gamma^y=e_a^\mu\varepsilon_y^a\gamma^y$ using (17.2) and $G^{\mu\nu}=\eta^{\mu\nu}+\rho A_\gamma^\mu \rho A_\gamma^\nu=\varepsilon_y^\mu\varepsilon_z^\nu\eta^{yz}$ back to $G^{\mu\nu}=g^{\mu\nu}+\rho A_\gamma^\mu\rho A_\gamma^\nu=\mathrm{E}_y^\mu\mathrm{E}_z^\nu\eta^{yz}$ using (17.3). And finally, we turn the ordinary derivative in $\partial_\sigma\Gamma_{(e)}^\nu$ into a covariant derivative of the form $\partial_\sigma\Gamma_{(e)}^\nu\mapsto\partial_{;\sigma}\Gamma^\nu=\partial_\sigma\Gamma^\nu+\Gamma_{\mu\sigma}^\nu\Gamma^\mu$ because the now spacetime is curved. Therefore, the spin-covariant derivative (18.5) becomes:

$$\nabla_\sigma=\partial_\sigma+\tfrac{1}{4}\Gamma_\nu\partial_{;\sigma}\Gamma^\nu=\partial_\sigma+\tfrac{1}{4}\Gamma_\nu\partial_\sigma\Gamma^\nu+\tfrac{1}{4}\Gamma_{\mu\sigma}^\nu\Gamma_\nu\Gamma^\mu\,. \quad (18.7)$$

Finally, the complete hyper-canonical Dirac equation with gravitation and spin connection is:

$$\boxed{\left(i\hbar c\left(\Gamma^\sigma+\frac{q}{mc^2}A_\gamma^\sigma\right)\nabla_\sigma-mc^2\right)|\Psi\rangle=\left(i\hbar c\left(\Gamma^\sigma+\frac{q}{mc^2}A_\gamma^\sigma\right)\left(\partial_\sigma+\frac{1}{4}\Gamma_\nu\partial_{;\sigma}\Gamma^\nu\right)-mc^2\right)|\Psi\rangle=0}\,. \quad (18.8)$$

Using the relations $|\Psi\rangle\equiv\exp\left(-iH_\sigma x^\sigma/\hbar c\right)|U_0\rangle$ and $i\hbar c\partial_\sigma|\Psi\rangle=cp_\sigma|\Psi\rangle$ from prior to (13.6), to parallel (18.7) we may also define a spin-covariant momentum:

$$c\Pi_\sigma\equiv cp_\sigma+\tfrac{1}{4}i\hbar c\Gamma_\nu\partial_{;\sigma}\Gamma^\nu=cp_\sigma+\tfrac{1}{4}i\hbar c\Gamma_\nu\partial_\sigma\Gamma^\nu+\tfrac{1}{4}i\hbar c\Gamma_{\mu\sigma}^\nu\Gamma_\nu\Gamma^\mu\,, \quad (18.9)$$

and then convert (18.8) into momentum space to arrive at:

$$\boxed{\left(\left(\Gamma^\sigma+\frac{q}{mc^2}A_\gamma^\sigma\right)c\Pi_\sigma-mc^2\right)|U_0\rangle=\left(\left(\Gamma^\sigma+\frac{q}{mc^2}A_\gamma^\sigma\right)\left(cp_\sigma+\frac{1}{4}i\hbar c\Gamma_\nu\partial_{;\sigma}\Gamma^\nu\right)-mc^2\right)|U_0\rangle=0}\,. \quad (18.10)$$

Respectively, (18.8) and (18.10) are hyper-canonical Dirac equations with electrodynamics and gravitation, in configuration and momentum space, with Lorentz indexes hidden in $\Gamma_\nu=\mathrm{E}_\nu^y\gamma_y$.

If we contrast (18.8) to the usual Dirac equation $\left(i\hbar c\Gamma^{\sigma}_{(g)}\left(\partial_\sigma+\frac{1}{4}\Gamma_{(g)\nu}\partial_{;\sigma}\Gamma^{\nu}_{(g)}\right)-mc^2\right)\psi=0$ with gravitation, there are two main differences: First, in the usual Dirac equation $\Gamma^{\nu}_{(g)}=e^{\nu}_{a}\gamma^{a}$ couples only to the gravitational e^{ν}_{a}, whereas $\Gamma^{\nu}=\mathrm{E}^{\nu}_{y}\gamma^{y}=e^{\nu}_{a}\varepsilon^{a}_{y}\gamma^{y}$ in (18.8) contains both the electromagnetic ε^{a}_{y} derived in (14.9) and the gravitational e^{ν}_{a}. Thus, we replace $\Gamma^{\nu}_{(g)}\mapsto\Gamma^{\nu}$. Second, in the usual Dirac equation $\Gamma^{\sigma}_{(g)}$ stands alone contracting with $\nabla_\sigma=\partial_\sigma+\frac{1}{4}\Gamma_{(g)\nu}\partial_{;\sigma}\Gamma^{\nu}_{(g)}$, while in the hyper-canonical Dirac equation we find that $\Gamma^{\sigma}_{(g)}\mapsto\Gamma^{\sigma}+qA^{\sigma}_{\gamma}/mc^2$, adding an extra $qA^{\sigma}_{\gamma}/mc^2$ term. This is reminiscent (and in fact yet another downstream consequence) of how $\partial_\mu\mapsto\mathfrak{D}_\mu=\partial_\mu-iqA_\mu/\hbar c$ and $p^\mu\mapsto\pi^\mu=p^\mu+qA^\mu/c$ as a result of Weyl's Local U(1) Gauge Symmetry, as reviewed in section 1. These (18.8) and (18.10) are now in a form enabling Hamiltonian calculations to be carried out as simply as possible, which is our next undertaking.

PART IV: THE HYPER-CANONCIAL DIRAC HAMILTONIAN: MAGNETIC MOMENT ANOMALIES WITHOUT RENORMALIZATION

19. Preparing the Hyper-Canonical Dirac Equation for Calculating the Hamiltonian

To obtain the Dirac Hamiltonian, we start with (18.10) which is in momentum space. Because our interest is in the electrodynamic Hamiltonian and particularly showing how (18.10) naturally contains the magnetic moment anomaly obviating any need for renormalization, we shall eliminate gravitation and work in flat spacetime by setting $g_{\mu\nu}=\eta_{\mu\nu}$. This also means that we replace $\Gamma^{\nu}=\mathrm{E}^{\nu}_{y}\gamma^{y}=e^{\nu}_{a}\varepsilon^{a}_{y}\gamma^{y}$ with $\Gamma^{\nu}_{(e)}=\varepsilon^{\nu}_{y}\gamma^{y}$ because $e^{\nu}_{a}=\delta^{\nu}_{a}$, and that we replace $\partial_{;\sigma}\Gamma^{\nu}$ with the ordinary derivative $\partial_\sigma\Gamma^{\nu}$ because in flat spacetime the connections $\Gamma^{\nu}_{\mu\sigma}=0$. With these changes, and using $\rho=q/mc^2$ for compactness, (18.10) becomes:

$$\left(\left(\Gamma^{\sigma}_{(e)}+\rho A^{\sigma}_{\gamma}\right)c\Pi_{(e)\sigma}-mc^2\right)\left|U_0\right\rangle=\left(\left(\Gamma^{\sigma}_{(e)}+\rho A^{\sigma}_{\gamma}\right)\left(cp_\sigma+\frac{1}{4}i\hbar c\Gamma_{(e)\nu}\partial_\sigma\Gamma^{\nu}_{(e)}\right)-mc^2\right)\left|U_0\right\rangle=0. \qquad (19.1)$$

This will be our starting point for extracting the Hamiltonian, and it includes the spin connection term $\frac{1}{4}i\hbar c\Gamma_{(e)\nu}\partial_\sigma\Gamma^{\nu}_{(e)}$ which descends from $\frac{1}{4}i\hbar c\Gamma_{\nu}\partial_{;\sigma}\Gamma^{\nu}$ in (18.10) and does not disappear, in contrast to $\partial_\sigma\Gamma^{\nu}_{(g)}\to\partial_\sigma\gamma^{\nu}=0$ as reviewed after (18.2). This is because $\partial_{;\sigma}\Gamma^{\nu}\to\partial_\sigma\Gamma^{\nu}_{(e)}\neq 0$ here. As we shall see, this spin connection term is central to how (19.1) obviates renormalization. We have added the γ subscript to make clear that A^{σ}_{γ} is (15.3) for an individual photon

First, we obtain the four components of $\Gamma^{\sigma}_{(\varepsilon)}=\varepsilon^{\sigma}_{y}\gamma^{y}$ using ε^{σ}_{y} derived in (14.9), which in retrospective view of sections 15 and 16 contains what we now label as A^{k}_{γ}, as such:

$$\begin{aligned}\Gamma^0_{(\varepsilon)} &= \varepsilon^0_y\gamma^y = \varepsilon^0_0\gamma^0+\varepsilon^0_1\gamma^1+\varepsilon^0_2\gamma^2+\varepsilon^0_3\gamma^3 = \gamma^0-\rho A^1_\gamma\gamma^1-\rho A^2_\gamma\gamma^2-\rho A^3_\gamma\gamma^3=\gamma^0-\rho A^k_\gamma\gamma^k\\ \Gamma^1_{(\varepsilon)} &= \varepsilon^1_y\gamma^y=\varepsilon^1_0\gamma^0+\varepsilon^1_1\gamma^1=\gamma^1-\rho A^1_\gamma\gamma^0\\ \Gamma^2_{(\varepsilon)} &= \varepsilon^2_y\gamma^y=\varepsilon^2_0\gamma^0+\varepsilon^2_2\gamma^2=\gamma^2-\rho A^2_\gamma\gamma^0\\ \Gamma^3_{(\varepsilon)} &= \varepsilon^3_y\gamma^y=\varepsilon^3_0\gamma^0+\varepsilon^3_3\gamma^3=\gamma^3-\rho A^3_\gamma\gamma^0\end{aligned} \quad . \tag{19.2}$$

This may be consolidated into:

$$\Gamma^\sigma_{(\varepsilon)}=\begin{pmatrix}\Gamma^0_{(\varepsilon)} & \Gamma^k{}_{(\varepsilon)}\end{pmatrix}=\begin{pmatrix}\gamma^0-\rho A^k_\gamma\gamma^k & \gamma^k-\rho A^k_\gamma\gamma^0\end{pmatrix}=\begin{pmatrix}\Gamma^0_{(\varepsilon)} & \boldsymbol{\Gamma}_{(\varepsilon)}\end{pmatrix}=\begin{pmatrix}\gamma^0-\rho\mathbf{A}_\gamma\cdot\boldsymbol{\gamma} & \boldsymbol{\gamma}-\rho\mathbf{A}_\gamma\gamma^0\end{pmatrix}. \tag{19.3}$$

Let us also write down the γ^μ in the in the Dirac representation, which are:

$$\gamma^0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix};\quad \gamma^k=\boldsymbol{\gamma}=\begin{pmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix} \tag{19.4}$$

It will be helpful at various times in the upcoming calculations to make use of the Dirac relation $\gamma^j b^j\gamma^k c^k=(\boldsymbol{\gamma}\cdot\mathbf{b})(\boldsymbol{\gamma}\cdot\mathbf{c})=-I_{2\times2}(\boldsymbol{\sigma}\cdot\mathbf{b})(\boldsymbol{\sigma}\cdot\mathbf{c})$ easily apparent from (19.4) , where *I* is a unit matrix. Furthermore, the Pauli matrices satisfy the identity $(\boldsymbol{\sigma}\cdot\mathbf{b})(\boldsymbol{\sigma}\cdot\mathbf{c})=\mathbf{b}\cdot\mathbf{c}+i\boldsymbol{\sigma}\cdot(\mathbf{b}\times\mathbf{c})$ so that this Dirac relation becomes $(\boldsymbol{\gamma}\cdot\mathbf{b})(\boldsymbol{\gamma}\cdot\mathbf{c})=-I_{2\times2}\left(I_{2\times2}\mathbf{b}\cdot\mathbf{c}+i\boldsymbol{\sigma}\cdot(\mathbf{b}\times\mathbf{c})\right)$. In the special case where $\mathbf{b}=\mathbf{c}$ are the same vector and *not* sums of independent vectors (so that $\mathbf{b}\times\mathbf{b}=0$), this means that $(\boldsymbol{\gamma}\cdot\mathbf{b})^2(\boldsymbol{\gamma}\cdot\mathbf{b})=-I_{4\times4}\mathbf{b}^2$. And in the further special case where $\mathbf{b}=\mathbf{A}_\gamma$ is the photon three-vector potential for which $\mathbf{A}^2_\gamma=0$ via (14.8), we find that $(\boldsymbol{\gamma}\cdot\mathbf{A}_\gamma)^2=0$.

With this in mind we use (19.3) to calculate the spin connection term in (19.1), and find:

$$\begin{aligned}&\tfrac14 i\hbar c\Gamma_{(e)\nu}\partial_\sigma\Gamma^\nu_{(e)}=\tfrac14 i\hbar c\eta_{\mu\nu}\Gamma^\mu_{(e)}\partial_\sigma\Gamma^\nu_{(e)}=\tfrac14 i\hbar c\left(\Gamma^0_{(e)}\partial_\sigma\Gamma^0_{(e)}-\Gamma^k_{(e)}\partial_\sigma\Gamma^k_{(e)}\right)\\ &=\tfrac14 i\hbar c\left(\left(\gamma^0-\rho A^k_\gamma\gamma^k\right)\partial_\sigma\left(\gamma^0-\rho A^l_\gamma\gamma^l\right)-\left(\gamma^k-\rho A^k_\gamma\gamma^0\right)\partial_\sigma\left(\gamma^k-\rho A^k_\gamma\gamma^0\right)\right).\\ &=\tfrac14 i\hbar c\left(-\gamma^0\partial_\sigma\rho A^l_\gamma\gamma^l+\gamma^k\partial_\sigma\rho A^k_\gamma\gamma^0\right)=\tfrac12\gamma^k\gamma^0 cq_\sigma\rho A^k_\gamma\end{aligned} \tag{19.5}$$

In the above, we have reduced using $\partial_\sigma\gamma^\mu=0$, $\mathbf{A}_\gamma{}^2=0$, $(\boldsymbol{\gamma}\cdot\mathbf{A}_\gamma)^2=0$, $i\hbar\partial_\sigma A^\mu_\gamma=q_\sigma A^\mu_\gamma$ from (15.4), and $\gamma^k\gamma^0=-\gamma^0\gamma^k$.

The hyper-canonical Dirac equation (19.1) contains (19.5) in the form of $\left(\Gamma^\sigma_{(e)}+\rho A^\sigma_\gamma\right)\tfrac14 i\hbar c\Gamma_{(e)\nu}\partial_\sigma\Gamma^\nu_{(e)}$. However, $\rho A^\sigma_\gamma\tfrac14 i\hbar c\Gamma_{(e)\nu}\partial_\sigma\Gamma^\nu_{(e)}=\rho A^\sigma_\gamma\tfrac12\gamma^k\gamma^0 cq_\sigma\rho A^k_\gamma=0$ using (19.5) combined with $q_\sigma A^\sigma_\gamma=0$ from (15.5). Therefore, from (19.5) and again using (19.3) we calculate:

$$\begin{aligned}
&\left(\Gamma^{\sigma}_{(e)}+\rho A^{\sigma}_{\gamma}\right)\tfrac{1}{4}i\hbar c\Gamma_{(e)\nu}\partial_{\sigma}\Gamma^{\nu}_{(e)}=\tfrac{1}{2}\Gamma^{\sigma}_{(e)}\gamma^{k}\gamma^{0}cq_{\sigma}\rho A^{k}_{\gamma}=\tfrac{1}{2}\eta_{\mu\nu}\Gamma^{\mu}_{(e)}\gamma^{k}\gamma^{0}cq^{\nu}\rho A^{k}_{\gamma}\\
&=\tfrac{1}{2}\Gamma^{0}_{(e)}\gamma^{k}\gamma^{0}cq^{0}\rho A^{k}_{\gamma}-\tfrac{1}{2}\Gamma^{j}_{(e)}\gamma^{k}\gamma^{0}cq^{j}\rho A^{k}_{\gamma}\\
&=\tfrac{1}{2}\left(\gamma^{0}-\rho A^{j}_{\gamma}\gamma^{j}\right)\gamma^{k}\gamma^{0}cq^{0}\rho A^{k}_{\gamma}-\tfrac{1}{2}\left(\gamma^{j}-\rho A^{j}_{\gamma}\gamma^{0}\right)\gamma^{k}\gamma^{0}cq^{j}\rho A^{k}_{\gamma}\\
&=\tfrac{1}{2}\left(-\gamma^{k}cq^{0}\rho A^{k}_{\gamma}-\gamma^{0}\gamma^{j}\gamma^{k}cq^{j}\rho A^{k}_{\gamma}\right)\\
&=-\tfrac{1}{2}\left(cq^{0}\left(\boldsymbol{\gamma}\cdot\rho\mathbf{A}_{\gamma}\right)+\gamma^{0}\left(\boldsymbol{\gamma}\cdot c\mathbf{q}\right)\left(\boldsymbol{\gamma}\cdot\rho\mathbf{A}_{\gamma}\right)\right)
\end{aligned} \quad . \tag{19.6}$$

To reduce the above, we use $\gamma^k\gamma^0=-\gamma^0\gamma^k$, $\gamma^0\gamma^0=1$, $\left(\boldsymbol{\gamma}\cdot\mathbf{A}_\gamma\right)^2=0$, and $\mathbf{q}\cdot\mathbf{A}_\gamma=0$ from (15.5).

We therefore see that the spin connection contributes two additional terms that would be absent from (19.1) if it did not include the spin connection and had been left as is at (17.5). Given that the photon energy momentum vector $cq^\mu=\left(h\nu,c\mathbf{q}\right)$ where $cq^0=h\nu$ is the photon energy, the first spin term $-\tfrac{1}{2}cq^0\left(\boldsymbol{\gamma}\cdot\rho\mathbf{A}_\gamma\right)=-\tfrac{1}{2}h\nu\left(\boldsymbol{\gamma}\cdot\rho\mathbf{A}_\gamma\right)$ places the photon energy directly into the momentum space Dirac equation. As to the second term, we may use the identity $\left(\boldsymbol{\gamma}\cdot\mathbf{b}\right)\left(\boldsymbol{\gamma}\cdot\mathbf{c}\right)=-\left(\mathbf{b}\cdot\mathbf{c}+i\boldsymbol{\sigma}\cdot\left(\mathbf{b}\times\mathbf{c}\right)\right)$ (with I matrices implicit) as well as $i\hbar\partial_\sigma A^\mu_\gamma=q_\sigma A^\mu_\gamma$ from (15.4) and $\mathbf{q}\cdot\mathbf{A}_\gamma=0$ from (15.5) and the field strength $F_\gamma^{ij}=\partial^i A^j_\gamma-\partial^j A^i_\gamma$ to deduce that:

$$\begin{aligned}
&-\tfrac{1}{2}\gamma^{0}\left(\boldsymbol{\gamma}\cdot c\mathbf{q}\right)\left(\boldsymbol{\gamma}\cdot\rho\mathbf{A}_{\gamma}\right)=\tfrac{1}{2}i\gamma^{0}\boldsymbol{\sigma}\cdot\left(c\mathbf{q}\times\rho\mathbf{A}\gamma\right)=\tfrac{1}{2}i\gamma^{0}\varepsilon^{ijk}\sigma^{i}cq^{j}\times\rho A^{k}_{\gamma}=-\tfrac{1}{2}\hbar\gamma^{0}\varepsilon^{ijk}\sigma^{i}c\partial^{j}\rho A^{k}_{\gamma}\\
&=-\tfrac{1}{2}\gamma^{0}\left(\hbar\sigma^{1}\left(c\partial^{2}\rho A^{3}_{\gamma}-c\partial^{3}\rho A^{2}_{\gamma}\right)+\hbar\sigma^{2}\left(c\partial^{3}\rho A^{1}_{\gamma}-c\partial^{1}\rho A^{3}_{\gamma}\right)+\hbar\sigma^{3}\left(c\partial^{1}\rho A^{2}_{\gamma}-c\partial^{2}\rho A^{1}_{\gamma}\right)\right)\\
&=-\tfrac{1}{2}\gamma^{0}\hbar c\rho\left(\sigma^{1}F_{\gamma}^{23}+\sigma^{2}F_{\gamma}^{31}+\sigma^{3}F_{\gamma}^{12}\right)=\tfrac{1}{2}\gamma^{0}\hbar c\rho\left(\sigma^{1}B^{1}_{\gamma}+\sigma^{2}B^{2}_{\gamma}+\sigma^{3}B^{3}_{\gamma}\right)=\tfrac{1}{2}\gamma^{0}\hbar c\rho\boldsymbol{\sigma}\cdot\mathbf{B}_{\gamma}
\end{aligned} \quad . \tag{19.7}$$

The above also embeds $i\boldsymbol{\sigma}\cdot\left(\mathbf{q}\times\mathbf{A}_\gamma\right)=\hbar\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\right)=\hbar\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$, i.e., $\mathbf{B}_\gamma=\nabla\times\mathbf{A}_\gamma$ or $i\left(\mathbf{q}\times\mathbf{A}_\gamma\right)=\hbar\mathbf{B}_\gamma$. Because $\mathbf{B}_\gamma$ is the curl of $\mathbf{A}_\gamma$, this is, at present, the magnetic field of the individual photon. But because of the gauge symmetry of **B** reviewed at (15.8), the *form* in which **B** enters the Dirac Hamiltonian will be unchanged whether this is $\mathbf{B}_\gamma$ for a single photon, $\mathbf{B}_\mathrm{c}$ for an external classical magnetic field, or the total $\mathbf{B}=\mathbf{B}_\mathrm{c}+\mathbf{B}_\gamma\cong\mathbf{B}_\mathrm{c}$ which adds the classical and photon magnetic fields. So, at a suitable time we will be able to follow the same steps that took us from (16.11) to (16.13) and transform this into an external magnetic field, and thus transform and interpret $\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\rightarrow\boldsymbol{\sigma}\cdot\mathbf{B}$ as part of the fermion magnetic moment in an external magnetic field.

Now, we now replace the substitute variable $\rho=q/mc^2$ with the actual charge $q=-e$ of the charged leptons. Then we define a triplet of spin matrices $\mathbf{S}\equiv\tfrac{1}{2}\hbar\boldsymbol{\sigma}$ and the Dirac g-factor $g_D=2$ in the usual way, so that

$$\mu_D = -\frac{\hbar e}{2mc}\boldsymbol{\sigma} = -\mu_B\boldsymbol{\sigma} = -2\frac{e}{2mc}\mathbf{S} = -g_D\frac{e}{2mc}\mathbf{S} = -\frac{g_D}{2}\frac{\hbar e}{2mc}\boldsymbol{\sigma} = -\frac{g_D}{2}\mu_B\boldsymbol{\sigma} \tag{19.8}$$

is the $g_D = 2$ Dirac *g*-factor and $\mu_B = \hbar e / 2mc$ is the Bohr magneton. Now (19.7) becomes:

$$-\tfrac{1}{2}\gamma^0\left(\gamma\cdot c\mathbf{q}\right)\left(\gamma\cdot\rho\mathbf{A}_\gamma\right) = \tfrac{1}{2}\gamma^0\hbar c\rho\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma = -\gamma^0\frac{\hbar ce}{2mc^2}\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma = -\gamma^0\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma. \tag{19.9}$$

It is very important to keep in mind that $\mu_D\cdot\mathbf{B} = -\left(g_D/2\right)\mu_B\boldsymbol{\sigma}\cdot\mathbf{B} = -\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ is the Dirac magnetic moment for a Dirac *g*-factor $g_D/2 = 1$. This means that the magnetic moment *anomaly* would have to arise from some $g/2 \neq 1$ slightly larger than 1 being a coefficient of $\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$.

We then use all of this in (19.6) to finally obtain:

$$\left(\Gamma^\sigma_{(e)} + \rho A^\sigma_\gamma\right)\tfrac{1}{4}i\hbar c\Gamma_{(e)\nu}\partial_\sigma\Gamma^\nu_{(e)} = -\gamma^0\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma - \tfrac{1}{2}h\nu\left(\gamma\cdot\rho\mathbf{A}_\gamma\right) = \gamma^\mu\Sigma_\mu = \eta_{\mu\nu}\gamma^\mu\Sigma^\nu, \tag{19.10}$$

where, using $\rho = q/mc^2 = -e/mc^2$ and $\mu_B = \hbar e/2mc$ to write (15.11) as $\tfrac{1}{2}h\nu\rho\mathbf{A}_\gamma = i\mu_B\mathbf{E}_\gamma$, we define a spin-connection four-vector:

$$\Sigma^\mu \equiv \begin{pmatrix}-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma & \tfrac{1}{2}h\nu\rho\mathbf{A}_\gamma\end{pmatrix} = \mu_B\begin{pmatrix}-\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma & i\mathbf{E}_\gamma\end{pmatrix}. \tag{19.11}$$

It will become of importance that the magnetic moment $-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$ and the electric field $i\mu_B\mathbf{E}_\gamma$ transform as the time and space components of a four-vector. Because of $\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$, this implicitly contains a 2x2 matrix, and $\gamma^\mu\Sigma_\mu = \gamma^\mu I_{2\times 2}\Sigma_\mu$.

Returning to (19.1) and inserting (19.10) now produces the rather simplified:

$$0 = \left(\left(\Gamma^\sigma_{(e)} + \rho A^\sigma_\gamma\right)cp_\sigma + \gamma^\sigma\Sigma_\sigma - mc^2\right)\left|U_0\right\rangle. \tag{19.12}$$

As the final step prior extracting the Hamiltonian, we raise an index using $g_{\mu\nu} = \eta_{\mu\nu}$ and substitute (19.3) into the above. We also reduce using $A^0 = 0$ from (14.8) and insert the fermion energy-momentum $cp^\mu = \left(E, c\mathbf{p}\right)$ and the spin connection vector (19.11), so the above becomes:

$$\begin{aligned}0 &= \left(\Gamma^0_{(e)}cp^0 - \Gamma^k_{(e)}cp^k - \rho A^k_\gamma cp^k + \gamma^0\Sigma^0 - \gamma^k\Sigma^k - mc^2\right)\left|U_0\right\rangle \\ &= \left(\gamma^0 cp^0 - \gamma^k cp^k - \gamma^k\rho A^k_\gamma cp^0 + \left(\gamma^0 - 1\right)\rho A^k_\gamma cp^k + \gamma^0\Sigma^0 - \gamma^k\Sigma^k - mc^2\right)\left|U_0\right\rangle \\ &= \left(\gamma^0\left(cp^0 + \Sigma^0\right) - mc^2 - \gamma^k\left(cp^k + \rho A^k_\gamma cp^0 + \Sigma^k\right) + \left(\gamma^0 - 1\right)\rho A^k_\gamma cp^k\right)\left|U_0\right\rangle \\ &= \left(\gamma^0\left(E + \rho\mathbf{A}_\gamma\cdot c\mathbf{p} - \mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right) - \gamma\cdot\left(c\mathbf{p} + E\rho\mathbf{A}_\gamma + i\mu_B\mathbf{E}_\gamma\right) - \left(mc^2 + \rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)\right)\left|U_0\right\rangle\end{aligned}. \tag{19.13}$$

The term $c\mathbf{p}+\rho\mathbf{A}_\gamma E$ within the above reveals why we chose to use a minus sign rather than a plus sign back at (14.5) though either choice seemed permissible: Restoring $\rho = q/mc^2$, this term becomes $c\mathbf{p}+\rho\mathbf{A}_\gamma E = c\mathbf{p}+\left(E/mc^2\right)q\mathbf{A}_\gamma$. Now, from prior to (1.4), the usual canonical momentum $\pi^\mu = p^\mu + qA^\mu / c$ has space components $c\boldsymbol{\pi} = c\mathbf{p}+q\mathbf{A}$. So in the limiting case where $E/mc^2 = 1$ we have $c\mathbf{p}+\rho\mathbf{A}_\gamma E = c\mathbf{p}+q\mathbf{A}_\gamma = c\boldsymbol{\pi}_\gamma$, which is a single-photon canonical momentum. Note that $\rho\mathbf{A}_\gamma\cdot c\mathbf{p}$ represents a single interaction between a single fermion and a single photon, and as a scalar product, accounts for the *spatial angle* of this interaction.

So, to further simplify (19.13) and highlight this canonical momentum relation, for the terms having a scaler product with $\boldsymbol{\gamma}$ we define a "hyper-canonical momentum" vector:

$$c\boldsymbol{\Pi} = c\Pi^k \equiv c\mathbf{p}+E\rho\mathbf{A}_\gamma + i\mu_B\mathbf{E}_\gamma = cp + E\rho A_\gamma^k + i\mu_B E_\gamma^k . \tag{19.14}$$

In the limit $E \to mc^2$ and with $\mathbf{E}_\gamma = 0$ this reduces $\boldsymbol{\Pi}\to\boldsymbol{\pi}_\gamma$ to the U(1) canonical momentum using a single photon. Additionally, for the terms multiplied by γ^0 in (19.13) we define a "hyper-canonical energy":

$$\mathrm{E}_{(2\times 2)} \equiv E+\rho\mathbf{A}_\gamma\cdot c\mathbf{p} - \mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma \tag{19.15}$$

which because of $\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$ is also a 2x2 matrix. And, for the terms which have no γ^0 at all, we define a "hyper-canonical rest energy":

$$\mathrm{M}c^2 \equiv mc^2 + \rho\mathbf{A}_\gamma\cdot c\mathbf{p} . \tag{19.16}$$

It will be appreciated from (19.13) that (19.15) and (19.16) transform respectively as the time and space components of a four-vector in spacetime. Therefore, we additionally define a "hyper-canonical energy-momentum" vector and also employ (19.11) as follows:

$$\begin{aligned} c\mathrm{P}^\mu &\equiv \begin{pmatrix}\mathrm{E} & c\Pi_x & c\Pi_y & c\Pi_z\end{pmatrix} = \begin{pmatrix}\mathrm{E} & c\boldsymbol{\Pi}\end{pmatrix} \\ &\equiv \begin{pmatrix}E+\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma & c\mathbf{p}+E\rho\mathbf{A}_\gamma + i\mu_B\mathbf{E}_\gamma\end{pmatrix} = \begin{pmatrix}E+\rho\mathbf{A}_\gamma\cdot c\mathbf{p} & c\mathbf{p}+E\rho\mathbf{A}_\gamma\end{pmatrix} + \Sigma^\mu \end{aligned} . \tag{19.17}$$

Then making use of all of (19.14) through (19.17) with $g_{\mu\nu} = \eta_{\mu\nu}$, we compact (19.13) to:

$$\begin{aligned} 0 &= \left(\gamma^0\left(E+\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right) - \boldsymbol{\gamma}\cdot\left(c\mathbf{p}+\rho\mathbf{A}_\gamma\left(E+\tfrac{1}{2}h\nu\right)\right) - \left(mc^2+\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)\right)\left|U_0\right\rangle \\ &= \left(\gamma^0\mathrm{E} - \boldsymbol{\gamma}\cdot c\boldsymbol{\Pi} - \mathrm{M}c^2\right)\left|U_0\right\rangle = \left(\gamma^\sigma c\mathrm{P}_\sigma - \mathrm{M}c^2\right)\left|U_0\right\rangle \end{aligned} . \tag{19.18}$$

This final result, $0=\left(\gamma^{\sigma}c\mathrm{P}_{\sigma}-\mathrm{M}c^2\right)\left|U_0\right\rangle$, has exactly the same form as the free-particle Dirac equation $\left(\gamma^{\sigma}cp_{\sigma}-mc^2\right)u_0=0$, with the hyper-canonical substitutions $p^{\sigma}\mapsto\mathrm{P}^{\sigma}$, $m\mapsto\mathrm{M}$ and $u_0\mapsto\left|U_0\right\rangle$. With this, we are ready to calculate the hyper-canonical Dirac Hamiltonian.

