How to Effect a Composite Rotation of a Vector via Geometric (Clifford) Algebra

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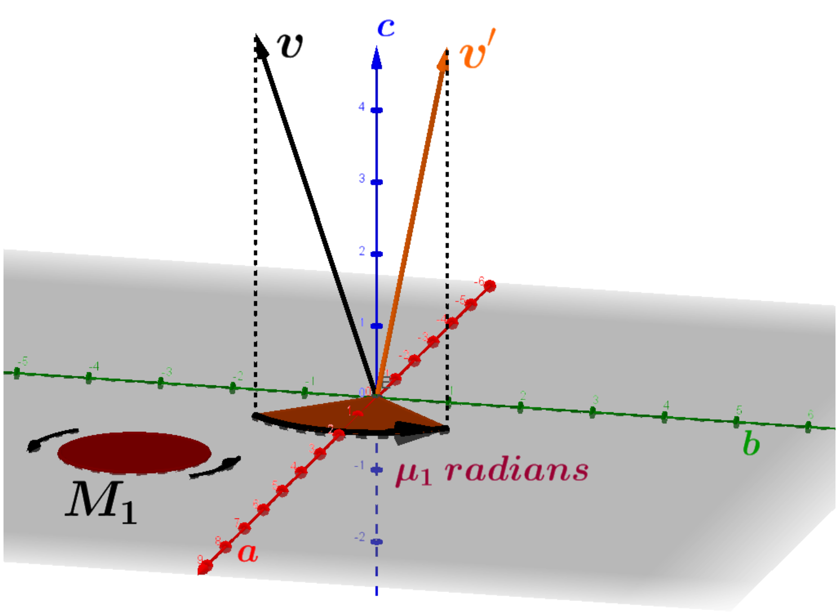
Abstract

We show how to express the representation of a composite rotation in terms that allow the rotation of a vector to be calculated conveniently via a spreadsheet that uses formulas developed, previously, for a single rotation. The work presented here (which includes a sample calculation) also shows how to determine the bivector angle that produces, in a single operation, the same rotation that is effected by the composite of two rotations.



"Rotation of the vector $\mathbf{v}$ *through the bivector angle* $\mathbf{M}_1\mu_1$*, to produce the vector* $\mathbf{v}'$*."*

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1 Introduction

Suppose that we rotate some vector $\mathbf{v}$ through the bivector angle $\mathbf{M}_1\mu_1$ to produce the vector that we shall call $\mathbf{v}'$ (Fig. 1), and that we then rotate $\mathbf{v}'$ through the bivector angle $\mathbf{M}_2\mu_2$ to produce the vector that we shall call $\mathbf{v}''$. That sequence of rotations is called the *composition* of the two rotations. It is equal to the rotation through some bivector angle $\mathbf{S}\sigma$ ([1], pp. 89-91). Geometric Algebra (GA) is a convenient and efficient tool for manipulating rotations—single as well as composite—as abstract symbols, but what form does a numerical calculation of a rotation take in a concrete situation? And how can we calculate the bivector angle $\mathbf{S}\sigma$?

Those are two of the questions that we will address in this document. Our procedure will make use of single-rotation formulas that were developed in [2]. We'll begin with a review of how a given vector can be rotated via GA. In that review, we'll discuss the important concept of the *representation* of a rotation, after which we'll present an formula that can be implemented in Excel for to calculate single rotations of a given vector.

Having finished that review, we'll see how to express the representation of a composite rotation in terms that can be substituted directly in the formula for single rotations. We'll then work a sample problem in which we'll calculate the results of successive rotations of a vector. We'll also calculate the bivector angle that produces the same rotation in a single operation. The method used for calculating that bivector angle is presented in the Appendix.