20. Calculating the Hyper-Canonical Dirac Hamiltonian Numerator

To derive the hyper-canonical Dirac Hamiltonian, we first split the four component Dirac spinor $\left|U_0\right\rangle$ into upper and lower components defined by $\left|U_0\right\rangle\equiv\left(\left|U_A\right\rangle \quad \left|U_B\right\rangle\right)^T$, and insert γ^{μ} into (19.18) using the Dirac representation (19.4). This produces:

$$0=\begin{pmatrix}\mathrm{E}-\mathrm{M}c^2 & -\boldsymbol{\sigma}\cdot c\boldsymbol{\Pi}\\ \boldsymbol{\sigma}\cdot c\boldsymbol{\Pi} & -\mathrm{E}-\mathrm{M}c^2\end{pmatrix}\begin{pmatrix}\left|U_{0A}\right\rangle\\ \left|U_{0B}\right\rangle\end{pmatrix}. \tag{20.1}$$

In the usual way, also with (19.14), we may now separate (20.1) into two equations, namely:

$$\begin{aligned}\left(\mathrm{E}-\mathrm{M}c^2\right)\left|U_{0A}\right\rangle&=\left(\boldsymbol{\sigma}\cdot c\boldsymbol{\Pi}\right)\left|U_{0B}\right\rangle\\ \left(\mathrm{E}+\mathrm{M}c^2\right)\left|U_{0B}\right\rangle&=\left(\boldsymbol{\sigma}\cdot c\boldsymbol{\Pi}\right)\left|U_{0A}\right\rangle\end{aligned}. \tag{20.2}$$

Dirac spinors may then be extracted in the usual way, using a two-component $\chi^{(s)}$, with $\chi^{(1)}=\left(1 \quad 0\right)^T$ and $\chi^{(2)}=\left(0 \quad 1\right)^T$, then multiplying through by the inverse 2x2 matrix $\left(\mathrm{E}+\mathrm{M}c^2\right)^{-1}$ in the bottom equation to extract the particle spinors and by $\left(\mathrm{E}-\mathrm{M}c^2\right)^{-1}$ in the top equation for the antiparticle spinors, where several signs are flipped by setting $E=-\left|E\right|$ for the $E<0$ antiparticle spinors using the Feynman-Stückelberg prescription. But our real interest is in extracting a Hamiltonian operator. To do so, also in the usual way, we combine both equations (20.2) to only keep the particle ket $\left|U_{0A}\right\rangle$, thus obtaining:

$$\left(\mathrm{E}-\mathrm{M}c^2\right)\left|U_{0A}\right\rangle=\left(\boldsymbol{\sigma}\cdot c\boldsymbol{\Pi}\right)\left(\mathrm{E}+\mathrm{M}c^2\right)^{-1}\left(\boldsymbol{\sigma}\cdot c\boldsymbol{\Pi}\right)\left|U_{0A}\right\rangle. \tag{20.3}$$

Now, the Hamiltonian operator H is ordinarily defined in relation to the work W and the energy $E=cp^0$ by $H\left|U_{0A}\right\rangle\equiv W\left|U_{0A}\right\rangle\equiv\left(E-mc^2\right)\left|U_{0A}\right\rangle$ with $W\equiv E-mc^2$. By these definitions the work constitutes the total physical energy in excess of rest energy mc^2, is obtained from eigenvalues of H. However, the expression $\left(\mathrm{E}-\mathrm{M}c^2\right)\left|U_{0A}\right\rangle=\left(E-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_{\gamma}-mc^2\right)\left|U_{0A}\right\rangle$ in (20.3) with (19.16) contains the hyper-canonical E in excess of the hyper-canonical $\mathrm{M}c^2$. Suppose that we were to define a hyper-canonical work by $\mathrm{W} \equiv \mathrm{E} - \mathrm{M}c^2 = E - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma - mc^2$. So-defined, this W would still be a total physical energy in excess of rest energy mc^2. But, this work would not be an energy-dimensioned number, but rather would be an *operator* because it contains the magnetic moment term $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$. Consequently, to obtain the eigenvalues of this work operator W, we would also have to define $W|U_{0A}\rangle \equiv \mathrm{W}|U_{0A}\rangle \equiv \left(E - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma - mc^2\right)|U_{0A}\rangle$, with this work eigenvalue W representing total energy in excess of rest energy. So, do we use this work definition which includes a work operator containing $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$, or do we simply use $W \equiv E - mc^2$?

Suppose we were to simply define $W \equiv E - mc^2$. Absent the $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ term from this definition, this work would only be a *kinetic* energy, and would exclude electromagnetic energy contributions from the eigenvalues of $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$. So, if we require the work W to be defined as the total energy in excess of mc^2, then lest we omit some electromagnetic energies, we are required to establish a work operator W as just described, and to obtain the work eigenvalues using $W|U_{0A}\rangle \equiv \mathrm{W}|U_{0A}\rangle$. If we further require that this work W be obtainable from the eigenvalues of H and thus define the Hamiltonian H accordingly, then making all of these definitions in combination, and connecting this with (20.3), we arrive at:

$$\begin{aligned} & W|U_{0A}\rangle \equiv H|U_{0A}\rangle \equiv \mathrm{W}|U_{0A}\rangle \equiv \left(\mathrm{E} - \mathrm{M}c^2\right)|U_{0A}\rangle = \left(E - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma - mc^2\right)|U_{0A}\rangle \\ & = \left(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi}\right)\left(\mathrm{E} + \mathrm{M}c^2\right)^{-1}\left(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi}\right)|U_{0A}\rangle \\ & = \left(\boldsymbol{\sigma} \cdot \left(c\mathbf{p} + E\rho\mathbf{A}_\gamma + i\mu_B \mathbf{E}_\gamma\right)\right)\left(E + mc^2 + 2\rho\mathbf{A}_\gamma \cdot c\mathbf{p} - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma\right)^{-1}\left(\boldsymbol{\sigma} \cdot \left(c\mathbf{p} + E\rho\mathbf{A}_\gamma + i\mu_B \mathbf{E}_\gamma\right)\right)|U_{0A}\rangle \end{aligned} \quad . \quad (20.4)$$

From this, we now pinpoint the hyper-canonical Hamiltonian, which has been defined above as:

$$\begin{aligned} & H \equiv \left(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi}\right)\left(\mathrm{E} + \mathrm{M}c^2\right)^{-1}\left(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi}\right) \\ & = \left(\boldsymbol{\sigma} \cdot \left(c\mathbf{p} + E\rho\mathbf{A}_\gamma + i\mu_B \mathbf{E}_\gamma\right)\right)\left(E + mc^2 + 2\rho\mathbf{A}_\gamma \cdot c\mathbf{p} - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma\right)^{-1}\left(\boldsymbol{\sigma} \cdot \left(c\mathbf{p} + E\rho\mathbf{A}_\gamma + i\mu_B \mathbf{E}_\gamma\right)\right) \end{aligned} \quad . \quad (20.5)$$

It is important to note that $\left(\mathrm{E} - \mathrm{M}c^2\right)|U_{0A}\rangle$ in (20.4) only contains the *time component* of the hyper-canonical $c\mathrm{P}^\mu = \left(\mathrm{E} \quad c\boldsymbol{\Pi}\right)$, and that because of these definitions, all the space components are segregated to $H|U_{0A}\rangle = \left(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi}\right)\left(\mathrm{E} + \mathrm{M}c^2\right)^{-1}\left(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi}\right)|U_{0A}\rangle = \left(\mathrm{E} - \mathrm{M}c^2\right)|U_{0A}\rangle$. This is analogous to writing the relativistic energy-momentum relation $c^2 p_\sigma p^\sigma = m^2 c^4$ as $c^2\mathbf{p}^2 = E^2 - m^2c^4$, and in fact, tracing back to the very start of section 1, this may be understood as a very-downstream consequence of this exact same relation. We also note that the above contains both magnetic $\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ and electric moments $\mu_B \boldsymbol{\sigma} \cdot \mathbf{E}_\gamma$. Now, we proceed to calculate this Hamiltonian.

First, we encounter $\left(\mathrm{E}+\mathrm{M}c^2\right)^{-1}=\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right)^{-1}$. This cannot be treated as an ordinary denominator because $\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$ term is a 2x2 matrix operator. So, we must first calculate this inverse. It simplifies the inverse calculation to briefly define a substitute variable $C\equiv E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}$ and write the inverse as $\left(\mathrm{E}+\mathrm{M}c^2\right)^{-1}=\left(C-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right)^{-1}$. Then we invert $C-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$ using the well-known inverse relation for a 2x2 matrix to obtain:

$$\begin{aligned}\left(\mathrm{E}+\mathrm{M}c^2\right)^{-1}&=\left(C-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right)^{-1}=\begin{pmatrix}C-\mu_B B_\gamma^3 & \mu_B B_\gamma^1+i\mu_B B_\gamma^2\\ \mu_B B_\gamma^1-i\mu_B B_\gamma^2 & C+\mu_B B_\gamma^3\end{pmatrix}^{-1}\\&=\frac{1}{\left|C-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right|}\begin{pmatrix}C+\mu_B B_\gamma^3 & B_\gamma^1-i\mu_B B_\gamma^2\\ \mu_B B_\gamma^1+i\mu_B B_\gamma^2 & C-\mu_B B_\gamma^3\end{pmatrix}=\frac{C+\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma}{\left|C-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right|}\end{aligned},\tag{20.6}$$

in which the matrix determinant is easily calculated to be:

$$\left|C-\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma\right|=C^2-\mu_B{}^2\mathbf{B}_\gamma{}^2=\left(E+mc^2+2\rho\mathbf{A}\cdot c\mathbf{p}\right)^2-\mu_B{}^2\mathbf{B}_\gamma{}^2=\left(E+mc^2+2\rho\mathbf{A}\cdot c\mathbf{p}\right)^2.\tag{20.7}$$

After the final equality we have set $\mathbf{B}_\gamma^2=0$ as discussed following (15.6), because the *magnitude* of the photon magnetic field is zero. However, to be as general as possible so that later on we can study the Hamiltonian behavior when external classical magnetic fields with non-zero magnitude are applied, we shall leave $\mathbf{B}_\gamma{}^2$ in place without zeroing it out. So, using (20.6) and (20.7) in (20.5) and replacing *C*, in both vector and index notation we obtain:

$$\begin{aligned}H&=\frac{\left(\boldsymbol{\sigma}\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\right)\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right)\left(\boldsymbol{\sigma}\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\right)}{\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)^2-\mu_B{}^2\mathbf{B}_\gamma{}^2}\\&=\frac{\left(\sigma^i cp^i+E\sigma^i\rho A_\gamma^i+i\mu_B\sigma^i E_\gamma^i\right)\left(E+mc^2+2\rho A_\gamma^j cp^j+\mu_B\sigma^j B_\gamma^j\right)\left(\sigma^k cp^k+E\sigma^k\rho A_\gamma^k+i\mu_B\sigma^k E_\gamma^k\right)}{\left(E+mc^2+2\rho A_\gamma^i\cdot cp^i\right)^2-\mu_B{}^2 B_\gamma^i B_\gamma^i}\end{aligned}.\tag{20.8}$$

Note that the numerator above is in dimensions of energy-cubed, the denominator is in dimensions of energy-squared, and thus the Hamiltonian is in dimensions of energy as is must be. Note also that each of the individual terms, namely *E*, mc^2, $c\mathbf{p}$, $E\rho\mathbf{A}_\gamma$, $\mu_B\mathbf{E}_\gamma$, $\rho\mathbf{A}_\gamma\cdot c\mathbf{p}$ and $\mu_B\mathbf{B}_\gamma$ all have dimensions of energy. Now we undertake to calculate this *H* in detail.

First, if all products are expanded, the numerator of *H* contains a total of 3x4x3=36 terms to start. This is each of the three terms in $\boldsymbol{\sigma}\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)$ on the left, times the four terms in $E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$ in the middle, times three more terms in the second $\boldsymbol{\sigma}\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)$ on the right. Twenty-seven (27) of these terms contain a double matrix product of the form $\sigma^i\sigma^k$ which may be expanded with the identity $\sigma^i\sigma^k=\delta^{ik}+i\varepsilon^{ikl}\sigma^l$, thus doubled into fifty-four (54) separate terms. The remaining nine (9) of these terms contain a triple matrix product expanded with $\sigma^i\sigma^j\sigma^k=\delta^{ij}\sigma^k+\delta^{jk}\sigma^i-\delta^{ki}\sigma^j+i\varepsilon^{ijk}$, which is thus quadrupled to thirty-six (36) terms. In total this is now ninety (90) terms, just to start.

Next, in (20.8) the Hamiltonian is represented in momentum space. In order to convert this to configuration space via the relation $\mathbf{p}\left|\Psi\right\rangle=-i\hbar\nabla\left|\Psi\right\rangle$, it is *essential* to commute every three-momentum **p** to the right of every $\mathbf{A}_\gamma(\mathbf{x})$ and $\mathbf{B}_\gamma(\mathbf{x})$ and $\mathbf{E}_\gamma(\mathbf{x})$ so as to butt directly against $\left|\Psi\right\rangle$ with no other intervening objects. But, in thirteen (13) of the 36 individual terms in (20.8) there is a **p** is situated to the left of at least one of $\mathbf{A}_\gamma(\mathbf{x})$ or $\mathbf{B}_\gamma(\mathbf{x})$ or $\mathbf{E}_\gamma(\mathbf{x})$ which are functions of the space coordinates **x**, and we cannot simply move these **p** over to the right, because of the Heisenberg canonical commutation relation which of course underlies the uncertainty principle. Rather, each time we commute p^i past any function $b(\mathbf{x})$ expansible as a Maclaurin series in **x**, we must use $[\mathbf{p},b]=-i\hbar\nabla b$, a.k.a. $\left[p^i,b\right]=i\hbar\partial^i b$ in index notation (note $\partial^i=-\nabla$ in flat spacetime). And if b is a vector **b**, this becomes $[\mathbf{p},\mathbf{b}]=-i\hbar\nabla\mathbf{b}$, or $\left[p^i,b^j\right]=i\hbar\partial^i b^j$ in index notation, while for any object $O(\mathbf{x})$ this generalizes to $\left[O,p^j\right]=-i\hbar\partial^j O$. The origin of this relation $\left[O,p^j\right]=-i\hbar\partial^j O$ in Heisenberg's relation $[p_x,x]=-i\hbar$ was reviewed in the paragraph following (7.10). What will become very important for the present development., is that each time we commute p^i to the right past a generalized vector $b^j(\mathbf{x})$, we further increase the number of terms in H, with the relation $p^i b^j=b^j p^i+i\hbar\partial^i b^j$ adding a partial derivative.

Also, while it is clear $\mathbf{A}_\gamma(\mathbf{x})$, $\mathbf{B}_\gamma(\mathbf{x})$ and $\mathbf{E}_\gamma(\mathbf{x})$ are functions of space and time, we must recall that the total fermion energy content E is also a function of space (and time). Specifically, we determined after (11.4) that $E=\Gamma mc^2=\gamma_v\gamma_g\gamma_{em}mc^2$, and that this holds for both classical and quantum systems. For the present flat spacetime calculation, this energy content relation is $E=mc^2\gamma_v\gamma_{em}$. Moreover, at (11.3) we obtained the electromagnetic time dilation $\gamma_{em}=1/\left(1-\rho\left\langle\phi_0\right\rangle\right)$ with $\rho=q/mc^2$. In the classical correspondence $\phi_0=\left\langle\phi_0\right\rangle$ this is simply $\gamma_{em}=1/\left(1-\rho\phi_0\right)$ first found at (5.8). Therefore $E=mc^2\gamma_v\gamma_{em}=mc^2\gamma_v/\left(1-\rho\phi_0\right)$. But because the proper potential $\phi_0(t,\mathbf{x})$ is a function of space and time, this means that the total energy $E(t,\mathbf{x})=E\left(\phi_0(t,\mathbf{x})\right)$ is a function of space and time, precisely since the total energy includes electromagnetic interaction energy which is a function of space and time (see also (6.2) and (6.3) involving the Coulomb potential). Thus, we must use $\left[p^i,E\right]=i\hbar\partial^i E=-i\hbar\nabla E$ whenever we commute **p** with the total fermion energy *E*. This will produce even more terms when we commute **p** to the right in (20.8). In fact, given $\left[p^i,E\right]=i\hbar\partial^i E$, we may deduce from (7.5) that:

$$\left[p^i,E\right]=i\hbar\partial^i E=-i\hbar\nabla E=i\hbar q\partial^j\phi=-i\hbar q\nabla\phi=i\hbar q\left(E^j+\dot{A}^j/c\right)=i\hbar q\left(\mathbf{E}+\dot{\mathbf{A}}/c\right). \tag{20.9}$$

Note that in the above, **E** and $\dot{\mathbf{A}}$ do *not* have a γ subscript. Owing to their origin from the total energy $E=mc^2\gamma_v\gamma_{em}=mc^2\gamma_v/\left(1-\rho\phi_0\right)$ of a *quantum* fermion in a *classical* scalar potential $\phi=\phi_0\gamma_v\gamma_{em}$, these enter (20.9) as *classical* external fields $\mathbf{E}=\mathbf{E}_{\text{c}}+\mathbf{E}_\gamma\cong\mathbf{E}_{\text{c}}$ and $\dot{\mathbf{A}}=\dot{\mathbf{A}}_{\text{c}}+\dot{\mathbf{A}}_\gamma\cong\dot{\mathbf{A}}_{\text{c}}$. All of this was reviewed in section 15 (see (15.8) and associated discussion of gauge transformations between luminous and classical potentials) and in section 16 (note especially the relations (16.14) which extend (15.4), (15.7) and (15.15) to classical fields). So, although (20.8) only displays $\mathbf{A}_\gamma$ for individual photons, and although $A_\gamma^0=\phi_\gamma=0$ for individual photons, the commutator $\left[p^i,E\right]$ and the very presence of a total energy *E* will end up smuggling some classical external $A_{\text{c}}^\mu=\left(\phi,\mathbf{A}_{\text{c}}\right)$ into the Hamiltonian, allowing us to study the effects of *classical* external potentials on individual *quantum* interactions between fermions and photons.

The general commutations $\left[O,p^j\right]=-i\hbar\partial^j O$ in some instances will produce many additional terms. The most extreme example of this, and a good example of what we must do with what turns out to be fourteen (14) of the 36 distinct terms in (20.8), is the single term $\left(\boldsymbol{\sigma}\cdot c\mathbf{p}\right)\left(2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)\left(\boldsymbol{\sigma}\cdot E\rho\mathbf{A}_\gamma\right)$ in (20.8) which contains two momenta **p** which need to commute to the right past two $\mathbf{A}_\gamma$ as well as an *E*. The commutations for this one of the 36 terms in (20.8), using $\left[p^i,b\right]=i\hbar\partial^i b$ including the variants $\left[p^i,\partial^j E\right]=i\hbar\partial^i\partial^j E$ and $\left[p^i,\partial^j A_\gamma^k\right]=i\hbar\partial^i\partial^j A_\gamma^k$ to move **p** all the way to the right, produces twelve (12) different terms. Then, we consolidate the commutation results using the product rule together with $\nabla^j A_\gamma^j=0$ from (15.5). We thus obtain:

$$\begin{aligned}&\left(\boldsymbol{\sigma}\cdot c\mathbf{p}\right)\left(2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)\left(\boldsymbol{\sigma}\cdot\rho\mathbf{A}_\gamma E\right)=\left(\sigma^i cp^i\right)\left(2\rho A_\gamma^j cp^j\right)\left(\sigma^k\rho A_\gamma^j E\right)\\
&=2\sigma^i\sigma^k\begin{bmatrix}\rho A_\gamma^k E\rho A_\gamma^k cp^i cp^j-i\hbar c\rho A_\gamma^j\nabla^j E\rho A_\gamma^k cp^i-i\hbar c\rho A_\gamma^j E\nabla^j\rho A_\gamma^k cp^i\\
-i\hbar c\nabla^i\rho A_\gamma^j E\rho A_\gamma^k cp^j-i\hbar c\rho A_\gamma^j\nabla^i E\rho A_\gamma^k cp^j-i\hbar c\rho A_\gamma^j E\nabla^i\rho A_\gamma^k cp^j\\
-\hbar^2c^2\nabla^i\rho A_\gamma^j\nabla^j E\rho A_\gamma^k-\hbar^2c^2\rho A_\gamma^j\nabla^i\nabla^j E\rho A_\gamma^k-\hbar^2c^2\rho A_\gamma^j\nabla^j E\nabla^i\rho A_\gamma^k\\
-\hbar^2c^2\nabla^i\rho A_\gamma^j E\nabla^j\rho A_\gamma^k-\hbar^2c^2\rho A_\gamma^j\nabla^i E\nabla^j\rho A_\gamma^k-\hbar^2c^2\rho A_\gamma^j E\nabla^i\nabla^j\rho A_\gamma^k\end{bmatrix}\\
&=2\sigma^i\sigma^k\begin{bmatrix}\rho A_\gamma^j E\rho A_\gamma^k cp^i cp^j\\
-i\hbar c\left(\nabla^i\left(\rho A_\gamma^j E\rho A_\gamma^k\right)cp^j+\nabla^j\left(\rho A_\gamma^j E\rho A_\gamma^k\right)cp^i\right)-\hbar^2c^2\nabla^i\nabla^j\left(\rho A_\gamma^j E\rho A_\gamma^k\right)\end{bmatrix}\end{aligned}. \tag{20.10}$$

And this is all before using $\sigma^i \sigma^k = \delta^{ik} + i\varepsilon^{ikl}\sigma^l$ which doubles the number of terms. Again, this particular term is an extreme example, but together with all of the foregoing, it highlights how the calculation to move all **p** to the right in (20.8) so that we can use $c\mathbf{p}|\Psi\rangle = -c\nabla|\Psi\rangle$ to convert the Hamiltonian operator into configuration space, is very complicated, produces a very large number of terms, and must be carefully managed. This also shows how it is virtually impossible at this stage to carry out the calculation without using index notation.

With all of this in mind, we begin the calculation to move all **p** to the very right of the Hamiltonian denominator in (20.8). But, rather than show this straightforward albeit tedious calculation in the main paper, we have placed this calculation in Appendix C. As a result of the detailed calculations in Appendix C, for the Hamiltonian denominator in (20.8), we deduce:

$$
\begin{aligned}
&\left(\boldsymbol{\sigma}\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\right)\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\right)\left(\boldsymbol{\sigma}\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\right)=\\
&\left.\begin{array}{l}
+\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)c\mathbf{p}\cdot c\mathbf{p}+2\left(E+mc^2\right)E\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
+\left(4E\rho\mathbf{A}_\gamma\cdot\rho\mathbf{A}_\gamma+2\left(c\mathbf{q}\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(c\mathbf{q}\times\rho\mathbf{A}_\gamma\right)\right)\right)\cdot c\mathbf{p}c\mathbf{p}\\
-2imc^2\mu_B\dot{\mathbf{A}}\cdot\mathbf{p}+\left(2E+4mc^2\right)\boldsymbol{\sigma}\cdot\left(\mu_B\dot{\mathbf{A}}\times\mathbf{p}\right)
\end{array}\right\}\\
&\left.\begin{array}{l}
-2\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma E\left(E+mc^2+4\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)-i\hbar\mu_B\boldsymbol{\sigma}\cdot\dot{\mathbf{B}}_\gamma\left(E+mc^2+4\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)\\
-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\left(\left(c\mathbf{p}+c\mathbf{q}\right)\cdot c\mathbf{p}+\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\cdot\left(2c\mathbf{p}+c\mathbf{q}\right)\right)\\
-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma\left(26e\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-26e\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma+2E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma-\mu_B\mathbf{E}_\gamma\cdot\mu_B\mathbf{E}_\gamma+4\pi\hbar c\mu_B\rho_{em\,\gamma}\right)
\end{array}\right\}\\
&\left.\begin{array}{l}
+\boldsymbol{\sigma}\cdot\left(c\mathbf{q}+2E\rho\mathbf{A}_\gamma+2i\mu_B\mathbf{E}_\gamma-2i\mu_B\mathbf{E}\right)\mu_B\mathbf{B}_\gamma\cdot c\mathbf{p}\\
+2\mu_B\mathbf{B}_\gamma\cdot\left(c\mathbf{p}+i\mu_B\mathbf{E}\right)\boldsymbol{\sigma}\cdot c\mathbf{p}+2\mu_B\mathbf{B}_\gamma\cdot\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\boldsymbol{\sigma}\cdot\left(c\mathbf{p}+c\mathbf{q}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\\
+2\mu_B\mathbf{B}_\gamma\cdot\left(e\mathbf{A}_\gamma\boldsymbol{\sigma}\cdot i\mu_B\mathbf{E}-e\mathbf{A}\boldsymbol{\sigma}\cdot i\mu_B\mathbf{E}_\gamma+i\mu_B\mathbf{E}\boldsymbol{\sigma}\cdot e\mathbf{A}_\gamma-i\mu_B\mathbf{E}_\gamma\boldsymbol{\sigma}\cdot e\mathbf{A}\right)\\
-2\mu_B\mathbf{B}_\gamma\cdot\left(\mu_B\mathbf{E}\times c\mathbf{p}-\left(e\mathbf{A}_\gamma+e\mathbf{A}\right)\times\mu_B\mathbf{E}\right)+2E\mu_B\mathbf{B}_\gamma\cdot\mu_B\mathbf{B}_\gamma\\
-2i\boldsymbol{\sigma}\cdot\mu_B\mathbf{A}\mu_B\dot{\mathbf{B}}_\gamma\cdot\mathbf{p}+2i\mu_B\dot{\mathbf{B}}_\gamma\cdot\mu_B\mathbf{A}\boldsymbol{\sigma}\cdot\mathbf{p}-2\mu_B\dot{\mathbf{B}}_\gamma\cdot\left(\mu_B\mathbf{A}\times\mathbf{p}\right)+i\hbar\mu_B\dot{\mathbf{B}}_\gamma\cdot\mu_B\mathbf{B}_\gamma
\end{array}\right\}\\
&+4\pi\hbar c\mu_B\rho_{em\,\gamma}\left(E+mc^2+4\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)+\left(4\pi\hbar\mu_B\mathbf{J}_\gamma+\hbar\mu_B\dot{\mathbf{E}}_\gamma\right)\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\}\\
&\left.\begin{array}{l}
+\left(4\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-4\rho\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma\right)c\mathbf{p}\cdot c\mathbf{p}+8\left(\rho\mathbf{A}_\gamma\cdot\boldsymbol{\sigma}\cdot\mu_B\mathbf{E}-\mu_B\mathbf{E}_\gamma\cdot\boldsymbol{\sigma}\cdot\rho\mathbf{A}\right)\times c\mathbf{p}c\mathbf{p}\\
+\left(2\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+2i\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma+4\rho\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma-4\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}\right)\cdot c\mathbf{p}c\mathbf{p}
\end{array}\right\}\\
&\left.\begin{array}{l}
+\left(4E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+8e\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-2\mu_B\mathbf{E}_\gamma\cdot\mu_B\mathbf{E}_\gamma\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
+4i\boldsymbol{\sigma}\cdot\left(E\rho\mathbf{A}_\gamma\times i\mu_B\mathbf{E}+E\rho\mathbf{A}\times i\mu_B\mathbf{E}_\gamma-e\mathbf{A}_\gamma\times i\mu_B\mathbf{E}+i\mu_B\mathbf{E}_\gamma\times i\mu_B\mathbf{E}-\mu_B\dot{\mathbf{E}}_\gamma\times\mu_B\mathbf{A}\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
-4i\left(E\rho\mathbf{A}_\gamma\cdot\mu_B\mathbf{E}-E\rho\mathbf{A}\cdot\mu_B\mathbf{E}_\gamma+e\mathbf{A}_\gamma\cdot\mu_B\mathbf{E}\right)i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times c\mathbf{p}\right)\\
+8\left(\mu_B\mathbf{E}_\gamma\cdot\mu_B\mathbf{E}+\mu_B\dot{\mathbf{E}}_\gamma\cdot\mu_B\mathbf{A}/c\right)i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}\times c\mathbf{p}\right)
\end{array}\right\}\\
&\left.\begin{array}{l}
+\left(2\left(E+mc^2\right)-4\left(2e\mathbf{A}_\gamma\cdot\rho\mathbf{A}-i\boldsymbol{\sigma}\cdot\left(e\mathbf{A}_\gamma\times\rho\mathbf{A}\right)\right)\right)i\mu_B\mathbf{E}_\gamma\cdot c\mathbf{p}-2imc^2\mu_B\mathbf{E}\cdot c\mathbf{p}\\
+4\left(\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-\rho\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma-e\mathbf{A}_\gamma\cdot\rho\mathbf{A}\right)\boldsymbol{\sigma}\cdot\left(\mu_B\mathbf{E}_\gamma\times c\mathbf{p}\right)\\
+\left(2\left(E+2mc^2\right)-8\left(\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-\rho\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma\right)\right)\boldsymbol{\sigma}\cdot\left(\mu_B\mathbf{E}\times c\mathbf{p}\right)
\end{array}\right\}\\
&\left.\begin{array}{l}
+2\left(E+mc^2\right)E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma-4e\mathbf{A}_\gamma\cdot\mu_B\mathbf{E}\left(\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\mu_B\mathbf{E}_\gamma\times\rho\mathbf{A}_\gamma\right)\right)\\
+2mc^2\left(e\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-e\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma+\mu_B\mathbf{A}\cdot\mu_B\dot{\mathbf{E}}_\gamma-i\boldsymbol{\sigma}\cdot\left(e\mathbf{A}_\gamma\times i\mu_B\mathbf{E}+e\mathbf{A}\times i\mu_B\mathbf{E}_\gamma-\mu_B\mathbf{A}\times\mu_B\dot{\mathbf{E}}_\gamma\right)\right)\\
-\left(E+mc^2-4\rho\mathbf{A}_\gamma\cdot e\mathbf{A}\right)\mu_B\mathbf{E}_\gamma\cdot\mu_B\mathbf{E}_\gamma+2mc^2\mu_B\mathbf{E}\cdot\mu_B\mathbf{E}_\gamma+\left(2mc^2+4\rho\mathbf{A}_\gamma\cdot e\mathbf{A}\right)i\boldsymbol{\sigma}\cdot\left(\mu_B\mathbf{E}\times\mu_B\mathbf{E}_\gamma\right)
\end{array}\right\}.
\end{aligned}
\tag{20.11}
$$

In the above, which is the numerator for the Hamiltonian (20.8) with all momenta **p** commuted to the very right, we have used right-brackets to segregate into eight groups of terms according to the following dominant physical characteristics. The first group contain terms which are entirely a function of the fermion total *E*, rest energy mc^2, momentum $c\mathbf{p}$ in energy dimensions, and charge e embedded in $\rho = -e / mc^2$; and also of the photon potential $\mathbf{A}_\gamma$ and momentum $c\mathbf{q}$. The combination $\mathbf{A}_\gamma \cdot \mathbf{p}$ contains data about the angle of interaction between the fermion and the photon. The terms on the second line of this group have the outer product $\mathbf{pp}$ all the way to the right and also affect the Hamiltonian based on the nature of the fermion / photon interaction. The third line incorporates the time derivative $\dot{\mathbf{A}} = \partial \mathbf{A} / \partial t = -i\omega \mathbf{A}$, not of the photon potential, but of any *external classical* potential applied to the fermion / photon interaction. Keep in mind, however, that the associated $\omega = i\partial_t$ is the radian frequency of a photon with energy $\hbar\omega$.