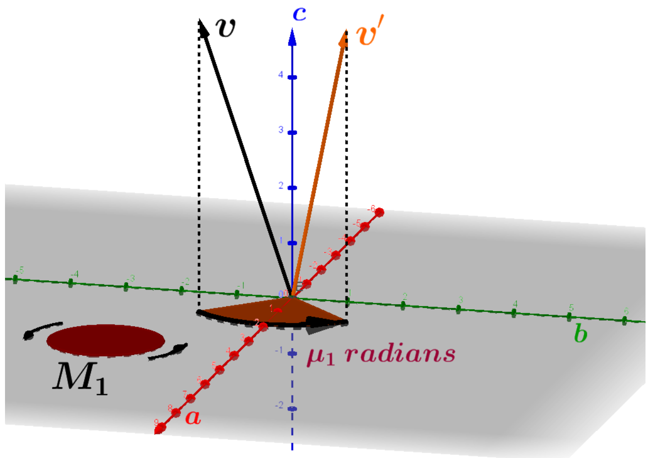


Figure 1: Rotation of the vector $\mathbf{v}$ through the bivector angle $\mathbf{M}_2\mu_2$, to produce the vector $\mathbf{v}'$.

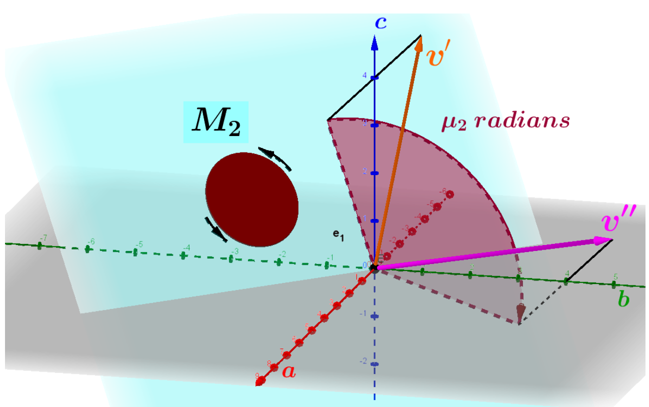


Figure 2: Rotation of the vector $\mathbf{v}'$ through the bivector angle $\mathbf{M}_2\mu_2$, to produce the vector $\mathbf{v}''$.

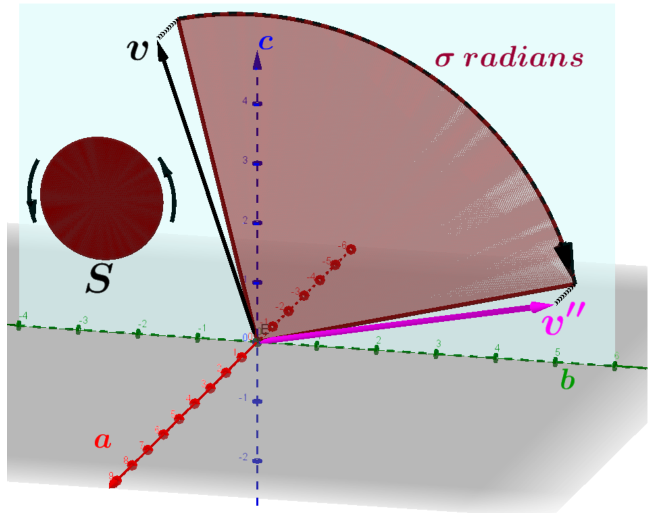


Figure 3: Rotation of **v** through the bivector angle $\mathbf{S}\sigma$, to produce the vector $\mathbf{v}''$ in a single operation.

2 A Brief Review of How a Rotation of a Given Vector Can be Effected via GA

References [3] (pp. 280-286) and [1] (pp. 89-91) derive and explain the following formula for finding the new vector, $\mathbf{w'}$, that results from the rotation of a vector **w** through the angle θ with respect to a plane that is parallel to the unit bivector **Q**:

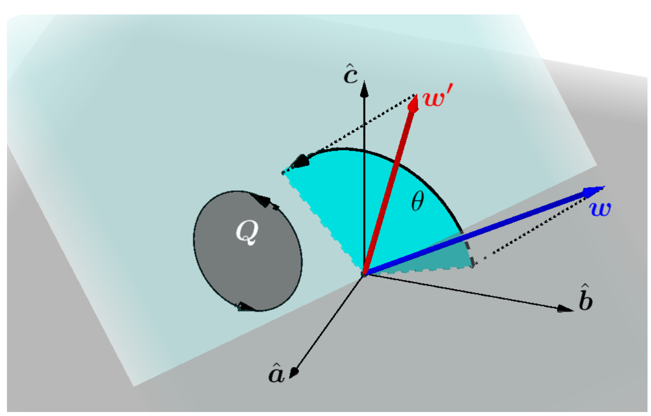


Figure 4: Rotation of the vector **w** through the bivector angle $\mathbf{Q}_1$, to produce the vector $\mathbf{w}'$.

$$\mathbf{w}' = \underbrace{\left[e^{-\mathbf{Q}\theta/2}\right][\mathbf{w}]\left[e^{\mathbf{Q}\theta/2}\right]}_{\text{Notation: } \mathsf{R}_{\mathbf{Q}\theta}(\mathbf{w})}. \tag{2.1}$$

Notation: $\mathsf{R}_{\mathbf{Q}\theta}(\mathbf{w})$ is the rotation of the vector $\mathbf{w}$ by the bivector angle $\mathbf{Q}\theta$.

For our convenience later in this document, we will follow Reference [1] (p. 89) in saying that the factor $e^{-\mathbf{Q}\theta/2}$ *represents* the rotation $\mathsf{R}_{\mathbf{Q}\theta}$. That factor is a quaternion, but in GA terms it is a multivector:

$$e^{-\mathbf{Q}\theta/2} = \cos\frac{\theta}{2} - \mathbf{Q}\sin\frac{\theta}{2}. \tag{2.2}$$

As further preparation for work that we'll do later, we'll mention that for any given right-handed reference system with orthonormal basis vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, we may express the unit bivector $\mathbf{Q}$ as a linear combination of the basis bivectors $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$:

$$\mathbf{Q} = \hat{\mathbf{a}}\hat{\mathbf{b}}q_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}q_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}q_{ac},$$

in which q_{ab}, q_{bc}, and q_{ac} are scalars, and $q_{ab}^2 + q_{bc}^2 + q_{ac}^2 = 1$.

To present a convenient way of calculating rotations via Excel spreadsheets, Ref. [2] built upon that idea to write $e^{-\mathbf{Q}\theta/2}$ as

$$e^{-\mathbf{Q}\theta/2} = f_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right), \tag{2.3}$$

with $f_o = \cos\frac{\theta}{2}$; $f_{ab} = q_{ab}\sin\frac{\theta}{2}$; $f_{bc} = q_{bc}\sin\frac{\theta}{2}$; and $f_{ac} = q_{ac}\sin\frac{\theta}{2}$. Similarly,

$$e^{\mathbf{Q}\theta/2} = f_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right). \tag{2.4}$$

Using these expressions for $e^{-\mathbf{Q}\theta/2}$ and $e^{\mathbf{Q}\theta/2}$, and writing $\mathbf{w}$ as $\mathbf{w} = \hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c$, Eq. (2.1) becomes

$$\mathbf{w}' = \left[f_o - \hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} - \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} - \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right]\left[\hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right]\left[f_o + \hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right].$$

By expanding and simplifying the right-hand, side we obtain

$$\begin{aligned}\mathbf{w}' = \hat{\mathbf{a}}&\left[w_a\left(f_o^2 - f_{ab}^2 + f_{bc}^2 - f_{ac}^2\right) + w_b\left(-2f_of_{ab} - 2f_{bc}f_{ac}\right) + w_c\left(-2f_of_{ac} + 2f_{ab}f_{bc}\right)\right] \\ +\, \hat{\mathbf{b}}&\left[w_a\left(2f_of_{ab} - 2f_{bc}f_{ac}\right) + w_b\left(f_o^2 - f_{ab}^2 - f_{bc}^2 + f_{ac}^2\right) + w_c\left(-2f_of_{bc} - 2f_{ab}f_{ac}\right)\right] \\ +\, \hat{\mathbf{c}}&\left[w_a\left(2f_of_{ac} + 2f_{ab}f_{bc}\right) + w_b\left(2f_of_{bc} - 2f_{ab}f_{ac}\right) + w_c\left(f_o^2 + f_{ab}^2 - f_{bc}^2 - f_{ac}^2\right)\right].\end{aligned} \tag{2.5}$$

Because this result can be implemented conveniently in (for example) a spreadsheet similar to Ref. [4], the sections that follow will show how to express the representation of a composite rotation in the form of Eq. (2.3).