The second group contains all of the magnetic moments $\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$, and on the first line also contains the time derivative $\mu_B \boldsymbol{\sigma} \cdot \dot{\mathbf{B}}_\gamma = -i\omega \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$. We will momentarily show how and why we can transform this magnetic field from $\mathbf{B}_\gamma$ for a photon to a classical external $\mathbf{B}$. Once we do so, we can see very clearly that $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B} = \boldsymbol{\mu}_D \cdot \mathbf{B}$ has a coefficient containing multiple physical objects and object combinations, which coefficient will not be equal to the Dirac $g_D / 2 = 1$. As discussed at (19.9), a coefficient of $\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ which is not equal to $g_D / 2 = 1$ but is slightly larger than 1, is the precise characteristic of the magnetic moment anomaly. And as we shall shortly see, it is the term combination $-2E\left(E + mc^2\right)\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ in (20.11) which, as part of the numerator in (20.8), will produce the magnetic moment anomaly without renormalization.

The third group of terms contains all remaining terms with a photon magnetic field $\mathbf{B}_\gamma$ which are not magnetic moment terms. The fourth group, which is a single line, contains all the source densities $\rho_{em\,\gamma}$ and $\mathbf{J}_\gamma$, but for the $\rho_{em\,\gamma}$ which is already part of the magnetic moment coefficient of the second group. Groups five through eight contain all remining electric field $\mathbf{E}$ terms not already in one of the first four groups. Group five contains the remaining double-momentum combinations which form either an inner product $\mathbf{p} \cdot \mathbf{p}$ or outer product $\mathbf{pp}$. Group six contains all remaining $\mathbf{A}_\gamma \cdot \mathbf{p}$ and $\mathbf{A}_\gamma \times \mathbf{p}$ terms which provide data about the angle of interaction between the individual fermion and photon quanta, and a $\mathbf{A} \times \mathbf{p}$ for the cross product between the fermion and any classical external potential. Group seven houses remaining $\mathbf{E} \cdot \mathbf{p}$ and $\mathbf{E} \times c\mathbf{p}$ terms for both classical and photonic $\mathbf{E}$. Group eight contains all remaining terms, which have only $\mathbf{E}$ and $\mathbf{A}$ for both classical and photonic fields. A number of time-derivatives $\partial_t = -i\omega$ of the various fields, designated by over-dots, also appear throughout the above, and there are some $\mathbf{A}_\gamma \cdot \mathbf{A} \neq 0$ which encodes the angle between the photon and any external potential.

The above (20.11) is obtained entirely by mathematical deduction from the numerator of the Hamiltonian (20.8), merely by commuting all $\mathbf{p}$ to the right of all other objects so that the conversion between momentum and configuration space via $\mathbf{p}\left|\Psi\right\rangle = -i\hbar\nabla\left|\Psi\right\rangle$ can be utilized at will. This fully reduces and consolidates the numerator of the Hamiltonian (20.8) as much as possible and enables the heuristic configuration space substitution $\mathbf{p} \mapsto -i\hbar\boldsymbol{\nabla}$ for every single momentum appearing in (20.11). But before we insert this back into (20.8) and begin to study the Hamiltonian, there is one final step that we now take. This step is not mathematically-deductive, but rather uses the gauge symmetries studied in section 15 and especially section 16 to populate the Hamiltonian throughout with classical external electric and magnetic fields, as well as classical external sources. This final step is the subject of the next section.

21. Transforming Gauge-Invariant Quantum Photon Fields to Classical External Fields, to obtain the Complete Hyper-Canonical Dirac Hamiltonian

It is important to observe that (20.11) contains a mix of classical external potentials **A** and quantum potentials $\mathbf{A}_\gamma$ for individual photons. At (15.2) and (15.3) we deduced as a downstream consequence of the conditions $\left\langle \mathfrak{D}_\sigma A^\sigma \right\rangle = 0$ and $\partial_\sigma \left\langle A^\sigma \right\rangle = 0$ in (9.4) and (9.5) which break gauge symmetry, that the photon potential must be $A_\gamma^\mu = A\varepsilon^\mu \exp\left(-iq_\sigma x^\sigma / \hbar\right)$ with a polarization $\varepsilon_{R,L}^\mu\left(\hat{z}\right) \equiv \begin{pmatrix} 0 & \mp 1 & -i & 0 \end{pmatrix} / \sqrt{2}$, when the *z* axis is chosen to align with the photon propagation. This means, as first deduced at (14.8), that the scalar potential for a photon is zero, $A_\gamma^0 = \phi_\gamma = 0$, and that the square magnitude of the photon potential is also zero $\mathbf{A}_\gamma \cdot \mathbf{A}_\gamma = 0$. These results are not new. They are well-established. It is merely the *covariant derivation* of these results which appears to be new. Simply put, as reviewed in section 16, this is all because the photon is a luminous massless particle that can *never* be placed at rest. At the same time, as reviewed at (16.10), a classical potential $A^\mu = \begin{pmatrix} \phi & \mathbf{A} \end{pmatrix}$ must always have a material source, and of course that source can *always* be placed at rest in which case $A^\mu = \begin{pmatrix} \phi_0 & \mathbf{0} \end{pmatrix}$ where ϕ_0 is the proper potential.

Contrasting the photon $\mathbf{A}_\gamma$ with the classical external **A**, the former is a complex object with imaginary components in $\boldsymbol{\varepsilon}$ and a Fourier kernel $\exp\left(-iq_\sigma x^\sigma / \hbar\right)$ and a proper scalar potential $\phi_0 = 0$ because there is no "proper" rest frame for a photon; while the latter is an entirely real object for which, at rest in its "proper" frame, the only *non-zero* component is $A^0 = \phi_0$. But, although $\mathbf{A}_\gamma$ and **A** have entirely different properties, they are still the same physics objects. This is because the only difference is that $\mathbf{A}_\gamma$ has had its gauge symmetry broken and turned into a luminous photon in the manner just reviewed, while **A** remains classical and can be placed at rest.

What is most important for the present development, as reviewed in depth in section 16, is that both the photon and the classical potentials are related to the field strength bivector by exactly the same invariant relation, with $F_\gamma^{\mu\nu} = \partial^\mu A_\gamma^\nu - \partial^\nu A_\gamma^\mu$ for the former and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ for the latter. Likewise, the Maxwell equations are the invariant $4\pi J_\gamma^\mu = \partial_\alpha F_\gamma^{\alpha\mu}$ and $\partial^\alpha F_\gamma^{\mu\nu} + \partial^\mu F_\gamma^{\nu\alpha} + \partial^\nu F_\gamma^{\alpha\mu} = 0$ for the former and $4\pi J^\mu = \partial_\alpha F^{\alpha\mu}$ and $\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0$ for the latter. The only difference, reviewed at (16.10), see also (15.10), is that $\phi_{0\,\gamma} = 0$, and as a direct consequence, so too $J_\gamma^\nu = 0$. Again, a scalar potential must have a material electrical source. Absent an electrical source we must have $\phi_{0\,\gamma} = 0$ for a luminous photon, and we are dealing with source-free electrodynamics. Which now brings us to (20.11).

Suppose that we are using (20.11) as the numerator of the Hamiltonian (20.8) to describe a physical situation in which one or more of a non-zero classical external $\mathbf{A} \neq 0$, $\mathbf{B} \neq 0$ or $\mathbf{E} \neq 0$ is applied to an individual fermion interacting with an individual photon, and / or in which that interaction occurs in a region of spacetime with a non-zero classical external charge density $\rho_{em} \neq 0$ which may also be in relative motion so that the current density $\mathbf{J} \neq 0$. Then we pose the question: how would we account for these classical external fields in (20.11)? First, as to $\mathbf{A}_\gamma$ and $\mathbf{A}$, we must leave these objects exactly as is. This is because the former has had its gauge symmetry broken to represent a luminous photon, while the latter is a classical external potential which may be transformed to a rest frame. And, this is because $\mathbf{A}$ in general is not a gauge-invariant object, because $qA_\mu \to qA'_\mu \equiv qA_\mu + \hbar c \partial_\mu \Lambda$ under a gauge transformation. However, far from being a difficulty, by having both $\mathbf{A}_\gamma$ and $\mathbf{A}$ in the same Hamiltonian (along with $\mathbf{p}$ and E and mc^2 and e for the fermion) we are enabled to inquire how the classical external potential $\mathbf{A}$ affects the individual interactions between a photon $\mathbf{A}_\gamma$ and a fermion Ψ. In this way, we are able to study the energy spectra of *quantum* interactions in *classical* fields.

But for **B**, **E**, ρ_{em} and **J**, we are not restricted in this way. Because $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $F_\gamma^{\mu\nu} = \partial^\mu A_\gamma^\nu - \partial^\nu A_\gamma^\mu$ have the same form, and because $F^{\mu\nu}$ in either event is invariant under gauge transformations, an applied **B** and an applied **E** will enter into (20.11) in exactly the same form as their photonic counterparts. Very importantly, this means we can substitute $\mathbf{B}_\gamma \mapsto \mathbf{B}$ and $\mathbf{E}_\gamma \mapsto \mathbf{E}$ everywhere these appear in (20.11). In fact, to be precise, because the magnetic and electric fields of a photon are both non-zero as seen at (15.6) and (15.11), these fields will simply add to whatever classical $\mathbf{B}_\text{c}$ and $\mathbf{E}_\text{c}$ are applied according to $F^{\mu\nu} = F_\text{c}^{\mu\nu} + F_\gamma^{\mu\nu} \cong F_\text{c}^{\mu\nu}$ as noted following (16.12), with the photon's $F_\gamma^{\mu\nu}$ swamped by the classical external $F_\text{c}^{\mu\nu}$ carried by innumerable photons. Thus, the square magnitude terms which are zero, $\mathbf{B}_\gamma \cdot \mathbf{B}_\gamma = 0$ and $\mathbf{E}_\gamma \cdot \mathbf{E}_\gamma = 0$ as also reviewed, are seen in (20.10) to have been placeholders which become *non-zero* then they are associated with a *materially-sourced* **B** and **E**, whereby $\mathbf{B}_\gamma \cdot \mathbf{B}_\gamma = 0 \mapsto \mathbf{B} \cdot \mathbf{B} \neq 0$ and $\mathbf{E}_\gamma \cdot \mathbf{E}_\gamma = 0 \mapsto \mathbf{E} \cdot \mathbf{E} \neq 0$. Moreover, these placeholders show us exactly where and how these square magnitude terms will enter the Hamiltonian when these external **B** and **E** fields are applied.

Finally, although $J_\gamma^\nu = \begin{pmatrix} c\rho_{em\,\gamma} & \mathbf{J}_\gamma \end{pmatrix} = 0$ from (15.10), here too, $4\pi J^\mu = \partial_\alpha F^{\alpha\mu}$ and $\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0$ have exactly the same invariant form whether applied to photons or to classical charge and current densities, with J_γ^ν also being invariant under gauge transformations.

The only difference is that for classical densities, $\rho_{em} \neq 0$. And if there is motion relative to the source, then $\mathbf{J} \neq 0$ as well. Thus, throughout (20.11) we may substitute $\rho_{em\,\gamma} = 0 \mapsto \rho_{em} \neq 0$ and $\mathbf{J}_\gamma = 0 \mapsto \mathbf{J} \neq 0$ everywhere these appear, with $\rho_{em\,\gamma}$ and $\mathbf{J}_\gamma$ – although zero for photons – having been a placeholder for the non-zero charge and current densities that exist when we go from source-free electrodynamics to electrodynamics with sources. In effect, the above merely restates section 16 where we reviewed Maxwell's equations for individual photons, now in the context of the hyper-canonical Dirac Hamiltonian (20.8) which has the numerator (20.11).

As a result of the foregoing symmetry considerations, we now proceed to substitute $\mathbf{B}_\gamma \mapsto \mathbf{B}$, $\mathbf{E}_\gamma \mapsto \mathbf{E}$, $\rho_{em\,\gamma} \mapsto \rho_{em}$ and $\mathbf{J}_\gamma \mapsto \mathbf{J}$ throughout (20.11) as well as $\mathbf{B}_\gamma \mapsto \mathbf{B}$ in the denominator of (20.8), then reduce and consolidate wherever possible. We denote all square magnitudes by the generalized $\mathbf{a} \cdot \mathbf{a} = \mathbf{a}^2$, and set some emergent $\mathbf{E} \times \mathbf{E} = 0$ by identity. Finally, because (20.11) is the numerator of the Hamiltonian (20.8), we substitute this back into (20.8), to find that with all $\mathbf{p}$ commuted to the right, the complete hyper-canonical Dirac Hamiltonian is:

$$
\boxed{
\begin{aligned}
&H = \\
&\frac{
\begin{array}{l}
\left.\begin{array}{l}
\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)c^2\mathbf{p}^2+2\left(E+mc^2\right)E\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
+\left(4E\rho\mathbf{A}_\gamma\cdot\rho\mathbf{A}_\gamma+2\left(c\mathbf{q}\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(c\mathbf{q}\times\rho\mathbf{A}_\gamma\right)\right)\right)\cdot c\mathbf{p}c\mathbf{p}\\
-2imc^2\mu_B\dot{\mathbf{A}}\cdot\mathbf{p}+\left(2E+4mc^2\right)\boldsymbol{\sigma}\cdot\left(\mu_B\dot{\mathbf{A}}\times\mathbf{p}\right)
\end{array}\right\}\\
\left.\begin{array}{l}
-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}\left(2E\left(E+mc^2\right)+c^2\mathbf{p}^2+c\mathbf{q}\cdot c\mathbf{p}+\left(10E\rho\mathbf{A}_\gamma+2i\mu_B\mathbf{E}\right)\cdot c\mathbf{p}+i\mu_B\mathbf{E}\cdot c\mathbf{q}\right)\\
-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}\left(\left(26e\left(\mathbf{A}_\gamma-\mathbf{A}\right)+2E\rho\mathbf{A}_\gamma\right)\cdot i\mu_B\mathbf{E}-\mu_B{}^2\mathbf{E}^2+4\pi\hbar c\mu_B\rho_{em}\right)\\
-i\hbar\mu_B\boldsymbol{\sigma}\cdot\dot{\mathbf{B}}\left(E+mc^2+4\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)
\end{array}\right\}\\
\left.\begin{array}{l}
+\boldsymbol{\sigma}\cdot\left(c\mathbf{q}+2E\rho\mathbf{A}_\gamma\right)\mu_B\mathbf{B}\cdot c\mathbf{p}+2\mu_B\mathbf{B}\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+2i\mu_B\mathbf{E}\right)\boldsymbol{\sigma}\cdot c\mathbf{p}\\
+2\mu_B\mathbf{B}\cdot\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}\right)\boldsymbol{\sigma}\cdot\left(c\mathbf{q}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}\right)\\
+2\mu_B\mathbf{B}\cdot\left(e\left(\mathbf{A}_\gamma-\mathbf{A}\right)\boldsymbol{\sigma}\cdot i\mu_B\mathbf{E}+i\mu_B\mathbf{E}\boldsymbol{\sigma}\cdot e\left(\mathbf{A}_\gamma-\mathbf{A}\right)\right)\\
-2\mu_B\mathbf{B}\cdot\left(\mu_B\mathbf{E}\times c\mathbf{p}-e\left(\mathbf{A}_\gamma+\mathbf{A}\right)\times\mu_B\mathbf{E}\right)+2E\mu_B{}^2\mathbf{B}^2\\
-2i\boldsymbol{\sigma}\cdot\mu_B\mathbf{A}\mu_B\dot{\mathbf{B}}\cdot\mathbf{p}+2i\mu_B\dot{\mathbf{B}}\cdot\mu_B\mathbf{A}\boldsymbol{\sigma}\cdot\mathbf{p}-2\mu_B\dot{\mathbf{B}}\cdot\left(\mu_B\mathbf{A}\times\mathbf{p}\right)+i\hbar\mu_B\dot{\mathbf{B}}\cdot\mu_B\mathbf{B}
\end{array}\right\}\\
+4\pi\hbar c\mu_B\rho_{em}\left(E+mc^2+4\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)+\left(4\pi\hbar\mu_B\mathbf{J}+\hbar\mu_B\dot{\mathbf{E}}\right)\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}\right)\}\\
\left.\begin{array}{l}
+4\rho\left(\mathbf{A}_\gamma-\mathbf{A}\right)\cdot i\mu_B\mathbf{E}c\mathbf{p}\cdot c\mathbf{p}+8\left(\rho\mathbf{A}_\gamma\cdot\boldsymbol{\sigma}\cdot\mu_B\mathbf{E}-\mu_B\mathbf{E}\cdot\boldsymbol{\sigma}\cdot\rho\mathbf{A}\right)\times c\mathbf{p}c\mathbf{p}\\
+\left(2\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}+2i\mu_B\mathbf{E}\cdot\rho\mathbf{A}_\gamma+4\rho\left(\mathbf{A}_\gamma-\mathbf{A}\right)\cdot i\mu_B\mathbf{E}\right)\cdot c\mathbf{p}c\mathbf{p}
\end{array}\right\}\\
\left.\begin{array}{l}
+\left(\left(4E\rho+8e\right)\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-2\mu_B{}^2\mathbf{E}^2\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
+4i\boldsymbol{\sigma}\cdot\left(E\rho\left(\mathbf{A}_\gamma+\mathbf{A}\right)\times i\mu_B\mathbf{E}-e\mathbf{A}_\gamma\times i\mu_B\mathbf{E}-\mu_B\dot{\mathbf{E}}\times\mu_B\mathbf{A}\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
-4i\left(E\rho\left(\mathbf{A}_\gamma-\mathbf{A}\right)\cdot\mu_B\mathbf{E}+e\mathbf{A}_\gamma\cdot\mu_B\mathbf{E}\right)i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times c\mathbf{p}\right)\\
+8\left(\mu_B{}^2\mathbf{E}^2+\mu_B\dot{\mathbf{E}}\cdot\mu_B\mathbf{A}/c\right)i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}\times c\mathbf{p}\right)
\end{array}\right\}\\
\left.\begin{array}{l}
+\left(2\left(E+mc^2\right)-4\left(2e\mathbf{A}_\gamma\cdot\rho\mathbf{A}-i\boldsymbol{\sigma}\cdot\left(e\mathbf{A}_\gamma\times\rho\mathbf{A}\right)\right)\right)i\mu_B\mathbf{E}\cdot c\mathbf{p}-2imc^2\mu_B\mathbf{E}\cdot c\mathbf{p}\\
+\left(2\left(E+2mc^2\right)-4\rho\left(\mathbf{A}_\gamma-\mathbf{A}\right)\cdot i\mu_B\mathbf{E}-4e\mathbf{A}_\gamma\cdot\rho\mathbf{A}\right)\boldsymbol{\sigma}\cdot\left(\mu_B\mathbf{E}\times c\mathbf{p}\right)
\end{array}\right\}\\
\left.\begin{array}{l}
+2\left(E+mc^2\right)E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-4e\mathbf{A}_\gamma\cdot\mu_B\mathbf{E}\left(\mu_B\mathbf{E}\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\mu_B\mathbf{E}\times\rho\mathbf{A}_\gamma\right)\right)\\
+2mc^2\left(e\left(\mathbf{A}_\gamma-\mathbf{A}\right)\cdot i\mu_B\mathbf{E}+\mu_B\mathbf{A}\cdot\mu_B\dot{\mathbf{E}}-i\boldsymbol{\sigma}\cdot\left(e\left(\mathbf{A}_\gamma+\mathbf{A}\right)\times i\mu_B\mathbf{E}-\mu_B\mathbf{A}\times\mu_B\dot{\mathbf{E}}\right)\right)\\
-\left(E-mc^2-4\rho\mathbf{A}_\gamma\cdot e\mathbf{A}\right)\mu_B{}^2\mathbf{E}^2
\end{array}\right\}
\end{array}
}{\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)^2-\mu_B{}^2\mathbf{B}^2}
\end{aligned}
}
. \qquad (21.1)
$$

22. Schrödinger's Equation, in the absence of Electromagnetic and Gravitational Interactions

From (20.4), we may calculate the work W, which is the total kinetic plus electromagnetic energy minus the fermion rest mass, from the eigenvalue equation $W|U_{0A}\rangle \equiv H|U_{0A}\rangle$. Therefore, (21.1) provides the ability to calculate the observed energy spectrum for individual fermion / photon interactions under a very broad domain of external conditions and combinations of conditions. We shall not explore all of these possible conditions, but rather, will focus on those of greatest interest, especially as regards the magnetic moment anomalies.

First, it will be seen that the majority of the terms in (21.1) arise when an external electrical field $\mathbf{E}$ is applied. Although this is of general interest and provides one avenue for experimental comparisons, we shall henceforth study the Hamiltonian (21.1) only in situations where there is no external electrical field applied, which we enforce by setting $\mathbf{E}=0$. Also, we will only study situations where there is no time-dependency in any field, whereby $\dot{\mathbf{A}}=0$, $\dot{\mathbf{B}}=0$ and $\dot{\mathbf{E}}=0$. (This is one reason we re-absorbed the residual $\omega=i\partial_t$ into the fields following (C.19).) And, we shall only study situations in which there is no charge density or current at the locale of the fermion / photon interaction, this setting $\rho_{em}=0$ and $\mathbf{J}=0$. With these restrictions on the external classical fields, (21.1) reduces to the greatly-simplified:

$$H=\frac{\left.\begin{array}{l}\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)c^2\mathbf{p}^2+2\left(E+mc^2\right)E\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\ +\left(4E\rho\mathbf{A}_\gamma\cdot\rho\mathbf{A}_\gamma+2\left(c\mathbf{q}\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(c\mathbf{q}\times\rho\mathbf{A}_\gamma\right)\right)\right)\cdot c\mathbf{p}c\mathbf{p}\\ -\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}\left(2E\left(E+mc^2\right)+c^2\mathbf{p}^2+c\mathbf{q}\cdot c\mathbf{p}+10E\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)\\ +\boldsymbol{\sigma}\cdot\left(c\mathbf{q}+2E\rho\mathbf{A}_\gamma\right)\mu_B\mathbf{B}\cdot c\mathbf{p}+2\mu_B\mathbf{B}\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma\right)\boldsymbol{\sigma}\cdot c\mathbf{p}\\ +2E\mu_B\mathbf{B}\cdot\rho\mathbf{A}_\gamma\boldsymbol{\sigma}\cdot\left(c\mathbf{q}+E\rho\mathbf{A}_\gamma\right)+2E\mu_B{}^2\mathbf{B}^2\end{array}\right\}}{\left(E+mc^2+2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\right)^2-\mu_B{}^2\mathbf{B}^2}. \tag{22.1}$$

Now, let us consider three further special cases of (22.1). In the first special case, let us turn off all electromagnetic interactions by setting $e=0$ thus $\rho=-e/mc^2=0$ and $\mathbf{B}=0$ above. Further, in this situation, the energy content relation $E=mc^2\Gamma=mc^2\left(dt/d\tau\right)=mc^2\gamma_v\gamma_g\gamma_{em}$ of (6.3) which we found following (12.4) remains intact for quantum as well as classical systems, with gravitation also turned off, becomes $E=mc^2\gamma_v$ with $\gamma_v=1/\sqrt{1-v^2/c^2}$, well-known from the Special Theory of Relativity. Using all of this in (22.1), and further showing the non-relativistic $v/c\to 0$ limit where $\gamma_v\to 1$ we obtain:

$$H=\frac{\left(E+mc^2\right)c^2\mathbf{p}^2}{\left(E+mc^2\right)^2}=\frac{c^2\mathbf{p}^2}{E+mc^2}=\frac{\mathbf{p}^2}{m\left(\gamma_v+1\right)}\underset{v/c\to 0}{\to}\frac{\mathbf{p}^2}{2m}=\frac{1}{2}m\mathbf{v}^2\,. \tag{22.2}$$

This correct result tells us that absent electromagnetic interactions and in the non-relativistic limit, the Hamiltonian approaches the Newtonian kinetic energy $E_{kin}=\frac{1}{2}m\mathbf{v}^2$.

Moreover, we may use the above to operate on a fermion wavefunction ket $\left|\Psi\right\rangle$, while also applying the eigenvalue relation $i\hbar\partial_\mu\left|\Psi\right\rangle=p_\mu\left|\Psi\right\rangle$ which separates into $E\left|\Psi\right\rangle=i\hbar\partial_t\left|\Psi\right\rangle$ and $\mathbf{p}\left|\Psi\right\rangle=-i\hbar\nabla\left|\Psi\right\rangle$. Also using $H\left|\Psi\right\rangle=\left(E-mc^2\right)\left|\Psi\right\rangle$ from the first line of (20.4) when $\mathbf{B}=0$ as it is here, then adding mc^2 inside of all terms, we obtain:

$$\left(H+mc^2\right)\left|\Psi\right\rangle=E\left|\Psi\right\rangle=i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle=\left(-\frac{\hbar^2c^2\nabla^2}{E+mc^2}+mc^2\right)\left|\Psi\right\rangle\underset{v/c\to 0}{\to}\left(-\frac{\hbar^2\nabla^2}{2m}+mc^2\right)\left|\Psi\right\rangle\,. \tag{22.3}$$

This will be recognized in the non-relativistic limit as the time-dependent Schrödinger equation with a potential energy $V=mc^2$ merely containing the rest energy. So, we see that absent electromagnetic and gravitational interactions, (21.1) produces the Schrödinger equation, as it must to accord with settled physics.

It is also helpful to rewrite (22.2) entirely using time dilations as, using the special relativistic $cp^\mu=mcv^\mu\gamma_v$ from which we obtain the space-component relation $\mathbf{p}^2=m^2\mathbf{v}^2\gamma_v{}^2$. We also use $\mathbf{v}^2/c^2=\left(\gamma_v{}^2-1\right)/\gamma_v{}^2=\left(\gamma_v+1\right)\left(\gamma_v-1\right)/\gamma_v{}^2$ which merely is a restatement of the square relation $\gamma_v{}^2=1/\left(1-\mathbf{v}^2/c^2\right)$. With these, (22.2) may be rewritten as:

$$H=\frac{\mathbf{p}^2}{m\left(\gamma_v+1\right)}=\frac{m\mathbf{v}^2\gamma_v{}^2}{\gamma_v+1}=mc^2\frac{\gamma_v{}^2}{\gamma_v+1}\frac{\mathbf{v}^2}{c^2}=mc^2\frac{\gamma_v{}^2}{\gamma_v+1}\frac{\left(\gamma_v+1\right)\left(\gamma_v-1\right)}{\gamma_v{}^2}=mc^2\left(\gamma_v-1\right), \tag{22.4}$$

This also contains the useful relation $\mathbf{p}^2=m^2c^2\left(\gamma_v-1\right)\left(\gamma_v+1\right)=m^2c^2\left(\gamma_v{}^2-1\right)$. As it must, $mc^2\left(\gamma_v-1\right)=mc^2\left(1/\sqrt{1-\mathbf{v}^2/c^2}-1\right)\underset{\mathbf{v}/c\to 0}{\to}\frac{1}{2}m\mathbf{v}^2$ matches the non-relativistic limit shown in (22.2).

23. Magnetic Moment Anomalies without Renormalization

For the second special case of (22.1), let us keep the electromagnetic interactions and the magnetic field $\mathbf{B}$ in (22.1), but let us observe the fermion at rest such that *any Lorentz boost is removed from the Dirac fermion* and thus $\mathbf{p}=0$. In this rest frame, (22.1) becomes:

$$H=\frac{-2E\left(E+mc^2\right)\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}+2E\mu_B\mathbf{B}\cdot\rho\mathbf{A}_\gamma\boldsymbol{\sigma}\cdot\left(c\mathbf{q}+E\rho\mathbf{A}_\gamma\right)+2E\mu_B{}^2\mathbf{B}^2}{\left(E+mc^2\right)^2-\mu_B{}^2\mathbf{B}^2}. \tag{23.1}$$

Now, as reviewed following (16.10) and (16.12), whenever an external materially-sourced classical field is applied, the effects of individual photons are completely swamped by the classical field. This was later used to obtain (21.1) by advancing all *gauge-invariant* fields for individual photons to materially-sourced classical external fields, that is, $\mathbf{B}_\gamma \mapsto \mathbf{B}$, $\mathbf{E}_\gamma \mapsto \mathbf{E}$, $\rho_{em\ \gamma} \mapsto \rho_{em}$ and $\mathbf{J}_\gamma \mapsto \mathbf{J}$, while leaving the gauge-dependent $\mathbf{A}_\gamma$ for individual photons as is.

Now, in (23.1) we are applying a classical external magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ with $\mathbf{A}$ containing the space components of the classical $A^\mu=\left(\phi \quad \mathbf{A}\right)$. And as we saw at (16.10), the time component ϕ can only be non-zero when there is a non-zero source density $J^\mu \neq 0$. So, in (23.1) the potentials $\mathbf{A}_\gamma$ for individual photons will be physically swamped by the innumerable photons in **A** carrying the classical field **B**. With $\mathbf{A}_\gamma$ swamped by **B** we may set $\mathbf{A}_\gamma=0$ in (23.1) to an extremely accurate approximation. Thus, (23.1) becomes:

$$H=\frac{-2E\left(E+mc^2\right)\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}+2E\mu_B{}^2\mathbf{B}^2}{\left(E+mc^2\right)^2-\mu_B{}^2\mathbf{B}^2}. \tag{23.2}$$

Let us now spend a moment on the denominator above. Turning again to the energy content relation (6.3) which in the present circumstance with the fermion at rest thus $\gamma_v=1$ becomes $E=mc^2\gamma_{em}$, the denominator (D) above may be rewritten as $\mathrm{D}=m^2c^4\left(\gamma_{em}+1\right)^2-\mu_B{}^2\mathbf{B}^2$. Noting that the numerator has dimensions of energy-cubed, we then compare the impact of the squared magnitude $\mathbf{B}^2$ of the magnetic field to that of the $m^2c^4\left(\gamma_{em}+1\right)^2$ by dividing through by m^2c^4, while also approximating $\gamma_{em}\cong 1$, thus obtaining $\mathrm{D}/m^2c^4 \cong 4-\mu_B{}^2\mathbf{B}^2/m^2c^4$. The salient ratio is then $\mu_B{}^2\mathbf{B}^2/4m^2c^4$, which has the square root $\mu_B\left|\mathbf{B}\right|/2mc^2$. The Bohr magneton $\mu_B=\hbar e/2mc$ for the electron with $m=m_e$ written in units of electron volts is $\mu_B \cong 5.788\times10^{-11}\ \mathrm{MeV/T}$, while the electron rest energy $m_ec^2 \cong .511\ \mathrm{MeV}$, so that $\mu_B/2m_ec^2 \cong 5.663\times10^{-11}\ 1/\mathrm{T}$. For the mu and tau leptons it is even smaller. So even for an extraordinary magnetic field $\left|\mathbf{B}\right|=100\ \mathrm{T}$ which is twice the size of the largest continuous magnetic field ever produced in a laboratory, the ratio $\mu_B\left|\mathbf{B}\right|/2m_ec^2=5.663\times10^{-9}\ 1/\mathrm{T}$ and $\mu_B{}^2\mathbf{B}^2/4m_e{}^2c^4=3.207\times10^{-17}$. Indeed, even a neutron star only has a magnetic field of about 10^6 T. So given that $\mu_B{}^2\mathbf{B}^2$ will at best affect the magnitude of the denominator a factor of 1 part in 10^{20} for an electron, and even less for the other leptons, we may safely neglect this term in the denominator and approximate it to zero. Thus, setting $\mu_B{}^2\mathbf{B}^2$ in the (23.2) denominator and reducing, we now have:

$$H=\frac{-2E\left(E+mc^2\right)\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}+2E\mu_B{}^2\mathbf{B}^2}{\left(E+mc^2\right)^2}=-\frac{2E}{E+mc^2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}+\frac{2E}{\left(E+mc^2\right)^2}\mu_B{}^2\mathbf{B}^2$$
$$=\left(-1+\frac{1}{E+mc^2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}\right)\frac{2E}{E+mc^2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}=\left(-1+\frac{\mu_B}{\left(\gamma_{em}+1\right)mc^2}\boldsymbol{\sigma}\cdot\mathbf{B}\right)\frac{2E}{E+mc^2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}\quad. \tag{23.3}$$

In the bottom line, we have also used the known property $\mathbf{B}^2=\left(\boldsymbol{\sigma}\cdot\mathbf{B}\right)^2$ of the Pauli matrices (generally $\mathbf{x}^2=\left(\boldsymbol{\sigma}\cdot\mathbf{x}\right)^2$ for any vector **x**), and then again used $E=mc^2\gamma_{em}$.

Next, we again use $\gamma_{em}\cong 1$ to note that $\mu_B/\left(\gamma_{em}+1\right)mc^2\cong\mu_B/2m_ec^2\cong 5.663\times 10^{-11}\ 1/\mathrm{T}$. Thus, even if **B** is very large, say, $\left|\mathbf{B}\right|=100\,\mathrm{T}$, the term $-1+\left(\mu_B/\left(\gamma_{em}+1\right)mc^2\right)\boldsymbol{\sigma}\cdot\mathbf{B}\cong -1$ will differ from -1 by only about one part in 10^{10} for an electron, and less for the other leptons. As a result, we may neglect $\left(\mu_B/\left(\gamma_{em}+1\right)mc^2\right)\boldsymbol{\sigma}\cdot\mathbf{B}$ in the above. Additionally using $E=mc^2\gamma_{em}=mc^2+E_{em}$, (23.3) reduces to:

$$H=-\frac{2E}{E+mc^2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}=-\frac{2mc^2+2E_{em}}{2mc^2+E_{em}}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}=-\frac{2\gamma_{em}}{\gamma_{em}+1}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}\,. \tag{23.4}$$

And now we are ready to review the magnetic moment anomaly and renormalization.

The term $-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ in the usual Dirac equation has a coefficient $g_D/2=1$ based on the Dirac *g*-factor $g_D=2$. But in (23.4), it has a coefficient $2E/\left(E+mc^2\right)=2\gamma_{em}/\left(\gamma_{em}+1\right)$. At the end of section 5 we reviewed how even for very powerful electromagnetic interactions, $\gamma_{em}\cong 1$. So with γ_{em} approximately but not exactly equal to 1 whenever there are electromagnetic interactions, this coefficient is likewise close to 1 but not exactly equal to 1. As such, this coefficient is at least suggestive of the magnetic moment anomaly. However, as will be reviewed more deeply in the next section, the magnetic moment anomaly is understood to arise *exclusively* from the electromagnetic (and much smaller hadronic and electroweak) *self-interactions* of a fermion *with itself*. This understanding is the genesis of the inordinate numbers of Feynman loop diagrams used to calculate magnetic moment anomalies.

Now, in (23.4), $E=mc^2\gamma_{em}=mc^2+E_{em}$ is the total energy of a charged body at rest in an *external* electromagnetic potential absent gravitation, i.e., its rest-plus-electromagnetic-interaction energy. (See again (6.1), (6.2) and (7.4), and see section 12 which establishes that the complete energy content relation $E=\Gamma mc^2=\gamma_v\gamma_g\gamma_{em}mc^2$ carries through from classical to quantum systems.) But it is well-established that the only interaction energies which go into the lepton magnetic moment anomalies, are these *self-interaction* (SI) energies, which we denote as E_{SI}. Again, this is why renormalization theory and explanations of the magnetic moment anomaly are built around a plethora of Feynman diagrams all containing various possible self-interaction loops, theoretically extending to infinite order, and calculable as a practical matter only through three or four orders. Therefore, if we wish to apply (23.4) to an *individual* charged lepton, E_{em} must represent the electromagnetic interaction energy of the lepton, *not in an external potential, but the self-interaction energy in its own potential*. Consequently, if we wish to apply (23.4) to the self-interactions of individual leptons, we must reinterpret $E_{em} \mapsto E_{SI}$.

Moreover, each charged lepton has an observed rest energy mc^2. But, part of this rest energy will naturally arise from the lepton's self-interaction energies E_{SI}, i.e., from all of its Feynman diagram self-interaction loops. Therefore, if we now employ $m_0c^2 \equiv mc^2 - E_{SI}$ (i.e. $mc^2 \equiv m_0c^2 + E_{SI}$) to define a lepton's *bare rest energy* defined its observed rest energy mc^2 less its self-interaction energy E_{SI}, then to apply (23.4) to an individual lepton, we must reinterpret $mc^2 \mapsto m_0c^2$ to be the bare rest energy of the lepton, and the energy content relation for a fermion at rest and absent gravitation becomes $mc^2 = m_0c^2\gamma_{em} = m_0c^2 + E_{SI}$. With the foregoing definitions and reinterpretations, (23.4) now becomes:

$$H = -\frac{2mc^2}{mc^2 + m_0c^2}\mu_B \boldsymbol{\sigma}\cdot\mathbf{B} = -\frac{2m_0c^2 + 2E_{SI}}{2m_0c^2 + E_{SI}}\mu_B \boldsymbol{\sigma}\cdot\mathbf{B} = -\frac{2\gamma_{em}}{\gamma_{em}+1}\mu_B \boldsymbol{\sigma}\cdot\mathbf{B}\,. \tag{23.5}$$

Now, in general, the coefficient of $\mu_B \boldsymbol{\sigma}\cdot\mathbf{B}$ in the Hamiltonian does correspond to one-half of the g-factor i.e. to $g/2$. So, it appears we might be able to associate $g/2$ with the coefficient $2\gamma_{em}/(\gamma_{em}+1)$ containing the time dilation. But first, we must account for one final matter.

The Particle Data Group in [14] provides a very thorough review of the muon anomalous magnetic moment. Although the numeric data developed in this review applies specifically to the muon, the theoretical principles exposited for analysis apply equally to the electron and to the tau lepton. For a given lepton, the complete standard model anomaly denoted in [14] as a_{SM}, which we simply denote here as $a = (g-2)/2$, is generally divided into three parts, namely, QED, hadronic and electroweak contributions to the lepton self-interaction. To leading order for all three leptons as first uncovered by Schwinger [15], $a \cong \alpha/2\pi$, where $\alpha = e^2/4\pi\varepsilon_0\hbar c$ is the running fine structure coupling which approaches the numerical value of $\alpha = 1/137.035999139$ [16] at low probe energies. These are then summed whereby $a = a_{\mathrm{QED}} + a_{\mathrm{Had}} + a_{\mathrm{EW}}$, see equation 4 and Figure 1 in [14]. This may also be written in terms of the g-factor and approximately tied via $g/2 = 1 + a = 1 + a_{\mathrm{QED}} + a_{\mathrm{Had}} + a_{\mathrm{EW}} \cong 1 + \alpha/2\pi = 1.00116140973242$ to Schwinger's first order anomaly $\alpha/2\pi$. Nonetheless, although each anomaly has these three contributions, the electromagnetic contribution dominates the other two by four or five orders of magnitude respectively. So up to this parts-per-greater-than-10^4 difference one may use the very close approximation $a \cong a_{\mathrm{QED}}$. Here, we denote a_{QED} as a_{em}. Thus, one may denote the electromagnetic self-interaction contribution to the g-factor as $g_{em}/2 \equiv 1+a_{em} \cong g/2$. The same qualitative considerations – though not the exact same numbers – apply to the electron and the tau lepton.

With this in mind, we see that the energy contributions in (23.5) arise only from the *electromagnetic* self-interactions of the charged leptons, and do not account for the comparatively tiny hadronic and electroweak contributions, with the electromagnetic contribution dominating by four or five orders of magnitude. Therefore, we cannot associate $2\gamma_{em}/(\gamma_{em}+1)$ in (23.5) with the complete $g/2 = 1+a_{em}+a_{\mathrm{Had}}+a_{\mathrm{EW}}$. Rather, to a parts-per-10^4 approximation we may only associate this with $g_{em}/2 \equiv 1+a_{em}$. Thus, we may finally identify the electromagnetic g-factor as:

$$\boxed{\frac{g_{em}}{2} = \frac{2mc^2}{mc^2+m_0c^2} = \frac{2m_0c^2+2E_{SI}}{2m_0c^2+E_{SI}} = \frac{2\gamma_{em}}{\gamma_{em}+1}}. \tag{23.6}$$

Next, we may place (23.6) into (23.5) to obtain:

$$H = -\frac{g_{em}}{2}\mu_B \boldsymbol{\sigma}\cdot\mathbf{B} = \boldsymbol{\mu}_{em}\cdot\mathbf{B}. \tag{23.7}$$

where, making note of (19.8) and particularly that the magnetic moment is defined so as to include the g factor, we define the electromagnetic portion of the magnetic moment as:

$$\boldsymbol{\mu}_{em} \equiv -\frac{g_{em}}{2}\mu_B\boldsymbol{\sigma} = -\frac{g_{em}}{2}\frac{\hbar e}{2mc}\boldsymbol{\sigma}, \tag{23.8}$$

with m being the mass of the charged lepton under consideration in any given situation.

Now, in (23.6) through (23.8) the magnetic moment anomaly – or at least the dominant electromagnetic contribution to the anomaly – has been obtained without any appearance of infinite quantities and thus without any need to resort to renormalization. The reason that the need to renormalize arises in the first place, is because in the usual Dirac equation the coefficient of $-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ is $g_D/2=1$, i.e., because $g=2$ exactly. But in the physical world, the empirical, observed g-factor is slightly larger than 2. And as noted already, to first order for all leptons, the observed $g/2 = 1+\alpha/2\pi = 1.00116140973242$ owing originally to Schwinger [15], and contains the fine structure number $\alpha = 1/137.035999139$ [16] for electromagnetic interaction strength at low probe energies. This discrepancy – a.k.a. "anomaly" – between what the Dirac equation predicts and what is actually observed, must be remedied. And at present, given that Dirac's equation merely produces a coefficient $g_D/2=1$, the only known remedy to address the fact that $g/2$ is the actual coefficient of $\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$, is infinite-number renormalization.

In other words, the reason why renormalization is needed in the first place, is because the ordinary canonical Dirac equation only predicts a term $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ in the Dirac Hamiltonian, or more precisely, a term $-(g_D/2)\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ with a Dirac *g*-factor $g_D = 2$. But in nature, what is observed is a $-(g/2)\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ with $g/2$ slightly larger than 1 – again, to first order – with $g/2 = 1 + \alpha/2\pi$. So the *raison d'être* for renormalization is to fill the gap from $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma \mapsto -(g/2)\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$ by producing a coefficient for $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ which is slightly larger than 1. But in (23.5), the hyper-canonical Dirac Hamiltonian has naturally provided us with a coefficient which is larger than 1, and indeed, only slightly larger than 1.

To directly illustrate why renormalization is no longer needed in view of (23.8), and how it is replaced, we now combine (20.4) using $|\Psi\rangle$ rather than $|U_{0A}\rangle$, with (23.8), and also use the relation $E = mc^2\gamma_{em}$ from (6.2) for a material body at rest and absent gravitation to write:

$$\begin{aligned} H|\Psi\rangle &= \left(E - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma - mc^2\right)|\Psi\rangle = \left(mc^2(\gamma_{em} - 1) - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma\right)|\Psi\rangle \\ &= (\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi})\left(\mathrm{E} + \mathrm{M}c^2\right)^{-1}(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi})|\Psi\rangle = -\frac{2\gamma_{em}}{\gamma_{em}+1}\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}|\Psi\rangle = -\frac{g_{em}}{2}\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}|\Psi\rangle \end{aligned} . \tag{23.9}$$

Now, $H = (\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi})\left(\mathrm{E} + \mathrm{M}c^2\right)^{-1}(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi})$ – which in its entirety is equal to (21.1) – becomes (23.8) when all of $\mathbf{p} = 0$, $\mathbf{E} = 0$, $\dot{\mathbf{A}} = 0$, $\dot{\mathbf{B}} = 0$, $\dot{\mathbf{E}} = 0$, $\rho_{em} = 0$ and $\mathbf{J} = 0$. But as noted after (20.5), $(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi})\left(\mathrm{E} + \mathrm{M}c^2\right)^{-1}(\boldsymbol{\sigma} \cdot c\boldsymbol{\Pi})|\Psi\rangle$ and $\left(E - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma - mc^2\right)|\Psi\rangle$ are merely the space and time-minus-rest energy components of a single equation, analogous to the relativistic energy-momentum relation $c^2\mathbf{p}^2 = E^2 - m^2c^4$. As a result, a magnetic moment term $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma$ with an implied coefficient $g_D/2$ residing in the *time components* of (23.9), is projected into a term $-(g_{em}/2)\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$ residing in the *space components* of (23.9). So, the gap from $-\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_\gamma \mapsto -(g_{em}/2)\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$ is filled, not by renormalization, but by *basic spacetime physics involving a relation between the time and space components of a single equation.* And as noted, this electromagnetic contribution to the overall g is greatly dominant over the hadronic and electroweak contributions. So in sum, the anomaly arises, physically, because a $\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$ term with $g = g_D = 2$ in the time components of (23.9) is empirically witnessed as a $\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$ term with $g_{em} = 4E/\left(E + mc^2\right) = 4\gamma_{em}/(\gamma_{em}+1)$ in the space components of (23.9). Thus (for the moment, to the dominant order of electromagnetic contributions), *the observed anomaly is merely one of the many consequences of the physical relation between space and time.* And the key parameter which determines the actual magnitude of the anomaly, is the electromagnetic time dilation γ_{em}.

We also point out that to obtain (23.6), the "reinterpretations" $E_{em} \mapsto E_{SI}$ and $mc^2 \mapsto m_0c^2$ and $m_0c^2 \equiv mc^2 - E_{SI}$ used to go from (23.4) to (23.5) are still a form of renormalization, albeit using only finite, not infinite energies. This is because in (23.5), we now have a finite bare quantity – namely a finite bare mass m_0 – being "renormalized" into a finite dressed / observed quantity – namely the finite observed rest mass m – along the lines of the approach advocated by, e.g., [17].

We also note in passing that the above results provide further validation to the substitutions $\mathbf{B}_\gamma \mapsto \mathbf{B}$, $\mathbf{E}_\gamma \mapsto \mathbf{E}$, $\rho_{em\,\gamma} \mapsto \rho_{em}$, $\mathbf{J}_\gamma \mapsto \mathbf{J}$ and $\mathbf{B}_\gamma \mapsto \mathbf{B}$ for the various gauge-invariant fields, used to advance (20.11) to (21.1). Specifically, even if one was to doubt the existence of electromagnetic time dilations and suppose they did not exist whereby $\gamma_{em} = 1$, or if one were to accept these time dilations but consider the $\gamma_{em} = 1$ approximation, in either event (23.6) would produce $g_{em}/2 = g_D/2 = 1$ and the magnetic moment term in (23.8) would reduce to $\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$. That this very term exists in the Dirac Hamiltonian, and that the $\mathbf{B}$ in *this term may be used to describe an externally-applied classical magnetic field and is not restricted solely to the magnetic field* $\mathbf{B}_\gamma$ *for a single photon*, is well-settled physics both theoretically and empirically. As such, this correspondence with well-settled physics in the $\gamma_{em} = 1$ limit also validates replacing gauge-invariant photon fields with classical external materially-sourced fields used to obtain (21.1).

In conclusion, despite the many critiques which have been leveled by the likes of Dirac [18] and Feynman [19] at the use of a renormalization technique which subtracts some infinites from other infinites to obtain finite answers, this standard method of renormalizing infinities is commendable and has exhibited remarkable staying power for one very important reason: the finite answers it produces are empirically correct, to extraordinarily high precision. But being a mere technique and not really a theory about nature, it suffers the theoretical fault of providing little insight into the fundamentally-intelligible order of nature. The results in highlighted in (23.9) hold out the prospect that infinite-number renormalization techniques can finally be replaced with the very deep insight that the magnetic moment anomaly is no more and no less than another consequence all physics occurring on the stage of spacetime, and of time dilations which directly drive the energy content of material bodies. But if the foregoing is to replace renormalization as the basis for understanding magnetic moment anomalies, it is important to show how this claim might be empirically tested. That is the purpose of the next two sections.

PART V: PROPOSED EXPERIMENTAL TESTS OF THE CONNECTION BETWEEN MAGNETIC MOMENT ANOMALIES AND ELECTROMAGNETIC TIME DILATIONS

24. Two Proposed Experimental Magnetic Moment Tests: Lepton Time Dilation, and Dressed versus Bare Rest Energies

We begin to study experimental tests of the hyper-canonical magnetic moment anomaly predicted in (23.6) by algebraically restructuring the *g*-factor relation (23.6) and also applying the finite renormalization relation $mc^2 = m_0 c^2 \gamma_{em} = m_0 c^2 + E_{SI}$ to obtain:

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{m}{m_0} = \frac{mc^2}{m_0 c^2} = \frac{m_0 c^2 + E_{SI}}{m_0 c^2} = \frac{g_{em}}{4 - g_{em}} = \frac{1 + a_{em}}{1 - a_{em}}. \tag{24.1}$$

We notice when placed into the form $\gamma_{em} = mc^2 / m_0 c^2$, that for an individual charged lepton the electromagnetic time dilation γ_{em} not only measures an electromagnetic time dilation $dt / d\tau$ intrinsic to that lepton, but it also measures the ratio m / m_0 of the observed "dressed" rest mass which includes electromagnetic self-interactions, to the bare rest mass which excludes electromagnetic self-interactions. In other words, the time dilation and the "dressed-to-bare" mass ratios for individual leptons are one and the same. Consequently, it is desirable to calculate this time dilation and the dressed versus bare masses for each of the three leptons.

Using (24.1), to an approximation that is valid within parts-per 10^4 as reviewed prior to (23.6), we may use the empirical values $g_e = 2.00231930436152$ and $g_\mu = 2.0023318418$ deduced from [20] and $g_\tau = 2.00235442$ from [21] for the three types of lepton to immediately obtain the approximate magnitude of γ_{em} for each of the three leptons. This calculation yields:

$$\gamma_{em\ (e)} \cong 1.00232199707049; \quad \gamma_{em\ (\mu)} \cong 1.0023345637; \quad \gamma_{em\ (\tau)} \cong 1.00235719. \tag{24.2}$$

Because $\gamma_{em} = dt / d\tau$, (24.2) is a prediction of *an electromagnetic time dilation intrinsically associated with each lepton*, due on the internal repulsive electromagnetic self-interaction energies of those leptons. Therefore, based on what was first discovered at (5.8) and (5.9) and further developed at (12.3) and (12.4), to the extent that an experiment can be designed to treat an individual lepton as a geometrodynamic clock emitting periodic signals, (24.2) tells the predicted time dilation of that lepton relative to a neutral laboratory clock, neglecting hadronic and electroweak contributions.

Now, let's take a closer look at the energies in (24.1). Because each of (24.2) via (24.1) also tells us the ratio m / m_0 of the dressed-to-bare rest masses and energies of each lepton, we may immediately calculate using the empirical rest energies $m_e c^2 = 0.5109989280\,\text{MeV}$; $m_\mu c^2 = 105.6583715\,\text{MeV}$ and $m_\tau c^2 = 1776.86\,\text{MeV}$ from [22] together with $\gamma_{em} = mc^2 / m_0 c^2$ from (24.1) and the three intrinsic lepton γ_{em} in (24.2), that:

$$\begin{aligned} &m_{e0}c^2 = 0.5098151387\,\text{MeV}; \quad m_{\mu 0}c^2 = 105.4122798\,\text{MeV}; \quad m_{\tau 0}c^2 = 1772.68\,\text{MeV} \\ &E_{SI\,e} = 0.0011837893\,\text{MeV}; \quad E_{SI\,\mu} = 0.2460917\,\text{MeV}; \quad E_{SI\,\tau} = 4.18\,\text{MeV} \end{aligned} \tag{24.4}$$

Consequently, to the extent that an experiment can be designed to separately determine how much of the total rest energy of each lepton arises from electromagnetic self-interactions and how much is a non-electromagnetic base energy, (24.4) predicts this energy division.

25. Three Additional Proposed Experimental Magnetic Moment Tests based on Relativistic and Nonrelativistic Lepton Kinetic Energies and Applied Magnetic Fields

Starting at (23.1) we set $\mathbf{p} = 0$ in (22.1) so that when we identified the charged lepton *g*-factor in (23.6), that *g*-factor depended only on the lepton electromagnetic self-interaction energies and not on the kinetic energy of the overall lepton. Naturally, when a fermion is observed in motion – whether relativistic or non-relativistic –the Hamiltonian and thus the energy eigenvalues will change. But *the g-factor itself must be independent of the state of motion of the lepton*, which is why the *g*-factor in (23.6) was defined based on observing the fermion at rest. Now that we have identified the *g*-factor (or precisely, the dominant electromagnetic contribution to the g-factor), we wish to study the Hamiltonian and associated energies when a charged lepton in an externally applied magnetic field is observed in motion, bot relativistic and non-relativistic. This will provide us with some additional experimental tests of these results.

Accordingly, we return to (22.1), but now we apply an external magnetic field and allow the fermion to be in motion. As discussed prior to (23.2), because $\mathbf{A}_\gamma$ for any individual photon will be swamped by the enormous number of photons carrying the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, we may likewise set $\mathbf{A}_\gamma = 0$ in (22.1) to obtain:

$$H = \frac{\begin{array}{c}\left(E + mc^2\right)c^2\mathbf{p}^2 - \mu_B \boldsymbol{\sigma}\cdot\mathbf{B}\left(2E\left(E+mc^2\right) + c^2\mathbf{p}^2 + c\mathbf{q}\cdot c\mathbf{p}\right) \\ +\boldsymbol{\sigma}\cdot c\mathbf{q}\mu_B \mathbf{B}\cdot c\mathbf{p} + 2\mu_B \mathbf{B}\cdot c\mathbf{p}\boldsymbol{\sigma}\cdot c\mathbf{p} + 2E{\mu_B}^2\mathbf{B}^2\end{array}}{\left(E+mc^2\right)^2 - {\mu_B}^2\mathbf{B}^2}. \tag{25.1}$$

Additionally, let us require that the applied magnetic field be a *constant* field with no spatial variation, i.e., that $\nabla\mathbf{B} = \nabla^i B^j = 0$. In this event, we may use $-i\hbar c\nabla\mathbf{B} = c\mathbf{q}\mathbf{B} = 0$ from (16.14) and the commutativity properties of **q** reviewed following (15.15) to also set to zero, the two terms above which contain **q**. Finally, for the reasons reviewed for going from (23.2) to (23.4), we may set ${\mu_B}^2\mathbf{B}^2 = 0$ above. As a result, separating terms, the above now becomes:

$$H = \frac{c^2\mathbf{p}^2}{E+mc^2} - \left(\frac{2E}{E+mc^2} + \frac{c^2\mathbf{p}^2}{\left(E+mc^2\right)^2} \right) \mu_B \boldsymbol{\sigma}\cdot\mathbf{B} + \frac{2\mu_B \mathbf{B}\cdot c\mathbf{p}\boldsymbol{\sigma}\cdot c\mathbf{p}}{\left(E+mc^2\right)^2}. \tag{25.2}$$

The first term $c^2\mathbf{p}^2/\left(E+mc^2\right)$ was already seen in (22.2). The coefficient $2E/\left(E+mc^2\right)$ was first encountered in (23.4) and was shown in (23.6) for a fermion at rest to be equal to $g_{em}/2$ following a reinterpretation of the energies when applied to individual leptons. So there two terms have previously been studied. The remaining terms with $\mathbf{p}\neq 0$ have not previously studied, and of course these capture the effects to motion which we shall now study here.

First, we will wish to renormalize (25.2) in the same way that we did (23.4) to get to (23.5). To do so, we start by renormalizing $mc^2 \mapsto m_0c^2$ to the bare rest energy of the leptons. Then we renormalize $c\mathbf{p} = mc\mathbf{v}\gamma_v\gamma_{em} \mapsto m_0 c\mathbf{v}\gamma_v\gamma_{em}$ thus $c^2\mathbf{p}^2 = m^2c^2\mathbf{v}^2{\gamma_v}^2{\gamma_{em}}^2 \mapsto {m_0}^2c^2\mathbf{v}^2{\gamma_v}^2{\gamma_{em}}^2$, and the total energy $E = mc^2\gamma_v\gamma_{em} \mapsto m_0c^2\gamma_{em}\gamma_v$. We create several $c\mathbf{v}/c = \mathbf{v}$ and then divide out a number of $m_0c^2/m_0c^2 = 1$ ratios. We also then use $\mathbf{v}^2/c^2 = \left({\gamma_v}^2-1\right)/{\gamma_v}^2$ Finally we apply the renormalization step $\gamma_{em}m_0c^2 = mc^2$. Following this finite-quantity renormalization we obtain:

$$H = \frac{\gamma_{em}\left({\gamma_v}^2-1\right)}{\gamma_{em}\gamma_v+1}mc^2 - \left(\frac{2\gamma_v\gamma_{em}}{\gamma_v\gamma_{em}+1} + \frac{{\gamma_{em}}^2\left({\gamma_v}^2-1\right)}{\left(\gamma_v\gamma_{em}+1\right)^2} \right) \mu_B\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{2{\gamma_v}^2{\gamma_{em}}^2}{\left(\gamma_{em}\gamma_v+1\right)^2}\mu_B\mathbf{B}\cdot\frac{\mathbf{v}}{c}\boldsymbol{\sigma}\cdot\frac{\mathbf{v}}{c}. \tag{25.3}$$

The terms linear in **B** turn out to be entirely unaffected by the renormalization because of an offset between numerator and denominator. For the first term with mc^2, the numerator drops by one order from ${\gamma_{em}}^2$ to γ_{em}, which results from applying $\gamma_{em}m_0c^2 = mc^2$.