3 Identifying the "Representation" of a Composite Rotation

Let's begin by defining two unit bivectors, $\mathbf{M}_1$ and $\mathbf{M}_2$:

$$\mathbf{M}_1 = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{1ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{1bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{1ac};$$
$$\mathbf{M}_2 = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{2ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{2bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{2ac}.$$

Now, write the rotation of a vector $\mathbf{v}$ by the bivector angle $\mathbf{M}_1\mu_1$ to produce the vector $\mathbf{v}'$:

$$\mathbf{v}' = \left[e^{-\mathbf{M}_1\mu_1/2}\right][\mathbf{v}]\left[e^{\mathbf{M}_1\mu_1/2}\right].$$

Next, we will rotate $\mathbf{v}'$ by the bivector angle $\mathbf{M}_2\mu_2$ to produce the vector $\mathbf{v}''$:

$$\mathbf{v}'' = \left[e^{-\mathbf{M}_2\mu_2/2}\right][\mathbf{v}']\left[e^{\mathbf{M}_2\mu_2/2}\right].$$

Combining those two equations,

$$\mathbf{v}'' = \left[e^{-\mathbf{M}_2\mu_2/2}\right]\left\{\left[e^{-\mathbf{M}_1\mu_1/2}\right][\mathbf{v}]\left[e^{\mathbf{M}_1\mu_1/2}\right]\right\}\left[e^{\mathbf{M}_2\mu_2/2}\right].$$

The vector $\mathbf{v}''$ was produced from $\mathbf{v}$ via the composition of the rotations by the bivector angles $\mathbf{M}_1\mu_1$ and $\mathbf{M}_2\mu_1$. The representation of that composition is the product $\left[e^{-\mathbf{M}_2\mu_1/2}\right]\left[e^{-\mathbf{M}_1\mu_1/2}\right]$. We'll rewrite the previous equation to make that idea clearer:

$$\mathbf{v}'' = \underbrace{\left\{\left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[e^{-\mathbf{M}_1\mu_1/2}\right]\right\}}_{\substack{\textit{Representation}\\ \textit{of the composition}}}[\mathbf{v}]\left\{\left[e^{\mathbf{M}_1\mu_1/2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right]\right\}.$$

There exists an identifiable bivector angle —we'll call it $\mathbf{S}\sigma$—through which $\mathbf{v}$ could have been rotated to produce $\mathbf{v}''$ in a single operation rather than through the composition of rotations through $\mathbf{M}_1\mu_1$ and $\mathbf{M}_2\mu_2$. (See the Appendix.) But instead of going that route, let's write $e^{-\mathbf{M}_1\mu_1/2}$ and $e^{-\mathbf{M}_2\mu_2/2}$ in a way that will enable us to use Eq. (2.3):

$$\begin{aligned} e^{-\mathbf{M}_1\mu_1/2} &= g_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}g_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}g_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}g_{ac}\right), \text{and} \\ e^{-\mathbf{M}_2\mu_2/2} &= h_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}h_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}h_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}h_{ac}\right), \end{aligned}$$

where $g_o = \cos\frac{\mu_1}{2}$; $g_{ab} = m_{1ab}\sin\frac{\mu_1}{2}$; $g_{bc} = m_{1bc}\sin\frac{\mu_1}{2}$; and $g_{ac} = m_{1ac}\sin\frac{\mu_1}{2}$, and $h_o = \cos\frac{\mu_2}{2}$; $h_{ab} = m_{2ab}\sin\frac{\mu_2}{2}$; $h_{bc} = m_{2bc}\sin\frac{\mu_2}{2}$; and $h_{ac} = m_{2ac}\sin\frac{\mu_2}{2}$. Now, we write the representation of the the composition as

$$\underbrace{\left[h_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}h_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}h_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}h_{ac}\right)\right]}_{e^{-\mathbf{M}_2\mu_2/2}}\underbrace{\left[g_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}g_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}g_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}g_{ac}\right)\right]}_{e^{-\mathbf{M}_1\mu_1/2}}.$$

After expanding that product and grouping like terms, the representation of the composite rotation can be written in a form identical to Eq. (2.3):

$$\mathcal{F}_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right), \tag{3.1}$$

with

$$\begin{aligned} \mathcal{F}_o &= \langle e^{-\mathbf{M}_2\mu_2/2}e^{-\mathbf{M}_1\mu_1/2}\rangle_0 \\ &= h_o g_o - h_{ab}g_{ab} - h_{bc}g_{bc} - h_{ac}g_{ac}, \\ \mathcal{F}_{ab} &= h_o g_{ab} + h_{ab}g_o - h_{bc}g_{ac} + h_{ac}g_{bc}, \\ \mathcal{F}_{bc} &= h_o g_{bc} + h_{ab}g_{ac} + h_{bc}g_o - h_{ac}g_{ab}, \text{ and} \\ \mathcal{F}_{ac} &= h_o g_{ac} - h_{ab}g_{bc} + h_{bc}g_{ab} + h_{ac}g_o. \end{aligned} \tag{3.2}$$