For a fermion at rest with $\mathbf{v}=0$ and $\gamma_v = 1$ and using $g_{em}/2 = 2\gamma_{em}/\left(\gamma_{em}+1\right)$ from (23.6), the above reduces to:

$$H = -\frac{2\gamma_{em}}{\gamma_{em}+1}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B} = -\frac{g_{em}}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B} = \boldsymbol{\mu}_{em}\cdot\mathbf{B}. \tag{25.4}$$

This reproduces (23.7) for the magnetic moment, as it must. Conversely, in the extreme relativistic limit where the special relativistic time dilation factor $\gamma_v \to \infty$ and $|\mathbf{v}| \to c$, but the electromagnetic time dilations (24.1) and (24.2) remain as is slightly above 1, $\mathbf{v}/c \to \hat{\mathbf{u}}$ becomes a unit vector with a magnitude $\hat{\mathbf{u}}\cdot\hat{\mathbf{u}} = 1$ and orientated in the direction of the fermion motion. For example, for a z-propagating extreme-relativistic fermion, $\hat{\mathbf{u}}=(0,0,1)$. Accordingly, for extreme relativistic motion, where $\gamma_{em}\left(\gamma_v{}^2-1\right)/\left(\gamma_{em}\gamma_v+1\right)=\left(\gamma_v-1\right)\cdot\left(\left(\gamma_v+1\right)\gamma_{em}/\left(\gamma_{em}\gamma_v+1\right)\right)\to\gamma_v-1$ in (25.3), we find that:

$$\lim_{|\mathbf{v}|\to c} H=\left(\gamma_v-1\right)mc^2-3\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}+2\mu_B\mathbf{B}\cdot\hat{\mathbf{u}}\boldsymbol{\sigma}\cdot\hat{\mathbf{u}}\,. \tag{25.5}$$

Now, a hallmark of the Special Theory of Relativity is that the total energy of a material body – that is, its rest-plus-kinetic energy – approaches infinity as the relative velocity of that body approaches the speed of light, because of the time-dilation factor $\gamma_v=\sqrt{1-v^2/c^2}$. The impossibility of having infinite energy then likewise precludes that material body from ever reaching the speed of light. The term $\left(\gamma_v-1\right)mc^2$ in (25.5) shows that the kinetic energy of a charged lepton indeed approaches infinity in the usual way as its velocity approaches the speed of light. But the $-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ term exhibits an entirely different character: At rest as shown in (25.4), this term is $-\left(g_{em}/2\right)\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ with $g_{em}/2$ being slightly larger than 1. The complete observed g-factors (also with hadronic and electroweak contributions) for each charged lepton were shown prior to (24.2). But at extreme relativistic velocities, this term does *not* become infinite. Instead, the coefficient of $-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ grows from just over 1 in (25.4), to exactly 3 in (25.5), i.e., it effectively triples. Specifically, $2\gamma_v\gamma_{em}/\left(\gamma_v\gamma_{em}+1\right)$ in (25.3) grows from Schwinger's $g_{em}/2\cong 1+\alpha/2\pi$ just above 1 at rest, to exactly 2 at the relativistic extreme, while $\gamma_{em}{}^2\left(\gamma_v{}^2-1\right)/\left(\gamma_v\gamma_{em}+1\right)^2$ grows from zero at rest to exactly 1 at the relativistic extreme. So, while the ordinary kinetic energies of the charged leptons which appear also in the Schrödinger equations (22.2) and (22.3) grow without limit in the usual way, the Hamiltonian energy contributions from the magnetic moments reach an absolute upper limit whereby they approximately triple, growing from being proportional to $g_{em}/2\cong 1$ at rest, to being proportional to 3 for extreme motion.

This is a very important, and potentially very testable prediction: *The kinetic energies of lepton magnetic moments interacting in magnetic fields do not grow without limit as relativistic velocities are attained.* And in fact, even for extreme relativistic motion approaching the speed of light, the total energy of the magnetic moment interaction energy can only grow by a factor of just under 3, relative to the magnitude of this same interaction energy at rest. The magnetic moments of the leptons are perhaps the most precisely tested data in all of physics. So it would seem highly feasible to design experiments which can detect the $-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ energy contributions for each of the three leptons in highly relativistic settings, to establish that the growth of these energies is *not* governed by the usual $\gamma_v=1/\sqrt{1-\mathbf{v}^2/c^2}$ which precedes mc^2 in (25.5), but rather by the factor of 3 times the $-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ in (25.5).

Now let's consider fermions in relative motion, but at non-relativistic (NR) velocities for which we may approximate $\gamma_v = 1/\sqrt{1-\mathbf{v}^2/c^2} \cong 1+\tfrac{1}{2}\mathbf{v}^2/c^2$. For the first (Schrödinger) term in (25.3), we insert $\gamma_v \cong 1+\tfrac{1}{2}\mathbf{v}^2/c^2$ and $\gamma_v{}^2 \cong 1+\mathbf{v}^2/c^2$ and $1/(1+x) \cong 1-x$ for small x generally while discarding all terms with $\mathbf{v}^4/c^4$ and higher order as soon as they appear, and finally use $g_{em}/2 = 2\gamma_{em}/(\gamma_{em}+1)$ from (23.6), to obtain:

$$\begin{aligned}&\frac{\gamma_{em}\left(\gamma_v{}^2-1\right)}{\gamma_{em}\gamma_v+1}mc^2 = \frac{\gamma_{em}\left(\gamma_v+1\right)\left(\gamma_v-1\right)}{\gamma_{em}\gamma_v+1}mc^2 \cong \frac{\gamma_{em}\left(2+\tfrac{1}{2}\mathbf{v}^2/c^2\right)}{1+\gamma_{em}\left(1+\tfrac{1}{2}\mathbf{v}^2/c^2\right)}\frac{1}{2}m\mathbf{v}^2 \\ &\cong \frac{\gamma_{em}}{1+\gamma_{em}}\left(2+\tfrac{1}{2}\mathbf{v}^2/c^2\right)\left(1-\frac{\tfrac{1}{2}\gamma_{em}\mathbf{v}^2/c^2}{1+\gamma_{em}}\right)\frac{1}{2}m\mathbf{v}^2 \cong \frac{2\gamma_{em}}{1+\gamma_{em}}\frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}\left(\frac{g_{em}}{2}m\right)\mathbf{v}^2\end{aligned} . \tag{25.6}$$

For the magnetic moment term in (25.3) we restructure the first coefficient term into:

$$\frac{2\gamma_v\gamma_{em}}{\gamma_v\gamma_{em}+1} = \frac{\gamma_{em}+1}{\gamma_v\gamma_{em}+1}\gamma_v\frac{2\gamma_{em}}{\gamma_{em}+1} = \frac{\left(\gamma_{em}+1\right)\gamma_v}{\gamma_v\gamma_{em}+1}\frac{g_{em}}{2} . \tag{25.7}$$

We then use the above and the above with γ_v divided out, and make the all the same approximations and reductions as well as use $g_{em}/2 = 2\gamma_{em}/(\gamma_{em}+1)$, to calculate:

$$\begin{aligned}&\left(\frac{2\gamma_v\gamma_{em}}{\gamma_v\gamma_{em}+1}+\frac{\gamma_{em}{}^2\left(\gamma_v+1\right)\left(\gamma_v-1\right)}{\left(\gamma_v\gamma_{em}+1\right)^2}\right)\mu_B\boldsymbol{\sigma}\cdot\mathbf{B} \\ &= \left(\frac{\left(\gamma_{em}+1\right)\gamma_v}{\gamma_v\gamma_{em}+1}+\frac{\gamma_{em}\left(\gamma_v+1\right)\left(\gamma_v-1\right)}{2\left(\gamma_v\gamma_{em}+1\right)}\frac{\gamma_{em}+1}{\gamma_v\gamma_{em}+1}\right)\frac{g_{em}}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B} \\ &\cong \left(\frac{\left(\gamma_{em}+1\right)\left(1+\tfrac{1}{2}\mathbf{v}^2/c^2\right)}{\left(1+\tfrac{1}{2}\mathbf{v}^2/c^2\right)\gamma_{em}+1}+\frac{\gamma_{em}\left(2+\tfrac{1}{2}\mathbf{v}^2/c^2\right)}{2\left(\left(1+\tfrac{1}{2}\mathbf{v}^2/c^2\right)\gamma_{em}+1\right)}\frac{\gamma_{em}+1}{\left(1+\tfrac{1}{2}\mathbf{v}^2/c^2\right)\gamma_{em}+1}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\frac{g_{em}}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B} . \\ &\cong \left(\left(1+\tfrac{1}{2}\mathbf{v}^2/c^2\right)\left(1-\frac{\tfrac{1}{2}\gamma_{em}\mathbf{v}^2/c^2}{1+\gamma_{em}}\right)+\frac{\gamma_{em}}{1+\gamma_{em}}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\frac{g_{em}}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B} \\ &\cong \left(1+\frac{1}{1+\gamma_{em}}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}+\frac{\gamma_{em}}{1+\gamma_{em}}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\frac{g_{em}}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B} = \frac{g_{em}}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}+\frac{1}{2}\left(\frac{g_{em}}{2}\frac{\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}}{c^2}\right)\mathbf{v}^2\end{aligned} \tag{25.8}$$

For the final term in (25.3) we use (25.7) and the same low-velocity approximations $\gamma_v \cong 1+\tfrac{1}{2}\mathbf{v}^2/c^2$ and $\gamma_v{}^2 \cong 1+\mathbf{v}^2/c^2$. We drop any term with an order higher than $\mathbf{v}^2/c^2$ as soon as it appears. But $\left(\mu_B\mathbf{B}\cdot\mathbf{v}/c\right)\left(\boldsymbol{\sigma}\cdot\mathbf{v}/c\right)$ already contains a second order velocity term. Thus:

$$\begin{aligned}
&\frac{2\gamma_v^2\gamma_{em}^{\ 2}}{\left(\gamma_{em}\gamma_v+1\right)^2}\mu_B\mathbf{B}\cdot\frac{\mathbf{v}}{c}\boldsymbol{\sigma}\cdot\frac{\mathbf{v}}{c}=\frac{1}{2}\frac{\left(\gamma_{em}+1\right)^2}{\left(\gamma_v\gamma_{em}+1\right)^2}\gamma_v^{\ 2}\left(\frac{g_{em}}{2}\right)^2\mu_B\mathbf{B}\cdot\frac{\mathbf{v}}{c}\boldsymbol{\sigma}\cdot\frac{\mathbf{v}}{c}\\
&\cong\frac{1}{2}\frac{\left(\gamma_{em}+1\right)^2}{\left(\left(1+\tfrac{1}{2}\mathbf{v}^2/c^2\right)\gamma_{em}+1\right)^2}\left(1+\mathbf{v}^2/c^2\right)\left(\frac{g_{em}}{2}\right)^2\mu_B\mathbf{B}\cdot\frac{\mathbf{v}}{c}\boldsymbol{\sigma}\cdot\frac{\mathbf{v}}{c}\\
&\cong\frac{1}{2}\left(1-\frac{\gamma_{em}}{\gamma_{em}+1}\frac{\mathbf{v}^2}{c^2}\right)\left(1+\frac{\mathbf{v}^2}{c^2}\right)\left(\frac{g_{em}}{2}\right)^2\mu_B\mathbf{B}\cdot\frac{\mathbf{v}}{c}\boldsymbol{\sigma}\cdot\frac{\mathbf{v}}{c}\cong\frac{1}{2}\left(\frac{g_{em}}{2}\right)^2\mu_B\mathbf{B}\cdot\frac{\mathbf{v}}{c}\boldsymbol{\sigma}\cdot\frac{\mathbf{v}}{c}
\end{aligned}\quad . \tag{25.9}$$

Combining (25.6), (25.8) and (25.9) into (25.3) then produces the non-relativistic:

$$H_{NR}=\frac{g_{em}}{2}\left(\frac{1}{2}m\mathbf{v}^2\right)-\frac{g_{em}}{2}\left(\frac{1}{2}\frac{\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}}{c^2}\mathbf{v}^2\right)-\frac{g_{em}}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}+\frac{1}{2}\left(\frac{g_{em}}{2}\right)^2\mu_B\mathbf{B}\cdot\frac{\mathbf{v}}{c}\boldsymbol{\sigma}\cdot\frac{\mathbf{v}}{c}. \tag{25.10}$$

We of course recognize $\tfrac{1}{2}m\mathbf{v}^2$ to be the non-relativistic kinetic energy of a material body. And referring to (23.7) we see that $\left(g_{em}/2\right)\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}/c^2=\boldsymbol{\mu}_{em}\cdot\mathbf{B}/c^2$ is the mass-equivalent of the Hamiltonian energy operator for a magnetic moment interacting with an external magnetic field. Therefore, $\tfrac{1}{2}\left(\boldsymbol{\mu}_{em}\cdot\mathbf{B}/c^2\right)\mathbf{v}^2$ is a kinetic energy operator associated with the magnetic moment operator. So, it appears especially from the term $\left(g_{em}/2\right)\tfrac{1}{2}m\mathbf{v}^2$ that the kinetic energies of the leptons are enhanced by a factor of $g_{em}/2$ above and beyond the usual kinetic energies $\tfrac{1}{2}m\mathbf{v}^2$ of a material body, with this enhancement offset by the subtraction of $\tfrac{1}{2}\left(\boldsymbol{\mu}_{em}\cdot\mathbf{B}/c^2\right)\mathbf{v}^2$. Of course, H_{NR} is a 2x2 matrix operator because it contains the 2x2 Pauli matrices $\boldsymbol{\sigma}$. We now wish to determine the energy eigenvalues of this operator, because these can be observed.

Using the particle ket $\left|U_{0A}\right\rangle$ which we last used at (20.4) to extract the Hamiltonian, the energy eigenvalues E_{NR} of a fermion in non-relativistic motion as described by (25.10) will be given by $E_{NR}\left|U_{0A}\right\rangle=H_{NR}\left|U_{0A}\right\rangle$. Using (25.10), these are ascertained via the eigenvalue equation:

$$\begin{aligned}
0&=\left(H_{NR}-E_{NR}\right)\left|U_{0A}\right\rangle\\
&=\left[\frac{g_{em}}{2}\left(\frac{1}{2}m\mathbf{v}^2\right)-\frac{g_{em}}{2}\left(\frac{1}{2}\frac{\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}}{c^2}\mathbf{v}^2\right)-\frac{g_{em}}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}+\frac{1}{2}\left(\frac{g_{em}}{2}\right)^2\mu_B\mathbf{B}\cdot\frac{\mathbf{v}}{c}\boldsymbol{\sigma}\cdot\frac{\mathbf{v}}{c}-E_{NR}\right]\left|U_{0A}\right\rangle
\end{aligned}. \tag{25.11}$$

To write this explicitly using the Pauli matrices $\boldsymbol{\sigma}$, referring to (19.4), keep in mind that the γ matrices are structured such that, by convention, the +*z* axis is aligned in the same direction as the fermion propagation, that is, $\mathbf{v}=\begin{pmatrix}0 & 0 & |\mathbf{v}|\end{pmatrix}$. The fermion spin is then either up (same direction) or down (opposite direction) relative to this propagation. So within the fourth term on the bottom line of (25.11) $\mathbf{B}\cdot\mathbf{v}=B_z\left|\mathbf{v}\right|$, $\operatorname{diag}\left(\boldsymbol{\sigma}\cdot\mathbf{v}\right)=\left(\left|\mathbf{v}\right| \quad -\left|\mathbf{v}\right|\right)$ and $\left|\mathbf{v}\right|^2=\mathbf{v}^2$, so this term expands to:

$$\left(\mathbf{B}\cdot\mathbf{v}\right)\left(\boldsymbol{\sigma}\cdot\mathbf{v}\right)=\begin{pmatrix} B_z\mathbf{v}^2 & 0 \\ 0 & -B_z\mathbf{v}^2 \end{pmatrix}. \tag{25.12}$$

Denoting the components $\left|U_A\right\rangle^T\equiv\left(\uparrow \quad \downarrow\right)$, using (25.12) and also replacing $g_{em}/2-1=a_{em}$ with the electromagnetic anomaly contribution, (25.1) expands to:

$$0=\begin{pmatrix} -\left(1-a_{em}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\frac{g_{em}}{2}\mu_B B_z+\frac{g_{em}}{2}\frac{1}{2}m\mathbf{v}^2-E_{NR} & -\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\frac{g_{em}}{2}\mu_B\left(B_x-iB_y\right) \\ -\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\frac{g_{em}}{2}\mu_B\left(B_x+iB_y\right) & \left(1-a_{em}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\frac{g_{em}}{2}\mu_B B_z+\frac{g_{em}}{2}\frac{1}{2}m\mathbf{v}^2-E_{NR} \end{pmatrix}\begin{pmatrix}\uparrow \\ \downarrow\end{pmatrix}. \tag{25.13}$$

The eigenvalues are extracted by setting the determinant of this matrix to zero, that is, by setting $\det\left(H_{NR}-E_{NR}\right)=0$. It is readily seen from (25.13) that this will first yield:

$$\left(\frac{g_{em}}{2}\frac{1}{2}m\mathbf{v}^2-E_{NR}\right)^2=\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)^2\left(\frac{g_{em}}{2}\right)^2\mu_B{}^2\left(B_x{}^2+B_y{}^2\right)+\left(1-a_{em}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)^2\left(\frac{g_{em}}{2}\right)^2\mu_B{}^2B_z{}^2. \tag{25.14}$$

Because the coefficients of $B_x{}^2+B_y{}^2$ and $B_z{}^2$ are different, we discern that E_{NR} will be dependent not only on the magnitude $\left|\mathbf{B}\right|$ of the magnetic field and the velocity of the fermion, but also at the *angle* at which the magnetic field is applied. Note that this angular-dependency originated from the $\left(\mathbf{B}\cdot\mathbf{v}\right)\left(\boldsymbol{\sigma}\cdot\mathbf{v}\right)$ term in (25.10) via (25.12). Accordingly, if θ represents the polar angle (descending from the fermion propagation direction +*z*) at which the magnetic field is applied, then $B_z=\left|\mathbf{B}\right|\cos\theta$ and $\sqrt{B_x{}^2+B_y{}^2}=\left|\mathbf{B}\right|\sin\theta$. Also with $\left|\mathbf{B}\right|^2=\mathbf{B}^2$, we may rewrite (25.14), and then take its positive and negative square roots, and finally isolate E_{NR}, to obtain:

$$E_{NR}=\frac{g_{em}}{2}\frac{1}{2}m\mathbf{v}^2\mp\frac{g_{em}}{2}\mu_B\left|\mathbf{B}\right|\sqrt{\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)^2\sin^2\theta+\left(1-a_{em}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)^2\cos^2\theta}\,. \tag{25.15}$$

Now, we wish to ascertain which of the two eigenvalues denoted by the $\pm$ above is associated with spin-up versus spin-down. To do so, we first substitute (25.15) back into (25.13), while defining the substitute variables $a\equiv 1+\frac{1}{2}\mathbf{v}^2/c^2$ and $b\equiv 1-a_{em}\frac{1}{2}\mathbf{v}^2/c^2$ simply for compactness. Because (25.13) is equal to zero, it is possible to factor out an overall $\left(g_{em}/2\right)\mu_B$.

Additionally, using spherical coordinates we use $B_x = |\mathbf{B}|\sin\theta\cos\varphi$, $B_y = |\mathbf{B}|\sin\theta\sin\varphi$ and $B_z = |\mathbf{B}|\cos\theta$, which is consistent with what we did in (25.15). Then, $|\mathbf{B}|$ may also be factored out. Finally, in the off-diagonal elements, we use $\cos\varphi \pm i\sin\varphi = \exp(\pm i\varphi)$. The result is:

$$0 = \begin{pmatrix} -b\cos\theta \pm \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} & -a\sin\theta\exp(-i\varphi) \\ -a\sin\theta\exp(i\varphi) & b\cos\theta \pm \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} \end{pmatrix} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}. \tag{25.16}$$

Then, let us consider the special case where the magnetic field points toward the +*z* axis, whereby $\cos\theta = 1$ and $\sin\theta = 0$, and (25.16) reduces to the diagonal:

$$0 = \begin{pmatrix} -b\cos\theta \pm b\cos\theta & 0 \\ 0 & b\cos\theta \pm b\cos\theta \end{pmatrix} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}. \tag{25.17}$$

For a spin-up fermion which we may represent as $\begin{pmatrix} \uparrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ because the above is a diagonal matrix, (25.17) is true for the + value of $\pm$. For spin-down with $\begin{pmatrix} \uparrow & \downarrow \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}$ this is true for the – value of $\pm$. In this way, we establish that the plus sign in (25.16) generally applies to spin-up and the minus sign to spin-down.

Generally, (25.16) is not a diagonal matrix, and separates into the simultaneous equations:

$$\begin{aligned} &\left(-b\cos\theta \pm \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\right)\uparrow - a\sin\theta\exp(-i\varphi)\downarrow = 0 \\ &\left(b\cos\theta \pm \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\right)\downarrow - a\sin\theta\exp(i\varphi)\uparrow = 0 \end{aligned}. \tag{25.18}$$

To ascertain explicit eigenstate vectors generally, we first set $\uparrow = 1$ for spin-up, and then set $\downarrow = 1$ for spin-down, subject to normalization. Given the eigenvalue determination from (25.17), for up and down respectively we use the bottom and top equations (25.7) to deduce:

$$|U_{A\uparrow}\rangle = N \begin{pmatrix} 1 \\ \dfrac{a\sin\theta\exp(i\varphi)}{b\cos\theta + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \end{pmatrix}; \quad |U_{A\downarrow}\rangle = N \begin{pmatrix} -\dfrac{a\sin\theta\exp(-i\varphi)}{b\cos\theta + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \\ 1 \end{pmatrix}, \tag{25.19}$$

where *N* is a normalization factor which may be fixed for example, by the condition $U_A{}^\dagger U_A = 1$, noting also that $\exp(i\varphi)\exp(-i\varphi) = 1$. For $\theta = 0$ (**B** parallel to fermion motion) we find $N = 1$, and $\left|U_{A\uparrow}\right\rangle^T = \begin{pmatrix}1 & 0\end{pmatrix}$ and $\left|U_{A\downarrow}\right\rangle^T = \begin{pmatrix}0 & 1\end{pmatrix}$. For $\theta = \pi/2$ (**B** orthogonal to fermion motion) we find $N = 1/\sqrt{2}$, and $\left|U_{A\uparrow}\right\rangle = \begin{pmatrix}1 & \exp(i\varphi)\end{pmatrix}/\sqrt{2}$ and $\left|U_{A\downarrow}\right\rangle = \begin{pmatrix}-\exp(-i\varphi) & 1\end{pmatrix}/\sqrt{2}$.

Now that we know which eigenvalues are associated with spins up and down, we may return to (25.15) which tells us the observable energies which should be experimentally detectable through experimentation. For spin-up and spin-down respectively, there are:

$$\begin{aligned} E_{NR}(\uparrow) &= \frac{g_{em}}{2}\frac{1}{2}m\mathbf{v}^2 + \frac{g_{em}}{2}\mu_B|\mathbf{B}|\sqrt{\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)^2\sin^2\theta + \left(1-a_{em}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)^2\cos^2\theta} \\ E_{NR}(\downarrow) &= \frac{g_{em}}{2}\frac{1}{2}m\mathbf{v}^2 - \frac{g_{em}}{2}\mu_B|\mathbf{B}|\sqrt{\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)^2\sin^2\theta + \left(1-a_{em}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)^2\cos^2\theta} \end{aligned}. \tag{25.20}$$

So, the energies detected for a fermion propagating with non-relativistic motion along the $+z$ axis will be slightly higher for a spin-up fermion than for a spin-down fermion. This sort of spin-based splitting is a form of Zeeman effect which is well-known, and it arises from the very same term $\boldsymbol{\mu}_{em}\cdot\mathbf{B}$ contained in (25.10). But it is of interest to find that the Newtonian kinetic energy $\frac{1}{2}m\mathbf{v}^2$ is enhanced by the factor $g_{em}/2$, and then either supplemented (spin-up) or offset (spin-down) by the Zeeman-type splitting shown in (25.20). This $g_{em}/2$ enhancement provides yet another prediction which should be testable by experiment. More generally, given that the corresponding kinetic energy term in (25.5) for extreme relativistic motion $|\mathbf{v}|\to c$ is the usual $(\gamma_v - 1)mc^2$, we discover that the $g_{em}/2$ kinetic enhancement in (25.20) is a decidedly non-relativistic phenomenon, which gradually diminishes as the motion becomes more relativistic, and vanishes entirely for extreme relativistic motion. So, any experiments to test for this enhancement should also test for its diminution as the fermion velocities are increased.

As to the angle of the magnetic field, it should also be noted that for $\theta = 0$ or $\theta = \pi$ (motion-aligned **B**), for the spin correspondences $\substack{\uparrow\\\downarrow} \Leftrightarrow \pm$, the above becomes:

$$E_{NR}\left(\substack{\uparrow\\\downarrow}, \theta = 0 \text{ or } \pi\right) = \frac{g_{em}}{2}\frac{1}{2}m\mathbf{v}^2 \pm \frac{g_{em}}{2}\mu_B|\mathbf{B}|\left(1 - a_{em}\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right). \tag{25.21}$$

Conversely, for $\theta = \pi/2$ (motion-orthogonal **B**), (25.20) becomes:

$$E_{NR}\left(\substack{\uparrow\\\downarrow}, \theta = \pi/2\right) = \frac{g_{em}}{2}\frac{1}{2}m\mathbf{v}^2 \pm \frac{g_{em}}{2}\mu_B|\mathbf{B}|\left(1 + \frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right). \tag{25.22}$$

Given as reviewed following (23.5) that $a \cong a_{em} \cong \alpha / 2\pi = 0.00116140973242$ is a rather small number, as well as the minus sign in (25.21), what we learn contrasting (25.21) with (25.22) is that when the magnetic field is applied with an orthogonal component to the fermion motion, there is a more-pronounced splitting of the energies versus when there is a lesser or no orthogonal component. This provides yet another avenue for experimental testing, wherein the angle at which the magnetic field is applied is varied in relation to the fermion motion, with the prediction that the splitting is minimized for parallel alignment and maximized for antiparallel alignment.

So, to summarize the results of this section, we have now proposed three additional empirical tests to add to the two tests from the last section. In the last section it was proposed to test for the time dilations (24.2) and the bare-versus dressed rest and self-interaction energies (24.4), for all three charged leptons. In this section, it was proposed based on the $-3\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$ term in (25.5) to test for the approximate tripling of the kinetic energies associated with the magnetic moments for extreme relativistic motion, versus the unlimited energy increases for the rest mass kinetic energy from the $(\gamma_v - 1) mc^2$ term in that same equation. Further, from (25.20) it was proposed not only to test for the usual splitting of energies based on spin direction relative to motion, but to also test for the $g_{em} / 2$ enhancement of the Newtonian kinetic energy $\frac{1}{2} m\mathbf{v}^2$, and the gradual diminishment of this enhancement at relativistic speeds, based again on the $(\gamma_v - 1) mc^2$ term in (25.20). Finally, based on (25.21) and (25.22), it was proposed to test for larger spin-based energy splitting when the magnetic field is applied orthogonally to the motion versus when it is applied parallel to the motion. Now let us turn to a sixth experimental test, based on what we shall define and refer to as the "statistical diameters" of the three charged leptons.

26. A Sixth Proposed Experimental Magnetic Moment Test: Charged Lepton Statistical Diameters

Above, we have reviewed five possible tests of the magnetic moment interaction and *g*-factor predicted by (22.6) through (23.8): First, time dilation measurements based on (24.1) and (24.2). Second, given in (24.4), the proportions of electromagnetic and non-electromagnetic energies which constitute the complete rest masses of the leptons. Third, the $\boldsymbol{\mu} \cdot \mathbf{B}$ interaction energy in extreme relativistic experiments which, based on (25.5), does not approach infinity but merely triples as the relative velocity approached the speed of light. Fourth and fifth, from (25.10), the $g_{em} / 2$ enhancement of the rest mass at non-relativistic velocities, and the subtraction from this, of the kinetic energy of the mass-dimensioned $\mu_B \boldsymbol{\sigma} \cdot \mathbf{B} / c^2$ also with a $g_{em} / 2$ enhancement. A sixth possible experimental test which will now be reviewed, is based on the "statistical diameters" of *free* charged leptons, "free" meaning leptons which are not bound in atomic orbits.

In the early days of quantum theory the notion was entertained that an electron might be distributed with a *charge density* ρ just like the classical charge distribution contained in the current four-vector $J^\mu = (\rho, \mathbf{J})$ sourcing Maxwell's charge equation $J^\mu = \partial_\sigma F^{\sigma\mu}$. But it has long-since been recognized that electrons and other leptons are observed as structureless point particles, and that ρ is a *probability density* for finding the structureless lepton at a given spatial position when an experiment is performed to "collapse" the lepton wavefunction and thus detect the lepton. In fact, this ρ is now understood to be the time component $J^0 = \rho = \psi^\dagger \psi$ of a conserved (continuous) Dirac current $J^\mu = \overline{\psi}\gamma^\mu \psi = (\rho, \mathbf{J})$ with $\partial_{;\mu} J^\mu = 0$. The experimental test to now be proposed, centers around the statistical diameter of ρ, which is to say, the "average draw separation" over large numbers of experimental trials which "collapse" the lepton wavefunctions.

In (12.4), we already have a "statistical inverse radius" $\langle 1/r \rangle$ which naturally emerged from the examination of the Heisenberg / Ehrenfest equations in section 8. And as noted after (12.3), $\langle 1/r \rangle \geq 1/\langle r \rangle$ for any positive random variable *r*, with the only distribution having $\langle 1/r \rangle = 1/\langle r \rangle$ being the Dirac delta $\delta(r)$. So we already have some information about a lower bound on a statistical radius $\langle r \rangle$. Now, we simply define a "statistical diameter" $\langle d \rangle \equiv 2\langle r \rangle$ to be twice the statistical radius. This statistical diameter is also known as the "average draw separation." Then, we begin by using (12.3) and (12.4) in (23.6) to deduce that:

$$g_{em} = \frac{4\gamma_{em}}{1+\gamma_{em}} = \frac{4}{2 - \dfrac{q\langle \phi_0 \rangle}{mc^2}} = \frac{4}{2 - \dfrac{k_e Qq}{mc^2}\left\langle \dfrac{1}{r} \right\rangle} = \frac{4}{2 - \dfrac{k_e Qq}{mc^2}\left\langle \dfrac{2}{d} \right\rangle} . \tag{26.1}$$

Now, because g_{em} due to its origin in (23.6) is the (electromagnetically-contributed) *g*-factor for an individual fermion, let us take that fermion to be a charged lepton which has a charge of $-e$. Because magnetic moment anomalies are understood to be the result of lepton self-interaction, we must regard Qq not as an interaction between two separate charges, but as the *self-interaction* between different "parts" of the same probability density $\rho = \psi^\dagger \psi$ with charge $-e$. So, for example, we may split ρ into two portions each with $Q = q = -\frac{1}{2}e$ to determine that $Qq = \frac{1}{4}e^2$. Or, for better precision, we may split the charge density ρ into three portions $Q = q = -\frac{1}{3}e$. But now, there are also 3 pairwise interactions, so the sum of these is $\Sigma Qq = 3\frac{1}{9}e^2 = \frac{1}{3}e^2$. For more precision, we split into four portions $Q = q = -\frac{1}{4}e$, but now there will be $C(4,2) = 4 \cdot 3/2 = 6$ pairwise combinations. So the sum $\Sigma Qq = C(4,2)\frac{1}{16}e^2 = \frac{3}{8}e^2$. In general, for *N* subdivisions, the number of pairwise combinations is $C(N,2) = N(N-1)/2$, and so $\Sigma Qq = \left(C(N,2)/N^2\right)e^2$. When we take the calculus limit as the number of split portions becomes infinite, we find that:

$$Qq = \sum_{N\to\infty} Qq = \lim_{N\to\infty} \frac{C(N,2)}{N^2} e^2 = \lim_{N\to\infty} \frac{N^2 - N}{2N^2} e^2 = \frac{1}{2} e^2 . \tag{26.2}$$

Substituting the above into (26.1), we then obtain:

$$g_{em} = \frac{4\gamma_{em}}{1+\gamma_{em}} = \frac{4}{2-\frac{k_e Qq}{mc^2}\left\langle \frac{2}{d} \right\rangle} = \frac{4}{2-\frac{k_e e^2}{mc^2}\left\langle \frac{1}{d} \right\rangle}. \tag{26.3}$$

Now, $k_e e^2 = \hbar c\alpha$ with $k_e = 1/4\pi\varepsilon_0$ is simply the running fine structure coupling which approaches the numerical value of $\alpha = 1/137.035999139$ [16] at low probe energies. Also, to provide a length dimension as a standard of reference, given that *m* is the mass $m = m_L$ of the self-interacting lepton, we may use the Compton wavelength $\lambda_L = h/m_L c$ to replace the mass. We should also write $g_{em} \mapsto g_{emL}$ so that this now denotes the *g*-factor of the specific lepton. Making these substitutions and also using $\hbar = h/2\pi$ the above becomes:

$$g_{emL} = \frac{4\gamma_{em}}{1+\gamma_{em}} = \frac{4}{2-\frac{k_e e^2}{m_L c^2}\left\langle \frac{1}{d} \right\rangle} = \frac{4}{2-\frac{\hbar c\alpha}{hc}\lambda_L\left\langle \frac{1}{d} \right\rangle} = \frac{4}{2-\frac{\alpha}{2\pi}\left\langle \frac{\lambda_L}{d} \right\rangle}. \tag{26.4}$$

An appearance is now made by $a_S = \alpha/2\pi = .00116140973242$ which is Schwinger's (subscript S) one-loop contribution to the anomalous magnetic moment of all three charged leptons. [15]

Next, we rearrange the above to isolate $\langle \lambda_L/d \rangle$. Then, because $\langle 1/d \rangle \ge 1/\langle d \rangle$ for any positive random variable *d* with only the delta $\delta(d)$ having $\langle 1/d \rangle = 1/\langle d \rangle$, we obtain:

$$\frac{4\pi}{\alpha}\left(2-\frac{4}{g_{emL}}\right) = \left\langle \frac{2\lambda_L}{d} \right\rangle \ge \frac{2\lambda_L}{\langle d \rangle}. \tag{26.5}$$

Finally, denoting $d \mapsto d_L$ so that the statistical radial diameter $\langle d \rangle$ is also associated with each lepton type, we rewrite the above as an *inequality* for $\langle d_L \rangle/\lambda_L$, namely:

$$\frac{\langle d \rangle}{\lambda_L} \ge \frac{\alpha}{2\pi}\frac{g_{emL}}{2}\frac{1}{g_{emL}-2} = \frac{\alpha}{4\pi}\frac{g_{emL}}{\left(g_{emL}-2\right)}. \tag{26.6}$$

The expression on the right sets a lower bound on $\langle d_L \rangle/\lambda_L$. So we now use $\langle d_L \rangle_{\min}$ to denote the *minimum* value of the statistical diameter $\langle d_L \rangle$, then set $\langle d_L \rangle_{\min}$ equal to the term to the right of the inequality. Now, as noted prior to (24.1), the magnetic moment anomalies of the charged leptons are generally divided into electromagnetic, hadronic and electroweak contributions [14]. So as in the last section, to an approximation valid within parts-per 10^4, we may use the empirical g-factors of the three charged leptons, as well as $\alpha = 1/137.035999139$, to calculate from (26.6) the ratio of the minimum statistical diameters $\langle d_L \rangle_{\min}$ of each lepton, to their Compton wavelengths λ_L. These are:

$$\frac{\langle d_e \rangle_{\min}}{\lambda_e} = 0.501338497456; \quad \frac{\langle d_\mu \rangle_{\min}}{\lambda_\mu} = 0.4986461107; \quad \frac{\langle d_\tau \rangle_{\min}}{\lambda_\tau} = 0.49386981. \tag{26.7}$$

For example, the Compton wavelength of the electron is $\lambda_e = 2.4263102367 \times 10^{-12}$ m [23], so (26.7) would tell us that $\langle d_e \rangle \geq 1.21640272842929 \times 10^{-12}$ m. This is in line with prevailing understandings of the smallest space that can be occupied by an electron probability density given the "underlying physical picture of the spin as due to a circulating energy flow in the Dirac field," [24] together with the speed of light as an upper material limit. With the statistical diameter of the tau lepton probability density scaled to 1, these ratios then progress relative to one another, as:

$$\frac{\langle d_\tau \rangle_{\min}}{\lambda_\tau} : \frac{\langle d_\mu \rangle_{\min}}{\lambda_\mu} : \frac{\langle d_e \rangle_{\min}}{\lambda_e} = 1 : 1.00967118 : 1.01512279. \tag{26.8}$$

Now, as an example to help interpret (26.8), for a Gaussian distribution represented along a single dimension labeled x, it is well-known that $\langle x \rangle = 2\sigma_x / \sqrt{\pi} \cong 1.128379\sigma_x$ is the weighted average draw separation, and is directly related to the standard deviation σ_x by a $2/\sqrt{\pi}$ coefficient. And in general, for *any* particular type of distribution, the statistical average draw separation $\langle d \rangle$ is directly proportional to the standard deviation σ of that distribution, $\langle d \rangle \propto \sigma$. Therefore, assuming that the underlying probability distributions for the three leptons all have the same character – Gaussian or otherwise – each of the lepton statistical draw separations will be directly proportional to the standard deviations of the lepton probability densities, $\langle d_L \rangle \propto \sigma_L$. So what (26.7) informs us, is that in relation to the Compton wavelength of each lepton, assuming the underlying probability distributions are all of the same type, the standard deviation of the muon probability density is about 1% larger than that of the tau lepton, while the standard deviation of the electron probability density is about 1.5% larger than that of the tau lepton.

These predictions in (26.7) and (26.8) suggest an experiment to confirm whether (23.6) is in fact a correct expression for the magnetic moment anomaly g-factor: Generate a large number of *free* leptons (not electrons in atoms), "collapse" them by having the strike a detector, and record their spatial strike positions. From these strikes, determine the probability distributions $\rho_L = \psi_L{}^\dagger \psi_L$ for each type of lepton (L). Use each ρ_L to ascertain the $\langle d_L \rangle$ which is the average draw separation, and a proportional $\sigma_L \propto \langle d_L \rangle$ which is the standard deviation of each ρ_L. What

(26.7) tells us is that the average draw separations will under all circumstances be approximately half the Compton wavelength of each lepton, i.e., that the Compton half-wavelengths $\lambda_L/2$ establishes approximate lower boundaries for the average draw separation of each ρ_L. For the electron the lower bound is slightly larger than its Compton wavelength, while for the mu and tau leptons the lower bound is slightly less than their Compton wavelengths. This in turn stems from the electron anomaly being slightly smaller than the Schwinger anomaly $a \cong \alpha/2\pi$ and the mu and tau anomalies being slightly larger than $a \cong \alpha/2\pi$. And what (26.8) tells us is that in relation to their respective Compton wavelengths, ρ_τ is more densely concentrated than ρ_μ, and ρ_μ in turn is denser than ρ_e, by the proportionalities indicated in (26.8). Finally, the precise numbers in (26.7) and (26.8) come with a caveat that they are derived using empirical values for g_L which naturally encompass electroweak and hadronic contributions, while the theoretical calculation used to arrive at (26.7) and (26.8) accounts (so far) only for electromagnetic effects. Therefore, these precise results are expected to be off by about one part per 10^4 because the hadronic contribution is about 10^{-4} times as large as the electromagnetic contribution.

APPENDIXES

Appendix A: Review of Derivation of the Gravitational Geodesic Motion from a Variation

To derive (1.3) from (1.2) we first apply δ to the (1.2) integrand and then use (1.1) to clear the denominator but keep the factor .5 arising from differentiating the square root, yielding:

$$0 = \delta \int_A^B d\tau = \frac{1}{2}\int_A^B d\tau \delta \left(g_{\mu\nu} \frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} \right). \tag{A.1}$$

The variation symbol δ commutes with the derivative symbol *d* such that $\delta d = d\delta$, and operates in the same way as *d* and so distributes via the product rule according to:

$$0 = \delta \int_A^B d\tau = \frac{1}{2}\int_A^B d\tau \left(\delta g_{\mu\nu} \frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + g_{\mu\nu}\frac{d\delta x^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} + g_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{d\delta x^\nu}{cd\tau} \right). \tag{A.2}$$

Now, one can use the chain rule in the small variation $\delta \to \partial$ limit to show that $\delta g_{\mu\nu} = \delta x^\alpha \partial_\alpha g_{\mu\nu}$. Indeed, the generic calculation for any field ϕ (taking $\delta \cong \partial$), is:

$$\delta x^\alpha \partial_\alpha \phi = \delta x^\alpha \frac{\partial \phi}{\partial x^\alpha} \cong \partial x^\alpha \frac{\delta \phi}{\partial x^\alpha} = \frac{\partial x^\alpha}{\partial x^\alpha}\delta\phi = \delta\phi. \tag{A.3}$$

Additionally, we may use the symmetry of $g_{\mu\nu}$ to combine the second and third term inside the parenthesis in (A.2). Thus, (A.2) becomes:

$$0=\delta\int_A^B d\tau=\frac{1}{2}\int_A^B d\tau\left(\delta x^\alpha\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+2g_{\mu\nu}\frac{d\delta x^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}\right). \quad \text{(A.4)}$$

The next step is to integrate by parts. From the product rule, we may obtain:

$$\frac{d}{cd\tau}\left(\delta x^\mu g_{\mu\nu}\frac{dx^\nu}{cd\tau}\right)=g_{\mu\nu}\frac{d\delta x^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+\delta x^\mu\frac{d}{cd\tau}\left(g_{\mu\nu}\frac{dx^\nu}{cd\tau}\right). \quad \text{(A.5)}$$

It will be recognized that the first term after the equality in (A.5) is the same as the final term in (A.4) up to the factor of 2. So we use (A.5) in (A.4) to write:

$$0=\delta\int_A^B d\tau=\frac{1}{2}\int_A^B d\tau\left(\delta x^\alpha\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}+2\frac{d}{cd\tau}\left(\delta x^\mu g_{\mu\nu}\frac{dx^\nu}{cd\tau}\right)-2\delta x^\mu\frac{d}{cd\tau}\left(g_{\mu\nu}\frac{dx^\nu}{cd\tau}\right)\right). \quad \text{(A.6)}$$

The middle term in the above, which is a total integral, is equal to zero because of the boundary conditions on the variation. Specifically, this middle term is:

$$\int_A^B d\tau\frac{d}{cd\tau}\left(\delta x^\mu g_{\mu\nu}\frac{dx^\nu}{cd\tau}\right)=\frac{1}{c}\int_A^B d\left(\delta x^\mu g_{\mu\nu}\frac{dx^\nu}{cd\tau}\right)=\frac{1}{c}g_{\mu\nu}\frac{dx^\nu}{cd\tau}\delta x^\mu\bigg|_A^B=0. \quad \text{(A.7)}$$

This definite integral is zero because the two worldlines intersect at the boundary events *A* and *B* but have a slight variational difference between *A* and *B* otherwise, so that $\delta x^\sigma(A)=\delta x^\sigma(B)=0$ while $\delta x^\sigma\neq 0$ elsewhere. Therefore we may zero out the middle term and rewrite (A.6) as:

$$0=\delta\int_A^B d\tau=\frac{1}{2}\int_A^B d\tau\left(\delta x^\alpha\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-2\delta x^\mu\frac{d}{cd\tau}\left(g_{\mu\nu}\frac{dx^\nu}{cd\tau}\right)\right). \quad \text{(A.8)}$$

Next, in the final term above, we distribute the $d/cd\tau$ via the product rule to each of $g_{\mu\nu}$ and $dx^\nu/cd\tau$, so that this becomes:

$$0=\delta\int_A^B d\tau=\frac{1}{2}\int_A^B d\tau\left(\delta x^\alpha\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-2\delta x^\mu\frac{dg_{\mu\nu}}{cd\tau}\frac{dx^\nu}{cd\tau}-2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{c^2d\tau^2}\right). \quad \text{(A.9)}$$

For the first time, we see an acceleration $d^2x^\nu/d\tau^2$. It is then straightforward to apply the chain rule to deduce $dg_{\mu\nu}/cd\tau=\partial_\alpha g_{\mu\nu}\left(dx^\alpha/cd\tau\right)$, which is a special case of the generic relation for any field ϕ given by:

$$\frac{d\phi}{cd\tau}=\frac{\partial\phi}{\partial x^\alpha}\frac{dx^\alpha}{cd\tau}=\partial_\alpha\phi\frac{dx^\alpha}{cd\tau}. \tag{A.10}$$

As a result, (A.9) now becomes:

$$0=\delta\int_A^B d\tau=\frac{1}{2}\int_A^B d\tau\left(\delta x^\alpha\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-2\delta x^\mu\partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{cd\tau}\frac{dx^\nu}{cd\tau}-2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{c^2d\tau^2}\right). \tag{A.11}$$

At this point we have a coordinate variation in front of all terms, but the indexes are not the same. So we need to re-index to be able to factor out the same coordinate variation from all terms. We thus rename the summed indexes $\mu\leftrightarrow\alpha$ in the second and third terms and factor out the resulting δx^α from all three terms. And we also use the symmetry of $g_{\mu\nu}$ to split the middle term into two, then cycle all indexes, then factor out all the terms containing derivatives of $g_{\mu\nu}$. The result of all this re-indexing, also moving the outside coefficient of ½ into the integrand, is:

$$0=\delta\int_A^B d\tau=\int_A^B\delta x^\alpha d\tau\left(\frac{1}{2}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-g_{\alpha\nu}\frac{d^2x^\nu}{c^2d\tau^2}\right). \tag{A.12}$$

Now we are ready for the final steps. Because the worldlines under consideration are for material particles, the proper time $d\tau\neq 0$. Likewise, while $\delta x^\sigma(A)=\delta x^\sigma(B)=0$ at the boundaries, between these boundaries where the variation occurs, $\delta x^\sigma\neq 0$. Therefore, for the overall expression (A.12) to be equal to zero, the expression inside the large parenthesis must be zero. Consequently:

$$0=\frac{1}{2}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}-g_{\alpha\nu}\frac{d^2x^\nu}{c^2d\tau^2}. \tag{A.13}$$

From here, we multiply through by $g^{\beta\alpha}$, apply $-\Gamma^\beta{}_{\mu\nu}=\tfrac{1}{2}g^{\beta\alpha}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)$ for the Christoffel symbols, flip the sign, and segregate the acceleration term to obtain the final result:

$$\frac{d^2x^\beta}{d\tau^2}=-\Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{A.14}$$

Appendix B: Review of Derivation of Time Dilations in Special and General Relativity

To derive time dilations in the Special Theory of Relativity, we begin with the flat spacetime metric $c^2d\tau^2=\eta_{\mu\nu}dx^\mu dx^\nu$ which using a squared velocity $v^2=\left(dx^k/dt\right)\left(dx^k/dt\right)$ is easily restructured with the chain rule into $1=\left(dt/d\tau\right)^2\left(1-v^2/c^2\right)$, then into the familiar

$\gamma_v \equiv dt / d\tau = 1/\sqrt{1-v^2/c^2}$, with γ_v defined as the motion-induced time dilation. In the General Theory we start with the line element $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ in which the metric tensor $g_{\mu\nu}$ contains the gravitational field. We isolate gravitation from motion by setting $dx^k = 0$ to place the clock at rest in the gravitational field. This is just as we did to place the test charge at rest in the electromagnetic potential to reach (4.1) and (5.3) above, isolating electromagnetic effects from motion effects. The line element then becomes $c^2 d\tau^2 = g_{00} dx^0 dx^0 = g_{00} c^2 dt^2$ which rearranges to $dt^2 / d\tau^2 = 1/g_{00}$. We then take the positive square root $\gamma_g \equiv dt / d\tau = 1/\sqrt{g_{00}}$ so that this approaches 1 in the flat spacetime $g_{00} = \eta_{00} = 1$ limit, with γ_g defined as the gravitationally-induced time dilation. For motion dt is the coordinate time element in the rest frame of the observer and $d\tau$ is the proper time element ticked off by a g-clock in motion relative to the observer. For gravitation dt is the coordinate time element in the frame of an observer outside the gravitational field and $d\tau$ is the proper time element ticked off by a g-clock inside the gravitational field.

Appendix C: Detailed Calculation of the Hyper-Canonical Dirac Hamiltonian Numerator

To calculate the complete hyper-canonical Hamiltonian (20.8), we extract and work with the numerator in (20.8). Before using the term-doubling or term-quadrupling expansions $\sigma^i \sigma^k = \delta^{ik} + i\varepsilon^{ikl}\sigma^l$ and $\sigma^i \sigma^j \sigma^k = \delta^{ij}\sigma^k + \delta^{jk}\sigma^i - \delta^{ki}\sigma^j + i\varepsilon^{ijk}$, we perform all the Heisenberg-based commutations using $\left[p^i, b \right] = i\hbar \partial^i b$, until all **p** have are moved to the very right, as was done in the (20.10) example. In some cases, such (20.10), more than one round of commutation is required before all **p** reach the very right. Commutator variants employed during this step are $\left[cp^i, E \right] = i\hbar c \partial^i E$, $\left[cp^j, A_\gamma^k \right] = i\hbar c \partial^j A_\gamma^k$, $\left[cp^i, B_\gamma^j \right] = i\hbar c \partial^i B_\gamma^j$, $\left[cp^j, \mu_B E^k \right] = i\hbar c \partial^j \mu_B E_\gamma^k$, $\left[cp^i, \partial^j E_\gamma^k \right] = i\hbar c \partial^i \partial^j E_\gamma^k$, and $\left[cp^i, \partial^j A_\gamma^k \right] = i\hbar c \partial^i \partial^j A_\gamma^k$. We then consolidate terms to reassemble several occurrences of $E + mc^2$ and $cp^i + E\rho A_\gamma^i + i\mu_B E_\gamma^i$, and then we group together certain sets of terms. Following all of the foregoing, for the numerator in (20.8), using $\nabla^i = -\partial^i$, we obtain:

$$
\begin{aligned}
&\left(\sigma^i cp^i + E\sigma^i \rho A^i_\gamma + i\mu_B \sigma^i E^i_\gamma\right)\left(E + mc^2 + 2\rho A^j_\gamma cp^j + \mu_B \sigma^j B^j_\gamma\right)\left(\sigma^k cp^k + E\sigma^k \rho A^k_\gamma + i\mu_B \sigma^k E^k_\gamma\right)\\
&\left.\begin{aligned}
&= \left(\sigma^i\sigma^k\right)\left(E+mc^2\right)\left(cp^i + E\rho A^i_\gamma + i\mu_B E^i_\gamma\right)cp^k\\
&+\left(\sigma^i\sigma^k\right)\left(E+mc^2\right)\left(E\rho A^k_\gamma + i\mu_B E^k_\gamma\right)\left(cp^i + E\rho A^i_\gamma + i\mu_B E^i_\gamma\right)\\
&+2\left(\sigma^i\sigma^k\right)\rho A^j_\gamma\left(cp^i + E\rho A^i_\gamma + i\mu_B E^i_\gamma\right)cp^j cp^k\\
&+2\left(\sigma^i\sigma^k\right)\rho A^j_\gamma\left(E\rho A^k_\gamma + i\mu_B E^k_\gamma\right)\left(cp^i + E\rho A^i_\gamma + i\mu_B E^i_\gamma\right)cp^j\\
&+\left(\sigma^i\sigma^j\sigma^k\right)\left(\mu_B B^j_\gamma cp^i cp^k + E\rho A^i_\gamma \mu_B B^j_\gamma cp^k + i\mu_B E^i_\gamma \mu_B B^j_\gamma cp^k\right)\\
&+\left(\sigma^i\sigma^j\sigma^k\right)\mu_B B^j_\gamma\left(E\rho A^k_\gamma + i\mu_B E^k_\gamma\right)\left(cp^i + E\rho A^i_\gamma + i\mu_B E^i_\gamma\right)
\end{aligned}\right\}\mathbf{I}\\
&\left.\begin{aligned}
&-i\hbar c\left(\sigma^i\sigma^k\right)\nabla^i\left(E + 2\rho A^j_\gamma cp^j\right)cp^k\\
&-i\hbar c\left(\sigma^i\sigma^k\right)\nabla^i\left(\left(E+mc^2\right)\left(\left(E\rho A^k_\gamma\right) + i\mu_B E^k_\gamma\right)\right)\\
&-i\hbar c\left(\sigma^i\sigma^j\sigma^k\right)\left(\nabla^i \mu_B B^j_\gamma cp^k + \nabla^i\left(\mu_B B^j_\gamma\left(\left(E\rho A^k_\gamma\right) + i\mu_B E^k_\gamma\right)\right)\right)\\
&-2i\hbar c\left(\sigma^i\sigma^k\right)\left(\nabla^i\left(\rho A^j_\gamma\left(\left(E\rho A^k_\gamma\right) + i\mu_B E^k_\gamma\right)\right)cp^j + \nabla^j\left(\rho A^j_\gamma\left(\left(E\rho A^k_\gamma\right) + i\mu_B E^k_\gamma\right)\right)cp^i\right)\\
&-2i\hbar c\left(\sigma^i\sigma^k\right)\left(\nabla^j\left(\rho A^j_\gamma\left(\left(E\rho A^k_\gamma\right) + i\mu_B E^k_\gamma\right)\right)\left(E\rho A^i_\gamma + i\mu_B E^i_\gamma\right)\right)\\
&-2\hbar^2 c^2\left(\sigma^i\sigma^k\right)\nabla^i\nabla^j\left(\rho A^j_\gamma\left(\left(E\rho A^k_\gamma\right) + i\mu_B E^k_\gamma\right)\right)
\end{aligned}\right\}\mathbf{II}
\end{aligned}
\quad . \qquad \text{(C.1)}
$$

Right brackets are used to segregate **I**: a first set of terms without any gradients, from **II**: a second set of terms with a gradient. This is simply for calculation management.

Next, we work with the group **I** terms that have no gradient. We first expand these terms using $\sigma^i\sigma^k = \delta^{ik} + i\varepsilon^{ikl}\sigma^l$ and $\sigma^i\sigma^j\sigma^k = \delta^{ij}\sigma^k + \delta^{jk}\sigma^i - \delta^{ki}\sigma^j + i\varepsilon^{ijk}$. We then separate scalar from cross products and evaluate each. Except for those terms which contain two momentum terms that are self-commuted ($\left[p^j, p^k\right]$) or self-crossed ($\varepsilon^{ijk} p^j p^k$), all cross-product terms (those with ε^{ijk}) cancel by identity. The remaining such terms would cancel as well, but for the fact that $\left[p^j, p^k\right] \neq 0$ and $\mathbf{p}\times\mathbf{p} \neq 0$, as found and reviewed at (7.12) through (7.18). The cross-product terms which do drop out, do so either by cancellation (positive plus negative of an identical term) or via the identities $\mathbf{A}\times\mathbf{A} = 0$ and $\mathbf{E}\times\mathbf{E} = 0$. We also find occasion to use $\mathbf{A}_\gamma \cdot \mathbf{A}_\gamma = 0$ from (14.8) with photon subscript added. We then consolidate terms as much as possible, renaming indexes as needed to help do so. Following all of this, for the group **I** set terms in (C.1), we obtain:

$$\begin{aligned}
\mathbf{I} &= \left(E+mc^2-\sigma^j\mu_B B_\gamma^j\right)\left(cp^i cp^i + 2E\rho A_\gamma^i cp^i + 2i\mu_B E_\gamma^i cp^i + 2E\rho A_\gamma^i i\mu_B E_\gamma^i + i\mu_B E_\gamma^i i\mu_B E_\gamma^i\right) \\
&+2\left(\rho A_\gamma^j cp^i cp^j + 2E\rho A_\gamma^i \rho A_\gamma^j cp^j + \rho A_\gamma^i i\mu_B E_\gamma^j cp^j + i\mu_B E_\gamma^i \rho A_\gamma^j cp^j\right) cp^i \\
&+2\left(2E\rho A_\gamma^j i\mu_B E_\gamma^j - \mu_B E_\gamma^j \mu_B E_\gamma^j\right)\rho A_\gamma^i cp^i \\
&+\mu_B B_\gamma^j \sigma^i\left(cp^i cp^j + cp^j cp^i\right) + 2\mu_B B_\gamma^j\left(E\rho A_\gamma^j + i\mu_B E_\gamma^j\right)\sigma^i cp^i \\
&+2\sigma^j\left(E\rho A_\gamma^j + i\mu_B E_\gamma^j\right)\mu_B B_\gamma^i\left(cp^i + E\rho A_\gamma^i + i\mu_B E_\gamma^i\right) \\
&+i\varepsilon^{ijk}\left(\left(E+mc^2\right)\sigma^i - \mu_B B_\gamma^i\right)cp^j cp^k + 2i\varepsilon^{lik}\sigma^l \rho A_\gamma^j\left(E\rho A_\gamma^i + i\mu_B E_\gamma^i\right)\left[cp^j, cp^k\right] \\
&+2i\varepsilon^{lik}\sigma^l \rho A_\gamma^j cp^i cp^j cp^k
\end{aligned} \quad . \tag{C.2}$$

Note the minus sign in the magnetic moment term in $E+mc^2-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$ versus the positive sign originally appearing in $E+mc^2+2\rho\mathbf{A}\cdot c\mathbf{p}+\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ in (20.8). This emerges from the minus sign in $\sigma^i\sigma^j\sigma^k=\delta^{ij}\sigma^k+\delta^{jk}\sigma^i-\delta^{ki}\sigma^j+i\varepsilon^{ijk}$, because it is the term $-\delta^{ki}\sigma^j$ that produces the dot product in $\boldsymbol{\sigma}\cdot\mathbf{B}$. So, this restores the original signage of $\mathrm{E}+\mathrm{M}c^2=E+mc^2+2\rho\mathbf{A}\cdot c\mathbf{p}-\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$ that first appeared in (20.4) which was flipped when we took the inverse in (20.6). Note also that on the second, with order of operation proceeding from right to left, that all of the terms contain *outer products* which are then twice contracted to form a scalar number. So, for example, $\mathbf{A}_\gamma\cdot\mathbf{A}_\gamma\cdot\mathbf{pp}=A_\gamma^i A_\gamma^j p^j p^i$ is *not zero* via $\mathbf{A}_\gamma\cdot\mathbf{A}_\gamma=0$ from (14.8), despite the superficial appearance that it might be. Rather, one starts with the 3x3 tensor matrix **pp**, then contacts down to the 3-component vector $\mathbf{A}\cdot\mathbf{pp}$, then finally down to the spatial scalar $\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{pp}$. Likewise, for the remaining terms in that second line. Some of these terms look superficially as if they may combine further, but it is important to keep all occurrences of momentum **p** pinned to the right to enable conversion between momentum and configuration space. We will seek wherever possible to commute objects to convert the contracted outer product terms into scalar products, but are restricted by the need to fix all **p** on the very right. Note also that (C.2) contains a term $E_\gamma^i E_\gamma^i=\mathbf{E}_\gamma{}^2$ for the magnitude of the electric field of a photon, which we determined at (15.11) is equal to zero. However, as with $\mathbf{B}_\gamma{}^2$ at (20.7), we shall not zero this out. This will later enable us to consider the application of classical, external electric fields with non-zero magnitude.

To advance (C.2), we turn to the self-commutator and self-cross relations at (7.12) through (7.18). There are five distinct terms in (C.2) which contain these relations. First, for the triple momentum term $\rho A_\gamma^j cp^i cp^j cp^i$ we use $\left[cp^i, cp^j\right]=2i\left(\left(\mu_B\partial^j\phi\right)cp^i-\left(\mu_B\partial^i\phi\right)cp^j\right)$ from (7.13). We remain mindful that $\mu_B\partial^i\phi=-\mu_B\nabla^i\phi=\mu_B\left(E^i+\dot{A}^i/c\right)$ "smuggles" in an classical external electric field **E** and the time derivative of a classical external potential vector $\dot{\mathbf{A}}$. But to keep everything compact, for the moment we leave $\mu_B\partial^i\phi$ in the equations. Thus:

$$\begin{aligned}&\rho A_\gamma^j cp^i cp^j cp^i = \rho A_\gamma^j cp^j cp^i cp^i + 2i\rho A_\gamma^j \left(\mu_B \partial^j \phi\right) cp^i cp^i - 2i\left(\mu_B \partial^i \phi\right) \rho A_\gamma^j cp^j cp^i \\ &= \rho \mathbf{A}_\gamma \cdot c\mathbf{p} c\mathbf{p} \cdot c\mathbf{p} - 2i\rho \mathbf{A}_\gamma \cdot \mu_B \nabla \phi c\mathbf{p} \cdot c\mathbf{p} + 2i\mu_B \nabla \phi \cdot \rho \mathbf{A}_\gamma \cdot c\mathbf{p} c\mathbf{p}\end{aligned} . \tag{C.3}$$

The final term $2i\mu_B \nabla \phi \cdot \rho \mathbf{A}_\gamma \cdot c\mathbf{p} c\mathbf{p}$ must remain a contracted outer product, because of the requirement to keep all the momenta $\mathbf{p}$ on the very right.