Therefore, with these definitions of $\mathcal{F}_o$, $\mathcal{F}_{ab}$, $\mathcal{F}_{bc}$, and $\mathcal{F}_{ac}$, $\mathbf{v}''$ can be calculated from $\mathbf{v}$ (written as $\hat{\mathbf{a}}v_a + \hat{\mathbf{b}}v_b + \hat{\mathbf{c}}v_c$) via an equation that is analogous, term for term, with Eq. (2.5):

$$\begin{aligned}\mathbf{v}' = \hat{\mathbf{a}}&\left[v_a\left(\mathcal{F}_o^2 - \mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 - \mathcal{F}_{ac}^2\right) + v_b\left(-2\mathcal{F}_o\mathcal{F}_{ab} - 2\mathcal{F}_{bc}\mathcal{F}_{ac}\right) + v_c\left(-2\mathcal{F}_o\mathcal{F}_{ac} + 2\mathcal{F}_{ab}\mathcal{F}_{bc}\right)\right]\\ +\hat{\mathbf{b}}&\left[v_a\left(2\mathcal{F}_o\mathcal{F}_{ab} - 2\mathcal{F}_{bc}\mathcal{F}_{ac}\right) + v_b\left(\mathcal{F}_o^2 - \mathcal{F}_{ab}^2 - \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2\right) + v_c\left(-2\mathcal{F}_o\mathcal{F}_{bc} - 2\mathcal{F}_{ab}\mathcal{F}_{ac}\right)\right]\\ +\hat{\mathbf{c}}&\left[v_a\left(2\mathcal{F}_o\mathcal{F}_{ac} + 2\mathcal{F}_{ab}\mathcal{F}_{bc}\right) + v_b\left(2\mathcal{F}_o\mathcal{F}_{bc} - 2\mathcal{F}_{ab}\mathcal{F}_{ac}\right) + v_c\left(\mathcal{F}_o^2 + \mathcal{F}_{ab}^2 - \mathcal{F}_{bc}^2 - \mathcal{F}_{ac}^2\right)\right] .\end{aligned} \tag{3.3}$$

At this point, you may (and should) be objecting that I've gotten ahead of myself. Please recall that Eq. (2.5) was derived starting from the "rotation" equation ((2.1))

$$\mathbf{w}' = \left[e^{-\mathbf{Q}\theta/2}\right][\mathbf{w}]\left[e^{\mathbf{Q}\theta/2}\right] .$$

The quantities f_o, f_o, f_{ab}, f_{bc}, and f_{ac} in Eq. (2.5), for which

$$e^{-\mathbf{Q}\theta/2} = f_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right), \tag{3.4}$$

also meet the condition that

$$e^{\mathbf{Q}\theta/2} = f_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right). \tag{3.5}$$

We are not justified in using $\mathcal{F}_o$, $\mathcal{F}_{ab}$, $\mathcal{F}_{bc}$, and $\mathcal{F}_{ac}$ in Eq. (2.5) unless we first prove that these composite-rotation "$\mathcal{F}$'s", for which

$$\mathcal{F}_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right) = e^{-\mathbf{M}_2\mu_2/2}e^{-\mathbf{M}_1\mu_1/2} , \tag{3.6}$$

also meet the condition that

$$\mathcal{F}_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right) = e^{\mathbf{M}_1\mu_1/2}e^{-\mathbf{M}_2\mu_2/2} . \tag{3.7}$$

Although more-elegant proofs may well exist, "brute force and ignorance" gets the job done. We begin by writing $e^{\mathbf{M}_1\mu_1/2}e^{-\mathbf{M}_2\mu_2/2}$ in a way that is analogous to that which was presented in the text that preceded Eq. (3.1):

$$\underbrace{\left[g_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}g_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}g_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}g_{ac}\right)\right]}_{e^{\mathbf{M}_1\mu_1/2}}\underbrace{\left[h_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}h_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}h_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}h_{ac}\right)\right]}_{e^{\mathbf{M}_2\mu_2/2}} .$$

Expanding, simplifying, and regrouping, we fine that $e^{\mathbf{M}_1\mu_1/2}e^{-\mathbf{M}_2\mu_2/2}$ is indeed equal to $\mathcal{F}_o + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right)$, as required.