Next in (C.2) we find $\mu_B B_\gamma^j \sigma^i \left(cp^i cp^j + cp^j cp^i\right)$. We again use (7.13) to turn this into:

$$\begin{aligned}&\mu_B B_\gamma^j \sigma^i \left(cp^i cp^j + cp^j cp^i\right) = 2\mu_B B_\gamma^j \sigma^i \left(cp^j cp^i + i\left(\left(\mu_B \partial^j \phi\right) cp^i - \left(\mu_B \partial^i \phi\right) cp^j\right)\right) \\ &= 2\left(\mu_B B_\gamma^j cp^j \sigma^i cp^i + \mu_B B_\gamma^j \left(\mu_B \partial^j \phi\right) i\sigma^i cp^i - i\sigma^i \left(\mu_B \partial^i \phi\right) \mu_B B_\gamma^j cp^j\right) \\ &= 2\left(\mu_B \mathbf{B}_\gamma \cdot c\mathbf{p} \boldsymbol{\sigma} \cdot c\mathbf{p} - \mu_B \mathbf{B}_\gamma \cdot \mu_B \nabla \phi i \boldsymbol{\sigma} \cdot c\mathbf{p} + i\boldsymbol{\sigma} \cdot \mu_B \nabla \phi \mu_B \mathbf{B}_\gamma \cdot c\mathbf{p}\right)\end{aligned} . \tag{C.4}$$

Next in (C.2) is $i\varepsilon^{ijk}\left(\left(E + mc^2\right)\sigma^i - \mu_B B_\gamma^i\right) cp^j cp^k$. Here we utilize (7.16) which contains $\varepsilon^{ijk} cp^j cp^k = -2i\varepsilon^{ijk}\left(\mu_B \partial^j \phi\right) cp^k$. Therefore, we may obtain:

$$\begin{aligned}&i\varepsilon^{ijk}\left(\left(E + mc^2\right)\sigma^i - \mu_B B_\gamma^i\right) cp^j cp^k = 2\varepsilon^{ijk}\left(\left(E + mc^2\right)\sigma^i - \mu_B B_\gamma^i\right) \mu_B \partial^j \phi cp^k \\ &\quad = -2\left(\left(E + mc^2\right) \boldsymbol{\sigma} \cdot \left(\mu_B \nabla \phi \times c\mathbf{p}\right) - \mu_B \mathbf{B}_\gamma \cdot \left(\mu_B \nabla \phi \times c\mathbf{p}\right)\right)\end{aligned} . \tag{C.5}$$

Next we turn to $2i\varepsilon^{lik}\sigma^l \rho A_\gamma^j \left(E\rho A_\gamma^i + i\mu_B E_\gamma^i\right)\left[cp^j, cp^k\right]$ and again use (7.13) to obtain:

$$\begin{aligned}&2i\varepsilon^{lik}\sigma^l \rho A_\gamma^j \left(E\rho A_\gamma^i + i\mu_B E_\gamma^i\right)\left[cp^j, cp^k\right] \\ &= -4\varepsilon^{lik}\sigma^l \rho A_\gamma^j \left(E\rho A_\gamma^i + i\mu_B E_\gamma^i\right)\left(\left(\mu_B \partial^k \phi\right) cp^j - \left(\mu_B \partial^j \phi\right) cp^k\right) \\ &= -4\varepsilon^{lik}\sigma^l \rho A_\gamma^j \left(E\rho A_\gamma^i + i\mu_B E_\gamma^i\right)\left(\mu_B \partial^k \phi\right) cp^j + 4\varepsilon^{lik}\sigma^l \rho A_\gamma^j \left(E\rho A_\gamma^i + i\mu_B E_\gamma^i\right)\left(\mu_B \partial^j \phi\right) cp^k \\ &= -4\varepsilon^{lik}\sigma^l \left(E\rho A_\gamma^i + i\mu_B E_\gamma^i\right)\left(\mu_B \partial^k \phi\right) \rho A_\gamma^j cp^j + 4\rho A_\gamma^j \left(\mu_B \partial^j \phi\right) \varepsilon^{lik}\sigma^l \left(E\rho A_\gamma^i + i\mu_B E_\gamma^i\right) cp^k \\ &= 4\boldsymbol{\sigma} \cdot \left(\left(E\rho \mathbf{A}_\gamma + i\mu_B \mathbf{E}_\gamma\right) \times \mu_B \nabla \phi\right)\left(\rho \mathbf{A}_\gamma \cdot c\mathbf{p}\right) - 4\left(\rho \mathbf{A}_\gamma \cdot \mu_B \nabla \phi\right) \boldsymbol{\sigma} \cdot \left(\left(E\rho \mathbf{A}_\gamma + i\mu_B \mathbf{E}_\gamma\right) \times c\mathbf{p}\right)\end{aligned} . \tag{C.6}$$

The final self-commutator term in (C.2) is $2i\varepsilon^{lik}\sigma^l \rho A_\gamma^j cp^i cp^j cp^k$. This deceptively-simple term is actually very rich, because of its triple momentum and its cross product. The first step is simply to commute $\left[cp^i, cp^j\right]$ so we can separate a scalar product $\rho A_\gamma^j cp^j$ from the cross product. We again use (7.13) followed by (7.16) to initially obtain:

$$\begin{aligned}
&2i\varepsilon^{lik}\sigma^l\rho A_\gamma^j cp^i cp^j cp^k = 2i\varepsilon^{lik}\sigma^l\rho A_\gamma^j\left[cp^j cp^i + 2i\left(\left(\mu_B\partial^j\phi\right)cp^i - \left(\mu_B\partial^i\phi\right)cp^j\right)\right]cp^k \\
&= 2\rho A_\gamma^j cp^j i\sigma^l\varepsilon^{lik}cp^i cp^k - 4\rho A_\gamma^j\left(\mu_B\partial^j\phi\right)\sigma^l\varepsilon^{lik}cp^i cp^k + 4\varepsilon^{lik}\sigma^l\rho A_\gamma^j\left(\mu_B\partial^i\phi\right)cp^j cp^k \\
&= 4\rho A_\gamma^j cp^j\sigma^l\varepsilon^{lik}\left(\mu_B\partial^i\phi\right)cp^k + 8i\rho A_\gamma^j\left(\mu_B\partial^j\phi\right)\varepsilon^{lik}\sigma^l\left(\mu_B\partial^i\phi\right)cp^k + 4\varepsilon^{lik}\sigma^l\rho A_\gamma^j\left(\mu_B\partial^i\phi\right)cp^j cp^k
\end{aligned} \quad . \quad (C.7)$$

Now, in the first term $4\rho A_\gamma^j cp^j\sigma^l\varepsilon^{lik}\left(\mu_B\partial^i\phi\right)cp^k$ a function of spacetime has snuck in to the right of a momentum cp^j, which must now be commuted to the right to enable conversion between momentum and configuration space. From the general $\left[p^i, O\right] = i\hbar\partial^i O$ we form then use $\left[cp^j, \mu_B\partial^i\phi\right] = i\hbar c\partial^j\left(\mu_B\partial^i\phi\right)$. The second term is fine for the moment. In the third term we need to self-commute $\left[cp^j, cp^k\right]$ so that the cross-product terms tied together with ε^{lik} are all adjacent. This again uses (7.13). When we do all of this, a term with $\varepsilon^{lik}\sigma^l\left(\partial^i\phi\right)\left(\partial^k\phi\right)A_\gamma^j p^j = 0$ drops out via $\nabla\phi\times\nabla\phi = 0$ and two oppositely-signed terms with $A_\gamma^j\left(\partial^j\phi\right)\varepsilon^{lik}\sigma^l\left(\partial^i\phi\right)p^k$ cancel out. As a result, (C.7) advances to:

$$2i\varepsilon^{lik}\sigma^l\rho A_\gamma^j cp^i cp^j cp^k = 4\rho A_\gamma^j\sigma^l\varepsilon^{lik}\left(\mu_B\partial^i\phi\right)\left(cp^j cp^k + cp^k cp^j\right) + 4i\hbar c\rho A_\gamma^j\partial^j\varepsilon^{lik}\sigma^l\left(\mu_B\partial^i\phi\right)cp^k \; . \quad (C.8)$$

Finally, we use (7.13) on the $cp^j cp^k + cp^k cp^j$ term. Another term appears and then drops out with $\nabla\phi\times\nabla\phi = 0$. We finally end up with:

$$\begin{aligned}
&2i\varepsilon^{lik}\sigma^l\rho A_\gamma^j cp^i cp^j cp^k = 4\rho A_\gamma^j\sigma^l\varepsilon^{lik}\left(\mu_B\partial^i\phi\right)\left(cp^j cp^k + cp^k cp^j\right) + 4i\hbar c\rho A_\gamma^j\partial^j\varepsilon^{lik}\sigma^l\left(\mu_B\partial^i\phi\right)cp^k \\
&= 8\rho A_\gamma^j\sigma^l\varepsilon^{lik}\left(\mu_B\partial^i\phi\right)cp^k cp^j - 8i\rho A_\gamma^j\left(\mu_B\partial^j\phi\right)\varepsilon^{lik}\sigma^l\left(\mu_B\partial^i\phi\right)cp^k + 4i\hbar c\rho A_\gamma^j\partial^j\varepsilon^{lik}\sigma^l\left(\mu_B\partial^i\phi\right)cp^k \\
&= -8\rho\mathbf{A}_\gamma\cdot\boldsymbol{\sigma}\cdot\mu_B\nabla\phi\times c\mathbf{p}c\mathbf{p} - 8\rho\mathbf{A}_\gamma\cdot\mu_B\nabla\phi i\boldsymbol{\sigma}\cdot\left(\mu_B\nabla\phi\times c\mathbf{p}\right) + 4\rho\mathbf{A}_\gamma\cdot i\hbar c\nabla\boldsymbol{\sigma}\cdot\left(\mu_B\nabla\phi\times c\mathbf{p}\right)
\end{aligned} \; . \quad (C.9)$$

The first term retains an outer product of two **p** which cannot be commuted away, but the other two terms do segregate into scalar products, the latter being $\boldsymbol{\sigma}$ dotted with a cross product.

Now we return to (C.2) insert all of (C.3) through (C.6) and (C.9), consolidate, and convert from index to vector notation. For the group **I** of terms in (C.1), segregating into two subgroups **Ia** and **Ib** without and with $\mu_B\nabla\phi = -\mu_B\left(\mathbf{E} + \dot{\mathbf{A}}/c\right)$, we obtain:

$$
\begin{aligned}
&\left.\begin{aligned}
\mathbf{I} &= \left(E+mc^2-\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma\right)\left(c\mathbf{p}\cdot c\mathbf{p}+2E\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+2i\mu_B\mathbf{E}_\gamma\cdot c\mathbf{p}+2E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+i\mu_B\mathbf{E}_\gamma\cdot i\mu_B\mathbf{E}_\gamma\right)\\
&+\left(2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}c\mathbf{p}+4E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma\rho\mathbf{A}_\gamma+2i\mu_B\mathbf{E}_\gamma\cdot i\mu_B\mathbf{E}_\gamma\rho\mathbf{A}_\gamma\right)\cdot c\mathbf{p}\\
&+\left(4E\rho\mathbf{A}_\gamma\cdot\rho\mathbf{A}_\gamma+2\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+2i\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma\right)\cdot c\mathbf{p}c\mathbf{p}\\
&+2\mu_B\mathbf{B}_\gamma\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\boldsymbol{\sigma}\cdot c\mathbf{p}+2\boldsymbol{\sigma}\cdot\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\mu_B\mathbf{B}_\gamma\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)
\end{aligned}\right\}\mathbf{a}\\
&\left.\begin{aligned}
&-4i\rho\mathbf{A}_\gamma\cdot\mu_B\nabla\phi c\mathbf{p}\cdot c\mathbf{p}+4i\mu_B\nabla\phi\cdot\rho\mathbf{A}_\gamma\cdot c\mathbf{p}c\mathbf{p}\\
&-2\mu_B\mathbf{B}_\gamma\cdot\mu_B\nabla\phi i\boldsymbol{\sigma}\cdot c\mathbf{p}+2i\boldsymbol{\sigma}\cdot\mu_B\nabla\phi\mu_B\mathbf{B}_\gamma\cdot c\mathbf{p}\\
&-2\left(E+mc^2\right)\boldsymbol{\sigma}\cdot\left(\mu_B\nabla\phi\times c\mathbf{p}\right)+2\mu_B\mathbf{B}_\gamma\cdot\left(\mu_B\nabla\phi\times c\mathbf{p}\right)\\
&+4\boldsymbol{\sigma}\cdot\left(\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\times\mu_B\nabla\phi\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-4\rho\mathbf{A}_\gamma\cdot\mu_B\nabla\phi\boldsymbol{\sigma}\cdot\left(\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\times c\mathbf{p}\right)\\
&-8\rho\mathbf{A}_\gamma\cdot\boldsymbol{\sigma}\cdot\mu_B\nabla\phi\times c\mathbf{p}c\mathbf{p}-8\rho\mathbf{A}_\gamma\cdot\mu_B\nabla\phi i\boldsymbol{\sigma}\cdot\left(\mu_B\nabla\phi\times c\mathbf{p}\right)+4\rho\mathbf{A}_\gamma\cdot i\hbar c\nabla\boldsymbol{\sigma}\cdot\left(\mu_B\nabla\phi\times c\mathbf{p}\right)
\end{aligned}\right\}\mathbf{b}
\end{aligned}
\quad . \qquad \text{(C.10)}
$$

Next, we move on to the group **II** terms in (C.1) which do contain gradients. We use the product rule to distribute each ∇^i, and the only terms we eliminate are those for which $\nabla^i A^i_\gamma = 0$, via (14.5). Then, as with group **I**, we expand using $\sigma^i\sigma^k = \delta^{ik} + i\varepsilon^{ikl}\sigma^l$ and $\sigma^i\sigma^j\sigma^k = \delta^{ij}\sigma^k + \delta^{jk}\sigma^i - \delta^{ki}\sigma^j + i\varepsilon^{ijk}$, but keep all terms in a single such expansion still grouped together for easy tracking. These group **II** terms all contain at least one gradient, and produce multiple dot and cross products. Many of these can and will be reduced, but it is helpful to show *all* of the terms before reduction, then show what reductions are possible. In general, while we keep as many terms as possible in the form of $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$ which is the vector-notation statement of $\left(\sigma^i\sigma^k\right)a^i b^k = \left(\delta^{ik} + i\varepsilon^{ikl}\sigma^l\right)a^i b^k$, in two situations this is not possible and we instead end up with the form $\left(\sigma^i\sigma^k\right)a^k b^i = \left(\delta^{ik} + i\varepsilon^{ikl}\sigma^l\right)a^k b^i = \mathbf{a}\cdot\mathbf{b} - i(\mathbf{a}\times\mathbf{b})$. The first situation is where $b^i = p^i$ is a momentum vector which we must keep on the very right. Examples of this below are terms containing $\rho\mathbf{A}_\gamma\cdot c\mathbf{p} - i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times c\mathbf{p}\right)$. The second such situation is where $a^k = \nabla^k$ is a gradient which we must keep on the left of its operand. An example below is the term $\left(\rho\mathbf{A}_\gamma\cdot\nabla - i\boldsymbol{\sigma}\cdot\rho\mathbf{A}_\gamma\times\nabla\right)\rho\mathbf{A}_\gamma\cdot e\nabla\phi$. In some instances, both situations appear, such as in the term $\left(i\mu_B\mathbf{E}_\gamma\cdot\nabla - i\boldsymbol{\sigma}\cdot\left(i\mu_B\mathbf{E}_\gamma\times\nabla\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}$. We also commute some double gradients using $\left[\nabla^i,\nabla^j\right]=0$ which applies in flat spacetime where there is no Riemann curvature. And throughout, from (7.5) we substitute $\nabla E = q\nabla\phi = -e\nabla\phi$ using $q=-e$ for the charged leptons. Finally, we convert from index into vector notation. So, prior to any of the reductions which we shall next consider, the group **II** gradient terms in (C.1), in their entirety, are calculated to be:

$$
\begin{aligned}
\mathbf{II} = &-i\hbar c\left(-e\nabla\phi\cdot c\mathbf{p}-i\boldsymbol{\sigma}\cdot\left(e\nabla\phi\times c\mathbf{p}\right)+2\left(\nabla\cdot\rho\mathbf{A}_\gamma\cdot c\mathbf{p}c\mathbf{p}+i\boldsymbol{\sigma}\cdot\left(\nabla\times\rho\mathbf{A}_\gamma\cdot c\mathbf{p}c\mathbf{p}\right)\right)\right)\\
&-i\hbar c\begin{pmatrix}+\left(E+mc^2\right)\left(E\left(\nabla\cdot\rho\mathbf{A}_\gamma+i\underline{\underline{\boldsymbol{\sigma}\cdot\left(\nabla\times\rho\mathbf{A}_\gamma\right)}}\right)+\left(\underline{\nabla\cdot i\mu_B\mathbf{E}_\gamma}+i\boldsymbol{\sigma}\cdot\left(\underline{\nabla\times i\mu_B\mathbf{E}_\gamma}\right)\right)-\left(e\nabla\phi\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(e\nabla\phi\times\rho\mathbf{A}_\gamma\right)\right)\right)\\
-E\left(e\nabla\phi\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(e\nabla\phi\times\rho\mathbf{A}_\gamma\right)\right)+\left(e\nabla\phi\cdot i\mu_B\mathbf{E}_\gamma+i\boldsymbol{\sigma}\cdot\left(e\nabla\phi\times i\mu_B\mathbf{E}_\gamma\right)\right)\end{pmatrix}\\
&-i\hbar c\begin{pmatrix}+\underline{\nabla\cdot\mu_B\mathbf{B}_\gamma}\boldsymbol{\sigma}\cdot c\mathbf{p}+\boldsymbol{\sigma}\cdot\nabla\mu_B\mathbf{B}_\gamma\cdot c\mathbf{p}-\nabla\cdot\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}c\mathbf{p}+i\left(\underline{\nabla\times\mu_B\mathbf{B}_\gamma}\right)\cdot c\mathbf{p}\\
+E\left(\underline{\nabla\cdot\mu_B\mathbf{B}_\gamma}\boldsymbol{\sigma}\cdot\rho\mathbf{A}_\gamma+\boldsymbol{\sigma}\cdot\nabla\mu_B\mathbf{B}_\gamma\cdot\rho\mathbf{A}_\gamma-\rho\mathbf{A}_\gamma\cdot\nabla\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}+i\left(\underline{\nabla\times\mu_B\mathbf{B}_\gamma}\right)\cdot\rho\mathbf{A}_\gamma\right)\\
+\underline{\nabla\cdot\mu_B\mathbf{B}_\gamma}\boldsymbol{\sigma}\cdot i\mu_B\mathbf{E}_\gamma+\boldsymbol{\sigma}\cdot\nabla\mu_B\mathbf{B}_\gamma\cdot i\mu_B\mathbf{E}_\gamma-i\mu_B\mathbf{E}_\gamma\cdot\nabla\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}+i\left(\underline{\nabla\times\mu_B\mathbf{B}_\gamma}\right)\cdot i\mu_B\mathbf{E}_\gamma\\
-\mu_B\mathbf{B}_\gamma\cdot e\nabla\phi\boldsymbol{\sigma}\cdot\rho\mathbf{A}_\gamma-\mu_B\mathbf{B}_\gamma\cdot\rho\mathbf{A}_\gamma\boldsymbol{\sigma}\cdot e\nabla\phi+\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}e\nabla\phi\cdot\rho\mathbf{A}_\gamma-i\mu_B\mathbf{B}_\gamma\cdot\left(e\nabla\phi\times\rho\mathbf{A}_\gamma\right)\\
+E\left(\underline{\mu_B\mathbf{B}_\gamma\cdot\nabla}\boldsymbol{\sigma}\cdot\rho\mathbf{A}_\gamma+\boldsymbol{\sigma}\cdot\nabla\rho\mathbf{A}_\gamma\cdot\mu_B\mathbf{B}_\gamma-\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}\nabla\cdot\rho\mathbf{A}_\gamma-i\mu_B\mathbf{B}_\gamma\cdot\left(\underline{\nabla\times\rho\mathbf{A}_\gamma}\right)\right)\\
+\underline{\mu_B\mathbf{B}_\gamma\cdot\nabla}\boldsymbol{\sigma}\cdot i\mu_B\mathbf{E}_\gamma+\boldsymbol{\sigma}\cdot\nabla i\mu_B\mathbf{E}_\gamma\cdot\mu_B\mathbf{B}_\gamma-\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}\underline{\nabla\cdot i\mu_B\mathbf{E}_\gamma}-i\mu_B\mathbf{B}_\gamma\cdot\left(\underline{\nabla\times i\mu_B\mathbf{E}_\gamma}\right)\end{pmatrix}\\
&-2i\hbar c\begin{pmatrix}E\left(\rho\mathbf{A}_\gamma\cdot\nabla-i\boldsymbol{\sigma}\cdot\left(\underline{\rho\mathbf{A}_\gamma\times\nabla}\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+\left(\underline{i\mu_B\mathbf{E}_\gamma\cdot\nabla}-i\boldsymbol{\sigma}\cdot\left(\underline{i\mu_B\mathbf{E}_\gamma\times\nabla}\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
+E\left(\nabla\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\underline{\nabla\times\rho\mathbf{A}_\gamma}\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+\left(\underline{\nabla\cdot i\mu_B\mathbf{E}_\gamma}+i\boldsymbol{\sigma}\cdot\left(\underline{\nabla\times i\mu_B\mathbf{E}_\gamma}\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-\left(e\nabla\phi\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(e\nabla\phi\times\rho\mathbf{A}_\gamma\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
-\rho\mathbf{A}_\gamma\cdot e\nabla\phi\left(\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times c\mathbf{p}\right)\right)+E\rho\mathbf{A}_\gamma\cdot\nabla\left(\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times c\mathbf{p}\right)\right)+\rho\mathbf{A}_\gamma\cdot\nabla\left(i\mu_B\mathbf{E}_\gamma\cdot c\mathbf{p}-i\boldsymbol{\sigma}\cdot\left(i\mu_B\mathbf{E}_\gamma\times c\mathbf{p}\right)\right)\end{pmatrix}\\
&-2i\hbar c\begin{pmatrix}-\rho\mathbf{A}_\gamma\cdot e\nabla\phi\left(i\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(i\mu_B\mathbf{E}_\gamma\times\rho\mathbf{A}_\gamma\right)\right)-E\rho\mathbf{A}_\gamma\cdot e\nabla\phi\left(\rho\mathbf{A}_\gamma\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times\rho\mathbf{A}_\gamma\right)\right)\\
+EE\rho\mathbf{A}_\gamma\cdot\nabla\left(\rho\mathbf{A}_\gamma\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times\rho\mathbf{A}_\gamma\right)\right)+E\rho\mathbf{A}_\gamma\cdot\nabla\left(\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times i\mu_B\mathbf{E}_\gamma\right)\right)\\
+E\rho\mathbf{A}_\gamma\cdot\nabla\left(i\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(i\mu_B\mathbf{E}_\gamma\times\rho\mathbf{A}_\gamma\right)\right)+\rho\mathbf{A}_\gamma\cdot\nabla\left(i\mu_B\mathbf{E}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+i\boldsymbol{\sigma}\cdot\left(i\mu_B\mathbf{E}_\gamma\times i\mu_B\mathbf{E}_\gamma\right)\right)\end{pmatrix}\\
&-2\hbar^2c^2\begin{pmatrix}-\left(\nabla\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\underline{\nabla\times\rho\mathbf{A}_\gamma}\right)\right)\rho\mathbf{A}_\gamma\cdot e\nabla\phi-\left(\rho\mathbf{A}_\gamma\cdot\nabla-i\boldsymbol{\sigma}\cdot\left(\underline{\rho\mathbf{A}_\gamma\times\nabla}\right)\right)\rho\mathbf{A}_\gamma\cdot e\nabla\phi-\left(\rho\mathbf{A}_\gamma\cdot\nabla-i\boldsymbol{\sigma}\cdot\left(\underline{\rho\mathbf{A}_\gamma\times\nabla}\right)\right)e\nabla\phi\cdot\rho\mathbf{A}\\
+E\left(\nabla\cdot\rho\mathbf{A}_\gamma\cdot\nabla\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\nabla\times\rho\mathbf{A}_\gamma\cdot\nabla\rho\mathbf{A}_\gamma\right)\right)+\nabla\cdot\rho\mathbf{A}_\gamma\cdot\nabla i\mu_B\mathbf{E}_\gamma+i\boldsymbol{\sigma}\cdot\left(\nabla\times\rho\mathbf{A}_\gamma\cdot\nabla i\mu_B\mathbf{E}_\gamma\right)-\rho\mathbf{A}_\gamma\cdot\nabla\left(\rho\mathbf{A}_\gamma\cdot e\nabla\phi-i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times e\nabla\phi\right)\right)\\
+E\rho\mathbf{A}_\gamma\cdot\nabla\left(\nabla\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\underline{\nabla\times\rho\mathbf{A}_\gamma}\right)\right)+\rho\mathbf{A}_\gamma\cdot\nabla\left(\nabla\cdot i\mu_B\mathbf{E}_\gamma+i\boldsymbol{\sigma}\cdot\left(\underline{\nabla\times i\mu_B\mathbf{E}_\gamma}\right)\right)\end{pmatrix}
\end{aligned}
\quad . \quad (C.11)
$$

Above, other than having already applied $\nabla^i A^i = 0$ when expanding derivatives using the product rule, each of the six large parenthetical expressions (three preceded by $-i\hbar c$, two by $-2i\hbar c$ and one by $-2\hbar^2 c^2$) is the complete expansion of the six respective lines of group **II** terms in (C.1). As we shall shortly see, many of these terms are zeros, primarily as a consequence of $\mathbf{A}\cdot\mathbf{q}=0$ from (14.5). Multiple terms are underlined above, to highlight further reductions which will also be reviewed. Specifically, we highlight all terms which either contain a magnetic moment expression $\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$, or which contain (or will be shown to contain) a field configuration corresponding either with $\nabla\times\mathbf{A}_\gamma=\mathbf{B}_\gamma$, or with one of the four Maxwell equations. This includes a term which in full is $-i\hbar c\left(E+mc^2\right)Ei\boldsymbol{\sigma}\cdot\left(\nabla\times\rho\mathbf{A}_\gamma\right)$, highlighted by double underling. We shall ultimately establish that this double-underlined term is the primary term for fermion magnetic moments *including anomalies, without any need for renormalization*.

Taken together, (C.10) and (C.11) contain the complete expansion of all the terms in the numerator (C.1) of the Hamiltonian (20.8). For here, we shall reduce and consolidate and reorganize these expressions in a number of different ways. First, in (C.11) there are a few occurrences of $\nabla\cdot\mathbf{A}_\gamma=0$ which can immediately be zeroed out using (15.15). So too with $\mathbf{A}_\gamma\cdot\mathbf{A}_\gamma=0$ from (14.8) and $\mathbf{A}_\gamma\times\mathbf{A}_\gamma=0$ and $\mathbf{E}_\gamma\times\mathbf{E}_\gamma=0$ by identity. Further, there are multiple places we may use the heuristic substitution $i\hbar c\nabla\mapsto -c\mathbf{q}$ from (15.4), (15.7) and (15.15), followed by $\mathbf{A}_\gamma\cdot\mathbf{q}=0$ from (15.5). Without the various constants which are irrelevant to these calculations, various terms that zero out from (C.11) in this way include the scalar and cross products:

$$\begin{aligned}
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\mathbf{A}_\gamma\cdot\mathbf{p}=\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{A}_\gamma\cdot\mathbf{p}=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\mathbf{E}_\gamma\cdot\mathbf{p}=\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{E}_\gamma\cdot\mathbf{p}=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\mathbf{A}_\gamma\cdot\mathbf{E}_\gamma=\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{A}_\gamma\cdot\mathbf{E}_\gamma=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\mathbf{E}_\gamma\cdot\mathbf{A}_\gamma=\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{E}_\gamma\cdot\mathbf{A}_\gamma=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\mathbf{E}_\gamma\cdot\mathbf{E}_\gamma=\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{E}_\gamma\cdot\mathbf{E}_\gamma=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\mathbf{A}_\gamma\cdot\nabla\phi=\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{A}_\gamma\cdot\nabla\phi=0
\end{aligned}\;; \tag{C.12}$$

$$\begin{aligned}
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\mathbf{p}\right)=\mathbf{A}_\gamma\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\mathbf{p}\right)=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\boldsymbol{\sigma}\cdot\left(\mathbf{E}_\gamma\times\mathbf{p}\right)=\mathbf{A}_\gamma\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\left(\mathbf{E}_\gamma\times\mathbf{p}\right)=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\mathbf{E}_\gamma\right)=\mathbf{A}_\gamma\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\mathbf{E}_\gamma\right)=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\boldsymbol{\sigma}\cdot\left(\mathbf{E}_\gamma\times\mathbf{A}_\gamma\right)=\mathbf{A}_\gamma\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\left(\mathbf{E}_\gamma\times\mathbf{A}_\gamma\right)=0\\
&-i\hbar\mathbf{A}_\gamma\cdot\nabla\sigma\cdot\left(\mathbf{A}_\gamma\times\nabla\phi\right)=\mathbf{A}_\gamma\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\nabla\phi\right)=0
\end{aligned}\;. \tag{C.13}$$

Another type of very useful reduction makes use of the commutator $[\mathbf{x},\mathbf{q}]=\left[x^i,q^j\right]=0$ of a *luminous* photon momentum $\mathbf{q}$ with functions of spacetime $b(t,\mathbf{x})$, as reviewed following (15.15). Here, we use the heuristic relation $i\hbar c\nabla \mapsto -c\mathbf{q}$ from (15.4), (15.7) and (15.15) for photon fields and (16.14) for classical external fields, followed by commutations of $\mathbf{q}$, followed by a restoration of $-c\mathbf{q}\mapsto i\hbar c\nabla$ with ∇ operating on a different object in a different position (so long as that latter object also is subject to (15.4), (15.7), (15.15) or (16.14)), followed by a further reduction. This type of reduction is applied to the following terms in (C.11):

$$\begin{aligned}
&i\hbar\mathbf{E}_\gamma\cdot\nabla\mathbf{A}_\gamma\cdot\mathbf{p}=-\mathbf{E}_\gamma\cdot\mathbf{q}\mathbf{A}_\gamma\cdot\mathbf{p}=-\mathbf{q}\cdot\mathbf{E}_\gamma\mathbf{A}_\gamma\cdot\mathbf{p}=i\hbar\nabla\cdot\mathbf{E}_\gamma\mathbf{A}_\gamma\cdot\mathbf{p}\\
&i\hbar\mathbf{B}_\gamma\cdot\nabla\boldsymbol{\sigma}\cdot\mathbf{A}_\gamma=-\mathbf{B}_\gamma\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{A}_\gamma=-\mathbf{q}\cdot\mathbf{B}_\gamma\boldsymbol{\sigma}\cdot\mathbf{A}_\gamma=i\hbar\nabla\cdot\mathbf{B}_\gamma\boldsymbol{\sigma}\cdot\mathbf{A}_\gamma\\
&i\hbar\mathbf{B}_\gamma\cdot\nabla\boldsymbol{\sigma}\cdot\mathbf{E}_\gamma=-\mathbf{B}_\gamma\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{E}_\gamma=-\mathbf{q}\cdot\mathbf{B}_\gamma\boldsymbol{\sigma}\cdot\mathbf{E}_\gamma=i\hbar\nabla\cdot\mathbf{B}_\gamma\boldsymbol{\sigma}\cdot\mathbf{E}_\gamma\\
&-i\hbar\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\nabla\right)\mathbf{A}_\gamma\cdot\mathbf{p}=\left(\boldsymbol{\sigma}\cdot\mathbf{A}_\gamma\times\mathbf{q}\right)\mathbf{A}_\gamma\cdot\mathbf{p}=-\boldsymbol{\sigma}\cdot\left(\mathbf{q}\times\mathbf{A}_\gamma\right)\mathbf{A}_\gamma\cdot\mathbf{p}=i\hbar\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\right)\mathbf{A}_\gamma\cdot\mathbf{p}\\
&-i\hbar\boldsymbol{\sigma}\cdot\left(\mathbf{E}_\gamma\times\nabla\right)\mathbf{A}_\gamma\cdot\mathbf{p}=\boldsymbol{\sigma}\cdot\left(\mathbf{E}_\gamma\times\mathbf{q}\right)\mathbf{A}_\gamma\cdot\mathbf{p}=-\boldsymbol{\sigma}\cdot\left(\mathbf{q}\times\mathbf{E}_\gamma\right)\mathbf{A}_\gamma\cdot\mathbf{p}=i\hbar\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{E}_\gamma\right)\mathbf{A}_\gamma\cdot\mathbf{p}\\
&-i\hbar\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\nabla\right)\mathbf{A}_\gamma\cdot\nabla\phi=\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\mathbf{q}\right)\mathbf{A}_\gamma\cdot\nabla\phi=-\boldsymbol{\sigma}\cdot\left(\mathbf{q}\times\mathbf{A}_\gamma\right)\mathbf{A}_\gamma\cdot\nabla\phi=i\hbar\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\right)\mathbf{A}_\gamma\cdot\nabla\phi
\end{aligned}\quad . \tag{C.14}$$

The upshot of these reductions is that so long as ∇ is operating on any one of the objects $\mathbf{O}=\mathbf{A},\mathbf{E},\mathbf{B}$, we may commute $\mathbf{A}\cdot\nabla\mathbf{O}\mapsto\nabla\cdot\mathbf{A}\mathbf{O}=0$, $\mathbf{B}\cdot\nabla\mathbf{O}\mapsto\nabla\cdot\mathbf{B}\mathbf{O}$ and $\mathbf{E}\cdot\nabla\mathbf{O}\mapsto\nabla\cdot\mathbf{E}\mathbf{O}$, as well as $\mathbf{A}\times\nabla\mathbf{O}\to-\nabla\times\mathbf{A}\mathbf{O}$ and $\mathbf{E}\times\nabla\mathbf{O}\to-\nabla\times\mathbf{E}\mathbf{O}$. The latter use the generalized identity $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$. The benefit of these reductions is that we reveal additional magnetic moment terms $\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\right)=\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$, as well as terms $\nabla\cdot\mathbf{E}_\gamma=4\pi\rho_{em\gamma}$ for an electric charge density (which we do not set to zero using (15.10) but leave as is for now), $\nabla\times\mathbf{E}_\gamma=-\dot{\mathbf{B}}_\gamma/c$ for the magnetic field time derivative, and $\nabla\cdot\mathbf{B}_\gamma=0$ which is always equal to zero even for a classical external magnetic field. There are already some appearances in (C.11) of the field configuration for the remaining Maxwell equation, $\nabla\times\mathbf{B}_\gamma=\left(4\pi\mathbf{J}+\dot{\mathbf{E}}_\gamma\right)/c$.