4 A Sample Calculation

The vector $\mathbf{v} = \frac{4}{3}\hat{\mathbf{a}} - \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$ is rotated through the bivector angle $\hat{\mathbf{a}}\hat{\mathbf{b}}\pi/2$ radians to produce a new vector, $\mathbf{v}'$. That vector is then rotated through the bivector angle $\left(\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}}{\sqrt{3}} + \frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{3}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{3}}\right)\left(-\frac{2\pi}{3}\right)$ to produce vector $\mathbf{v}''$. Calculate

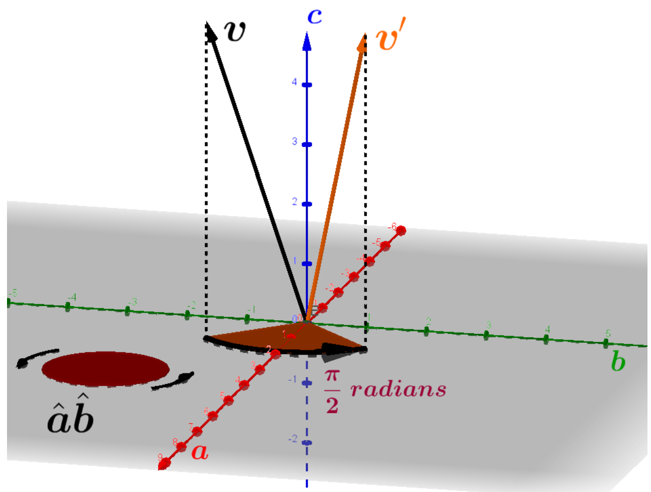


Figure 5: Rotation of $\mathbf{v}$ through the bivector angle $\hat{\mathbf{a}}\hat{\mathbf{b}}\pi/2$, to produce the vector $\mathbf{v}'$.

a The vectors $\mathbf{v}'$ and $\mathbf{v}''$, and

b The bivector angle $\mathbf{S}\sigma$ through which $\mathbf{v}$ could have been rotated to produce $\mathbf{v}''$ in a single operation.

We begin by calculating vector $\mathbf{v}'$. The rotation is diagrammed in Fig. 5

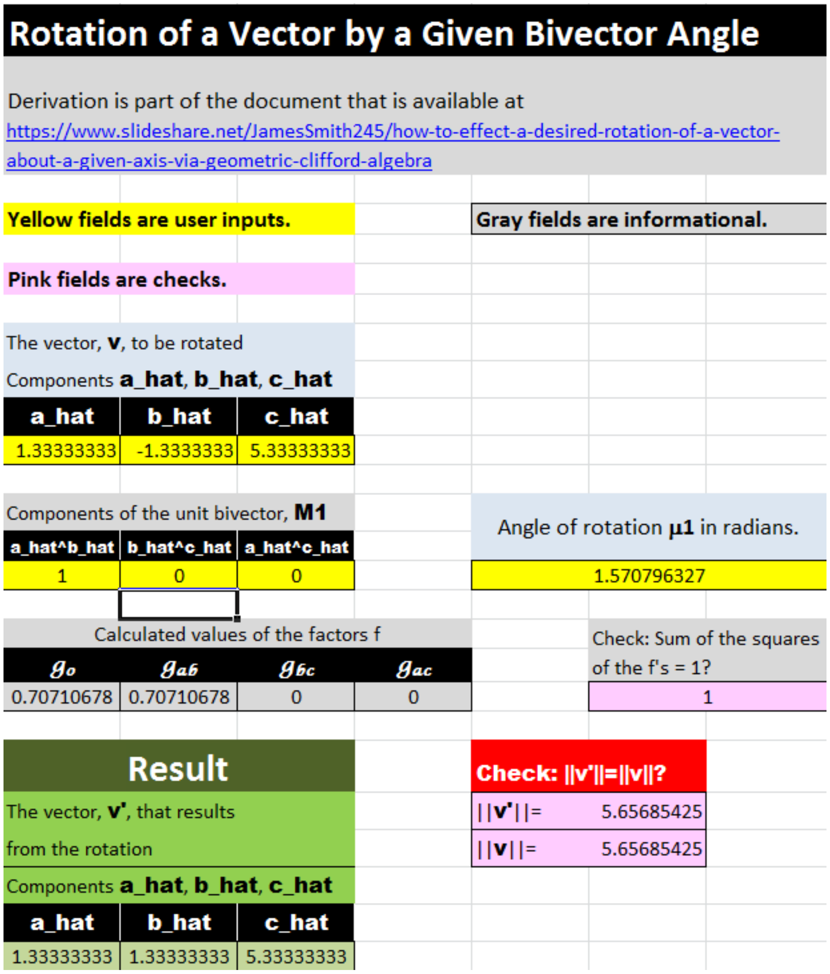
As shown in Fig. 6, $\mathbf{v}' = \frac{4}{3}\hat{\mathbf{a}} + \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$.

We'll calculate $\mathbf{v}''$ in two ways: as the rotation of $\mathbf{v}'$ by the bivector angle $\left(\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}}{\sqrt{3}} + \frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{3}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{3}}\right)\left(-\frac{2\pi}{3}\right)$, and as the result of the rotation by the composite of the two individual rotations. The rotation of $\mathbf{v}'$ by $\left(\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}}{\sqrt{3}} + \frac{\hat{\mathbf{b}}\hat{\mathbf{c}}}{\sqrt{3}} - \frac{\hat{\mathbf{a}}\hat{\mathbf{c}}}{\sqrt{3}}\right)\left(-\frac{2\pi}{3}\right)$ is diagrammed in Fig. 7. Fig. 8 shows that $\mathbf{v}'' = \frac{4}{3}\hat{\mathbf{a}} + \frac{16}{3}\hat{\mathbf{b}} + \frac{4}{3}\hat{\mathbf{c}}$.

As we can see from Fig. 9, that result agrees with that which was obtained by calculating $\mathbf{v}''$ in a single step, as the composition of the individual rotations. Fig. 9 also shows that the bivector angle $\mathbf{S}\sigma$ is $\hat{\mathbf{b}}\hat{\mathbf{c}}\,(-\pi/2)$, which we can also write as $\hat{\mathbf{c}}\hat{\mathbf{b}}\,(\pi/2)$. That rotation is diagrammed in Fig. 10.

5 Summary

We have seen how to express the representation of a composite rotation in terms that allow the rotation of a vector to be calculated conveniently via a spreadsheet that used formulas developed in [2] for a single rotation. The work presented here also shows how to determine the bivector angle that produces, in a single operation, the same rotation that is effected by the composite of two rotations.



Rotation of a Vector by a Given Bivector Angle

Derivation is part of the document that is available at
https://www.slideshare.net/JamesSmith245/how-to-effect-a-desired-rotation-of-a-vector-about-a-given-axis-via-geometric-clifford-algebra

Yellow fields are user inputs.

Gray fields are informational.

Pink fields are checks.

The vector, **v**, to be rotated

Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.33333333	-1.3333333	5.33333333

Components of the unit bivector, **M1**

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
1	0	0

Angle of rotation **μ1** in radians.

1.570796327

Calculated values of the factors f

g_o	g_{ab}	g_{bc}	g_{ac}
0.70710678	0.70710678	0	0

Check: Sum of the squares of the f's = 1?

1

Result

The vector, **v'**, that results from the rotation

Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.33333333	1.33333333	5.33333333

Check: ||v'||=||v||?

||**v'**||=	5.65685425
||**v**||=	5.65685425

Figure 6: A spreadsheet (Ref. [5]) that uses Eq. (2.5) to calculate $\mathbf{v}'$ as the rotation of $\mathbf{v}$ through the bivector angle $\hat{\mathbf{a}}\hat{\mathbf{b}}\pi/2$.