In the final group of two-gradient terms preceded by the coefficient $-2\hbar^2c^2$ in (C.11), there are four additional terms containing outer products, which nonetheless zero out with (15.5) following close inspection. Using index notation which is helpful for seeing this, we find that:

$$\begin{aligned}
&-\hbar^2\nabla\cdot\mathbf{A}_\gamma\cdot\nabla\mathbf{A}_\gamma=\mathbf{q}\cdot\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{A}_\gamma=q^iA_\gamma^jq^jA_\gamma^i=q^i\left(\mathbf{A}_\gamma\cdot\mathbf{q}\right)A_\gamma^i=0\\
&-\hbar^2\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\cdot\nabla\mathbf{A}_\gamma\right)=\boldsymbol{\sigma}\cdot\left(\mathbf{q}\times\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{A}_\gamma\right)=\varepsilon^{ikl}\sigma^lq^iA_\gamma^jq^jA_\gamma^k=\varepsilon^{ikl}\sigma^lq^i\left(\mathbf{A}_\gamma\cdot\mathbf{q}\right)A_\gamma^k=0\\
&-\hbar^2\nabla\cdot\mathbf{A}_\gamma\cdot\nabla\mathbf{E}_\gamma=\mathbf{q}\cdot\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{E}_\gamma=q^iA_\gamma^jq^jE_\gamma^i=q^i\left(\mathbf{A}_\gamma\cdot\mathbf{q}\right)E_\gamma^i=0\\
&-\hbar^2\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\cdot\nabla\mathbf{E}_\gamma\right)=\boldsymbol{\sigma}\cdot\left(\mathbf{q}\times\mathbf{A}_\gamma\cdot\mathbf{q}\mathbf{E}_\gamma\right)=\varepsilon^{ikl}\sigma^lq^iA_\gamma^jq^jE_\gamma^k=\varepsilon^{ikl}\sigma^lq^i\left(\mathbf{A}_\gamma\cdot\mathbf{q}\right)E_\gamma^k=0
\end{aligned}\quad . \tag{C.15}$$

Additionally in the $-2\hbar^2c^2$ group, one more terms is eliminated and another commuted using $\nabla\phi = -\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)$ and (16.14) for classical external fields, as such:

$$i\hbar\mathbf{A}_\gamma\cdot\nabla\nabla\phi\cdot\mathbf{A} = -i\hbar\mathbf{A}_\gamma\cdot\nabla\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\mathbf{A} = \left(\mathbf{A}_\gamma\cdot\mathbf{q}\right)\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\mathbf{A} = 0. \tag{C.16}$$

$$\begin{aligned}&i\hbar\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\nabla\right)\nabla\phi\cdot\mathbf{A} = -i\hbar\boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\nabla\right)\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\mathbf{A} = \boldsymbol{\sigma}\cdot\left(\mathbf{A}_\gamma\times\mathbf{q}\right)\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\mathbf{A}\\ &= -\boldsymbol{\sigma}\cdot\left(\mathbf{q}\times\mathbf{A}_\gamma\right)\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\mathbf{A} = -i\hbar\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\right)\nabla\phi\cdot\mathbf{A} = -i\hbar\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\right)\mathbf{A}\cdot\nabla\phi\end{aligned}. \tag{C.17}$$

As to the time derivative, the above makes use of $\left[\partial_\mu,\partial_\nu\right]=0$ in flat spacetime where the Reimann curvature tensor $R^\sigma{}_{\alpha\mu\nu}=0$. So, the upshot of (C.16) and (C.17) is that we may also commute $\mathbf{A}_\gamma\cdot\nabla\nabla\phi \mapsto \nabla\cdot\mathbf{A}_\gamma\nabla\phi = 0$ and $\mathbf{A}_\gamma\times\nabla\nabla\phi \to \nabla\times\mathbf{A}_\gamma\nabla\phi = \mathbf{B}_\gamma\nabla\phi$, with the operands in (C.14) extended to include $\mathbf{O}=\mathbf{A},\mathbf{E},\mathbf{B},\nabla\phi$. As a result, $\mathbf{A}_\gamma\cdot i\hbar\nabla\boldsymbol{\sigma}\cdot\left(\nabla\phi\times\mathbf{p}\right) = \mathbf{A}_\gamma\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\left(\nabla\phi\times\mathbf{p}\right)=0$, which is the final term in (C.10), also drops out.

Finally, keeping in mind all of section 16 which examined Maxwell's equations for individual photons, the final reduction step throughout (C.10) and (C.11) is to substitute the generally-covariant antisymmetric gauge-invariant relation $F^{\mu\nu}=\partial^\mu A^\nu - \partial^\nu A^\mu$ between gauge potentials and electric and magnetic fields, and the generally-covariant, gauge invariant Maxwell equations $4\pi J^\mu = \partial_\alpha F^{\alpha\mu}$ for electrical sources and $\partial^\alpha F^{\mu\nu}+\partial^\mu F^{\nu\alpha}+\partial^\nu F^{\alpha\mu}=0$ a.k.a. $\partial_\alpha {}^*F^{\alpha\mu}=0$ for (the non-existence of) magnetic sources. Given the terms which appear in (C.10) and (C.11), the potential / field relation breaks down into $\nabla\phi = -\mathbf{E}-\dot{\mathbf{A}}/c$ and $\nabla\times\mathbf{A}_\gamma = \mathbf{B}_\gamma$. The former contains the classical, external fields $\mathbf{E}$ and $\dot{\mathbf{A}}$ and not individual photon fields because $\nabla\phi$ enters as the gradient of a classical, external scalar potential. The latter contain the individual photon magnetic field $\mathbf{B}_\gamma$ because $\nabla\times\mathbf{A}_\gamma$ enters as the curl of the photon three-potential. Keeping in mind (15.6) and the related discussion that photons have non-zero magnetic fields with zero magnitude, this reveals several photon magnetic moment terms $\boldsymbol{\sigma}\cdot\mathbf{B}_\gamma$. The Maxwell equations all enter as those for individual photons because the field divergences and curls appearing in (C.11) contain the individual photons fields. Thus, we insert $\nabla\cdot\mathbf{E}_\gamma = 4\pi\rho_{em\,\gamma}$, $\nabla\times\mathbf{B}_\gamma = \left(4\pi\mathbf{J}_\gamma+\dot{\mathbf{E}}_\gamma\right)/c$, $\nabla\cdot\mathbf{B}_\gamma=0$ and $\nabla\times\mathbf{E}_\gamma = -\dot{\mathbf{B}}_\gamma/c$ wherever these appear. Although $\rho_{em\,\gamma}=0$ and $\mathbf{J}_\gamma=0$ as deduced in (15.10), mindful of section 16 and especially of (16.10) which reminds us that a material electrical source is what creates a scalar potential and there is never one without the other, we shall not zero out these sources, but shall keep them in place so we can later consider the effects of external classical sources on the hyper-canonical Dirac Hamiltonian.

Making all of the foregoing reductions and substitutions, (C.10) now advances to:

$$\left.\begin{aligned}\mathbf{I}=&\left(E+mc^2-\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma\right)\left(c\mathbf{p}\cdot c\mathbf{p}+2E\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+2i\mu_B\mathbf{E}_\gamma\cdot c\mathbf{p}+2E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+i\mu_B\mathbf{E}_\gamma\cdot i\mu_B\mathbf{E}_\gamma\right)\\&+\left(2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}c\mathbf{p}+4E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma\rho\mathbf{A}_\gamma+2i\mu_B\mathbf{E}_\gamma\cdot i\mu_B\mathbf{E}_\gamma\rho\mathbf{A}_\gamma\right)\cdot c\mathbf{p}\\&+\left(4E\rho\mathbf{A}_\gamma\cdot\rho\mathbf{A}_\gamma+2\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+2i\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma\right)\cdot c\mathbf{p}c\mathbf{p}\\&+2\mu_B\mathbf{B}_\gamma\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\boldsymbol{\sigma}\cdot c\mathbf{p}+2\boldsymbol{\sigma}\cdot\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\mu_B\mathbf{B}_\gamma\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\end{aligned}\right\}\mathbf{a}$$

$$\left.\begin{aligned}&+4i\rho\mathbf{A}_\gamma\cdot\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)c\mathbf{p}\cdot c\mathbf{p}-4i\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\rho\mathbf{A}_\gamma\cdot c\mathbf{p}c\mathbf{p}\\&+2\mu_B\mathbf{B}_\gamma\cdot\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)i\boldsymbol{\sigma}\cdot c\mathbf{p}-2i\boldsymbol{\sigma}\cdot\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\mu_B\mathbf{B}_\gamma\cdot c\mathbf{p}\\&+2\left(E+mc^2\right)\boldsymbol{\sigma}\cdot\left(\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times c\mathbf{p}\right)-2\mu_B\mathbf{B}_\gamma\cdot\left(\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times c\mathbf{p}\right)\\&-4\boldsymbol{\sigma}\cdot\left(\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\times\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\&+4\rho\mathbf{A}_\gamma\cdot\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\boldsymbol{\sigma}\cdot\left(\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\times c\mathbf{p}\right)\\&+8\rho\mathbf{A}_\gamma\cdot\boldsymbol{\sigma}\cdot\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times c\mathbf{p}c\mathbf{p}-8\rho\mathbf{A}_\gamma\cdot\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)i\boldsymbol{\sigma}\cdot\left(\mu_B\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times c\mathbf{p}\right)\end{aligned}\right\}\mathbf{b}\quad.\tag{C.18}$$

For (C.11) we keep the six large parenthetical expressions still grouped together to simplify comparison including with the group **II** terms in (C.1), and we continue to highlight the magnetic moment terms as well as a pair of emergent $\boldsymbol{\sigma}\cdot\dot{\mathbf{B}}_\gamma$ terms with the time derivative of the magnetic moment. As a result of everything laid out following (C.11), we obtain:

$$
\begin{aligned}
\mathbf{II} = & -i\hbar c\begin{pmatrix} e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot c\mathbf{p}+i\boldsymbol{\sigma}\cdot\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times c\mathbf{p}\right) \\ +2\left(\nabla\cdot\rho\mathbf{A}_\gamma\cdot c\mathbf{p}c\mathbf{p}+i\boldsymbol{\sigma}\cdot\left(\nabla\times\rho\mathbf{A}_\gamma\cdot c\mathbf{p}c\mathbf{p}\right)\right) \end{pmatrix} \\
& -i\hbar c\begin{pmatrix} \left(E+mc^2\right)\left(Ei\underline{\underline{\boldsymbol{\sigma}\cdot\rho\mathbf{B}_\gamma}}+4\pi i\mu_B\rho_{em\,\gamma}+\underline{\boldsymbol{\sigma}\cdot\mu_B\dot{\mathbf{B}}_\gamma/c}\right) \\ +\left(E+mc^2\right)\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times\rho\mathbf{A}_\gamma\right)\right) \\ -E\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times\rho\mathbf{A}_\gamma\right)\right) \\ +e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot i\mu_B\mathbf{E}_\gamma+i\boldsymbol{\sigma}\cdot\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times i\mu_B\mathbf{E}_\gamma\right) \end{pmatrix} \\
& -i\hbar c\begin{pmatrix} \boldsymbol{\sigma}\cdot\nabla\mu_B\mathbf{B}_\gamma\cdot c\mathbf{p}-\nabla\cdot\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}c\mathbf{p}+i\mu_B\left(4\pi\mathbf{J}_\gamma+\dot{\mathbf{E}}_\gamma\right)\cdot c\mathbf{p}/c \\ +E\left(\boldsymbol{\sigma}\cdot\nabla\mu_B\mathbf{B}_\gamma\cdot\rho\mathbf{A}_\gamma-\rho\mathbf{A}_\gamma\cdot\nabla\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}+i\mu_B\left(4\pi\mathbf{J}_\gamma+\dot{\mathbf{E}}_\gamma\right)\cdot\rho\mathbf{A}_\gamma/c\right) \\ +\boldsymbol{\sigma}\cdot\nabla\mu_B\mathbf{B}_\gamma\cdot i\mu_B\mathbf{E}_\gamma-i\mu_B\mathbf{E}_\gamma\cdot\nabla\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}+i\mu_B\left(4\pi\mathbf{J}_\gamma+\dot{\mathbf{E}}_\gamma\right)\cdot i\mu_B\mathbf{E}_\gamma/c \\ +\mu_B\mathbf{B}_\gamma\cdot e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\boldsymbol{\sigma}\cdot\rho\mathbf{A}_\gamma+\mu_B\mathbf{B}_\gamma\cdot\rho\mathbf{A}_\gamma\boldsymbol{\sigma}\cdot e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right) \\ -\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\rho\mathbf{A}_\gamma+i\mu_B\mathbf{B}_\gamma\cdot\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times\rho\mathbf{A}_\gamma\right) \\ +E\left(\boldsymbol{\sigma}\cdot\nabla\rho\mathbf{A}_\gamma\cdot\mu_B\mathbf{B}_\gamma-i\mu_B\mathbf{B}_\gamma\cdot\rho\mathbf{B}_\gamma\right) \\ +\boldsymbol{\sigma}\cdot\nabla i\mu_B\mathbf{E}_\gamma\cdot\mu_B\mathbf{B}_\gamma-\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}4\pi i\mu_B\rho_{em\,\gamma}-\mu_B\mathbf{B}_\gamma\cdot\mu_B\dot{\mathbf{B}}_\gamma/c \end{pmatrix} \\
& -2i\hbar c\begin{pmatrix} 2Ei\underline{\boldsymbol{\sigma}\cdot\rho\mathbf{B}_\gamma}\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+2\left(4\pi i\mu_B\rho_{em\,\gamma}+\underline{\boldsymbol{\sigma}\cdot\mu_B\dot{\mathbf{B}}_\gamma/c}\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p} \\ +\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\times\rho\mathbf{A}_\gamma\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p} \\ +\rho\mathbf{A}_\gamma\cdot e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\left(\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times c\mathbf{p}\right)\right) \end{pmatrix} \\
& -2i\hbar c\left(\rho\mathbf{A}_\gamma\cdot e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\left(i\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(i\mu_B\mathbf{E}_\gamma\times\rho\mathbf{A}_\gamma\right)\right)\right) \\
& -2\hbar^2c^2\left(3i\underline{\boldsymbol{\sigma}\cdot\rho\mathbf{B}_\gamma}\rho\mathbf{A}_\gamma\cdot e\left(\mathbf{E}+\dot{\mathbf{A}}/c\right)\right)
\end{aligned}
\quad . \tag{C.19}
$$

To reach the above from (C.11), a few terms combine, which is responsible for the coefficients 2 in the top line of the fourth group and 3 inside the sixth group. In the very top group, it should be noted that $\nabla\cdot\mathbf{A}_\gamma\cdot\mathbf{pp}=\nabla^i A_\gamma^j p^j p^i$ and $\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{A}_\gamma\cdot\mathbf{pp}\right)=\varepsilon^{ijk}\sigma^i\nabla^j A_\gamma^l p^l p^k$ are outer products, which is why we can neither set $\nabla\cdot\mathbf{A}_\gamma=0$ nor $\nabla\times\mathbf{A}_\gamma=\mathbf{B}_\gamma$ in those terms.

The next step in reducing the Hamiltonian is to convert every object in (C.18) and (C.19) containing a time derivative or a space gradient into energy-momentum space. It is best to first distribute all the $\hbar c$ factors to each of the terms inside the large parentheses. Then, from (15.4), (15.7) and (15.15) for the photon $\mathbf{A}_\gamma$, $\mathbf{B}_\gamma$ and $\mathbf{E}_\gamma$, and from (16.14) for the classical external counterparts **A**, **B** and **E**, converting to energy space means that for all the time derivatives designated by over-dots we substitute $\partial_t = -i\omega$, and converting to momentum space means that for all space gradients (which only appear in (C.19)) we substitute $i\hbar\nabla = -\mathbf{q}$. Once this is done, we have multiple $i\omega$ which are effectively time derivatives, that may be freely commuted to wherever we would like inside each discrete term, as well as several $\mathbf{q}$ which, as reviewed after (15.15), may also be freely commuted. So for any $i\omega$ from a classical external $\dot{\mathbf{A}} = -i\omega\mathbf{A}$ grouped in the same term as a photon $\mathbf{A}_\gamma$, we move the $i\omega$ left of the $\mathbf{A}_\gamma$ then apply $i\omega\mathbf{A}_\gamma = c\mathbf{E}_\gamma$ from (15.12). In effect, this converts any paired $\mathbf{A}_\gamma\dot{\mathbf{A}} = -i\omega\mathbf{A}_\gamma\mathbf{A} = -c\mathbf{E}_\gamma\mathbf{A}$, including scalar and cross products of these. In the process, we simultaneously reveal multiple additional photon electric fields $\mathbf{E}_\gamma$ paired with the same number of classical external potentials $\mathbf{A}$. Amongst these $\mathbf{q}$, for any which appear in an outer product term, wherever possible given the requirement to keep all **p** on the very right, we commute **q** to form a scalar (inner) or cross product.

Then, keeping in mind that we defined the substitute variable $\rho \equiv q/mc^2$ prior to (14.1) and that have been considering the circumstance where $q=-e$ is the quantized charge of a charged lepton and *m* is the lepton rest mass, we introduce the Bohr magneton by setting $\hbar c\rho = -2\mu_B$ and $\hbar ce = 2mc^2\mu_B$. In those circumstance where the latter creates the term combination $mc^2\rho$, we further set $mc^2\rho = -e$. This eliminates most, but not all, of the ρ from (C.18) and (C.19).

Finally, we consolidate terms. For (C.18) following consolidation we reorganize the **Ia** and **Ib** groupings into a separation of the terms containing only dots and no cross products, from terms containing cross products, as shown below:

$$\begin{aligned}\mathbf{I} = &\left.\begin{array}{l}\left(E+mc^2-\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma\right)\left(c\mathbf{p}\cdot c\mathbf{p}+2E\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+2i\mu_B\mathbf{E}_\gamma\cdot c\mathbf{p}+2E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+i\mu_B\mathbf{E}_\gamma\cdot i\mu_B\mathbf{E}_\gamma\right)\\ +\left(2\rho\mathbf{A}_\gamma\cdot c\mathbf{p}+4\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-4\rho\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma\right)c\mathbf{p}\cdot c\mathbf{p}+\left(4E\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+2i\mu_B\mathbf{E}_\gamma\cdot i\mu_B\mathbf{E}_\gamma\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\ +\left(4E\rho\mathbf{A}_\gamma\cdot\rho\mathbf{A}_\gamma+2\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}_\gamma+2i\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma+4\rho\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma-4\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}\right)\cdot c\mathbf{p}c\mathbf{p}\\ +2\mu_B\mathbf{B}_\gamma\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma+i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\boldsymbol{\sigma}\cdot c\mathbf{p}\\ +2\boldsymbol{\sigma}\cdot\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\mu_B\mathbf{B}_\gamma\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)-2\boldsymbol{\sigma}\cdot\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\mu_B\mathbf{B}_\gamma\cdot c\mathbf{p}\end{array}\right\}\bullet\\ &\left.\begin{array}{l}-2\left(E+mc^2\right)i\boldsymbol{\sigma}\cdot\left(\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\times c\mathbf{p}\right)+2i\mu_B\mathbf{B}_\gamma\cdot\left(\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\times c\mathbf{p}\right)\\ -4\boldsymbol{\sigma}\cdot\left(\left(E\left(\rho\mathbf{A}_\gamma\times\mu_B\mathbf{E}+\rho\mathbf{A}\times\mu_B\mathbf{E}_\gamma\right)+\mu_B\mathbf{E}_\gamma\times\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\ +4\left(\rho\mathbf{A}_\gamma\cdot\mu_B\mathbf{E}-\rho\mathbf{A}\cdot\mu_B\mathbf{E}_\gamma\right)\boldsymbol{\sigma}\cdot\left(\left(E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)\times c\mathbf{p}\right)\\ +8\left(\rho\mathbf{A}_\gamma\cdot\boldsymbol{\sigma}\cdot\mu_B\mathbf{E}-\mu_B\mathbf{E}_\gamma\cdot\boldsymbol{\sigma}\cdot\rho\mathbf{A}\right)\times c\mathbf{p}c\mathbf{p}\\ -8\left(\rho\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-\rho\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma\right)\boldsymbol{\sigma}\cdot\left(\mu_B\mathbf{E}\times c\mathbf{p}\right)\\ +8\mu_B\mathbf{E}_\gamma\cdot\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}\times c\mathbf{p}\right)\end{array}\right\}\times\end{aligned} \tag{C.20}$$

For (C.19), to be able to maintain a clear connection from the six lines of terms in group **II** of (C.1) through the six large parenthetical sets in (C.11) through the same groups in (C.19), we still keep these six groups separately displayed. This will be the last time we do so, because monetarily we will reorganize these terms from their current mathematical grouping, into one which is more physically-revealing. In the second group, an emergent $\mathbf{q}\cdot\mathbf{A}_\gamma = 0$ drops out via (15.5). In the fifth group, an emergent $\mathbf{E}_\gamma \times \mathbf{E}_\gamma = 0$ drops out by identity. With all of this, (C.19) advances to:

$$
\begin{aligned}
&\left.\begin{aligned}
&\mathbf{II} = -2mc^2\left(\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\cdot c\mathbf{p}+i\boldsymbol{\sigma}\cdot\left(\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\times c\mathbf{p}\right)\right)\\
&+2\left(c\mathbf{q}\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(c\mathbf{q}\times\rho\mathbf{A}_\gamma\right)\right)\cdot c\mathbf{p}c\mathbf{p}
\end{aligned}\right\}\\
&\left.\begin{aligned}
&-\left(E+mc^2\right)\left(\left(2E+\hbar\omega\right)\underline{\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}}-4\pi\hbar c\mu_B\rho_{em\,\gamma}\right)\\
&+2mc^2\left(e\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-e\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma-i\boldsymbol{\sigma}\cdot\left(e\mathbf{A}_\gamma\times i\mu_B\mathbf{E}\right)-i\boldsymbol{\sigma}\cdot\left(e\mathbf{A}\times i\mu_B\mathbf{E}_\gamma\right)\right)\\
&-2mc^2\left(\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\cdot i\mu_B\mathbf{E}_\gamma+i\boldsymbol{\sigma}\cdot\left(\left(i\mu_B\mathbf{E}+\left(\mu_B\omega/c\right)\mathbf{A}\right)\times i\mu_B\mathbf{E}_\gamma\right)\right)
\end{aligned}\right\}\\
&\left.\begin{aligned}
&+\boldsymbol{\sigma}\cdot c\mathbf{q}\mu_B\mathbf{B}_\gamma\cdot\left(c\mathbf{p}+2E\rho\mathbf{A}_\gamma+2i\mu_B\mathbf{E}_\gamma\right)\\
&-\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}\left(c\mathbf{q}\cdot c\mathbf{p}+c\mathbf{q}\cdot i\mu_B\mathbf{E}_\gamma+2e\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-2e\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma+4\pi\hbar c\mu_B\rho_{em\,\gamma}\right)\\
&+2\mu_B\mathbf{B}_\gamma\cdot\left(e\mathbf{A}_\gamma\boldsymbol{\sigma}\cdot i\mu_B\mathbf{E}-e\mathbf{A}\boldsymbol{\sigma}\cdot i\mu_B\mathbf{E}_\gamma+i\mu_B\mathbf{E}\boldsymbol{\sigma}\cdot e\mathbf{A}_\gamma-i\mu_B\mathbf{E}_\gamma\boldsymbol{\sigma}\cdot e\mathbf{A}+e\mathbf{A}_\gamma\times\mu_B\mathbf{E}+e\mathbf{A}\times\mu_B\mathbf{E}_\gamma\right)\\
&+\left(4\pi\hbar\mu_B\mathbf{J}_\gamma-i\hbar\omega\mu_B\mathbf{E}_\gamma\right)\cdot\left(c\mathbf{p}+E\rho\mathbf{A}_\gamma+i\mu_B\mathbf{E}_\gamma\right)+\left(2E+\hbar\omega\right)\mu_B\mathbf{B}_\gamma\cdot\mu_B\mathbf{B}_\gamma
\end{aligned}\right\}\\
&\left.\begin{aligned}
&-4\left(\left(2E+\hbar\omega\right)\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}-4\pi\hbar c\mu_B\rho_{em\,\gamma}\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}\\
&+4\left(\left(2e\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-i\boldsymbol{\sigma}\cdot\left(e\mathbf{A}_\gamma\times i\mu_B\mathbf{E}\right)\right)\rho\mathbf{A}_\gamma\cdot c\mathbf{p}-e\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}i\boldsymbol{\sigma}\cdot\left(\rho\mathbf{A}_\gamma\times c\mathbf{p}\right)\right)\\
&-4\left(\left(2e\mathbf{A}_\gamma\cdot\rho\mathbf{A}-i\boldsymbol{\sigma}\cdot\left(e\mathbf{A}_\gamma\times\rho\mathbf{A}\right)\right)i\mu_B\mathbf{E}_\gamma\cdot c\mathbf{p}-e\mathbf{A}_\gamma\cdot\rho\mathbf{A}i\boldsymbol{\sigma}\cdot\left(i\mu_B\mathbf{E}_\gamma\times c\mathbf{p}\right)\right)
\end{aligned}\right\}\\
&-4e\mathbf{A}_\gamma\cdot\mu_B\mathbf{E}\left(\mu_B\mathbf{E}_\gamma\cdot\rho\mathbf{A}_\gamma+i\boldsymbol{\sigma}\cdot\left(\mu_B\mathbf{E}_\gamma\times\rho\mathbf{A}_\gamma\right)\right)+4\rho\mathbf{A}_\gamma\cdot e\mathbf{A}\mu_B\mathbf{E}_\gamma\cdot\mu_B\mathbf{E}_\gamma\}\\
&-24\underline{\boldsymbol{\sigma}\cdot\mu_B\mathbf{B}_\gamma}\left(e\mathbf{A}_\gamma\cdot i\mu_B\mathbf{E}-e\mathbf{A}\cdot i\mu_B\mathbf{E}_\gamma\right)\}
\end{aligned}
\quad . \qquad (C.21)
$$

Although (C.20) and (C.21) are now in energy-momentum space because all time and space derivatives have been removed, there are a few photon frequencies ω which we were unable to turn into a photon electric field via $\mathbf{A}_\gamma\dot{\mathbf{A}} = -i\omega\mathbf{A}_\gamma\mathbf{A} = -c\mathbf{E}_\gamma\mathbf{A}$, because there was no $\mathbf{A}_\gamma$ paired with the classical external $\dot{\mathbf{A}}$. However, via $\omega = i\partial_t$ from all of (15.4), (15.7), (15.15) and (16.14), these can shuttle anywhere within their term and be turned into the time derivative of any field in that term. In those terms where there is an ω and a $\mathbf{B}_\gamma$ or $\mathbf{E}_\gamma$, we use $\omega\mathbf{B}_\gamma = i\dot{\mathbf{B}}_\gamma$ or $\omega\mathbf{E}_\gamma = i\dot{\mathbf{E}}_\gamma$. For the few remaining terms with neither, we reabsorb this into $\omega\mathbf{A} = i\mathbf{A}\cdot$ As a result, we still remain entirely in momentum space, but we do allow some time derivatives back into (C.20) and (C.21) and so convert out of a pure *energy*-momentum space. This is because it is important and has physical imminence, because it enables us to study the Hamiltonian energies when time-dependent fields are applied.

Finally, as just noted, (C.20) and (C.21) are still mathematical groupings. Now, we combine (C.20) and (C.21) into a single expression for the Hamiltonian numerator in (20.8), and segregate terms according to their dominant physical parameters, organized by the fields and field relations which each term contains. The result of all of this is the multi-term expression which is (20.11) of the main paper.

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