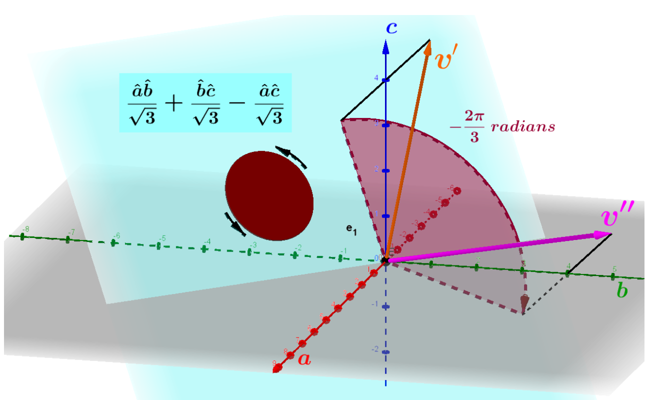
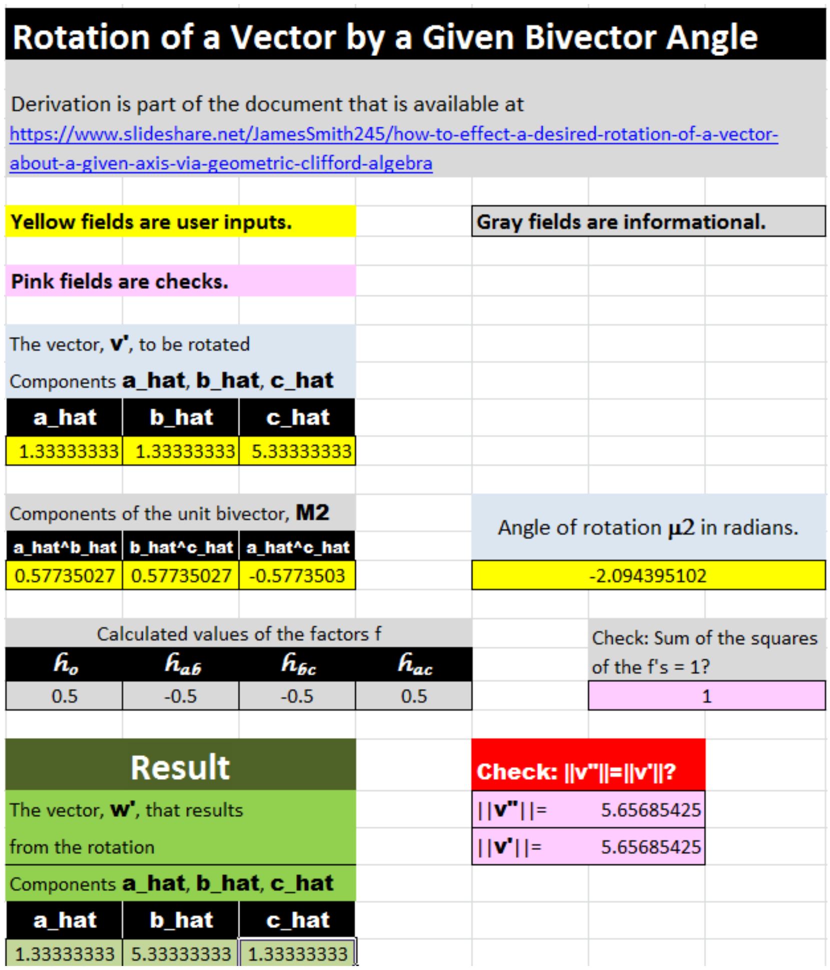


Figure 7: Rotation of $\mathbf{v}'$. Note the significance of the negative sign of the scalar angle: the direction in which $\mathbf{v}'$ is to be rotated is contrary to the orientation of the bivector. That significance is clearer in Fig. 10.



Rotation of a Vector by a Given Bivector Angle

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The vector, **v'**, to be rotated

Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.33333333	1.33333333	5.33333333

Components of the unit bivector, **M2**

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
0.57735027	0.57735027	-0.5773503

Angle of rotation **μ2** in radians.

-2.094395102

Calculated values of the factors f

f_o	f_{ab}	f_{bc}	f_{ac}
0.5	-0.5	-0.5	0.5

Check: Sum of the squares of the f's = 1?

1

Result

The vector, **w'**, that results from the rotation

Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.33333333	5.33333333	1.33333333

Check: ||v"||=||v'||?

||**v"**||=	5.65685425
||**v'**||=	5.65685425

Figure 8: A spreadsheet (Ref. [5]) that uses Eq. (2.5) to calculate $\mathbf{v}''$ as the rotation of $\mathbf{v}'$.

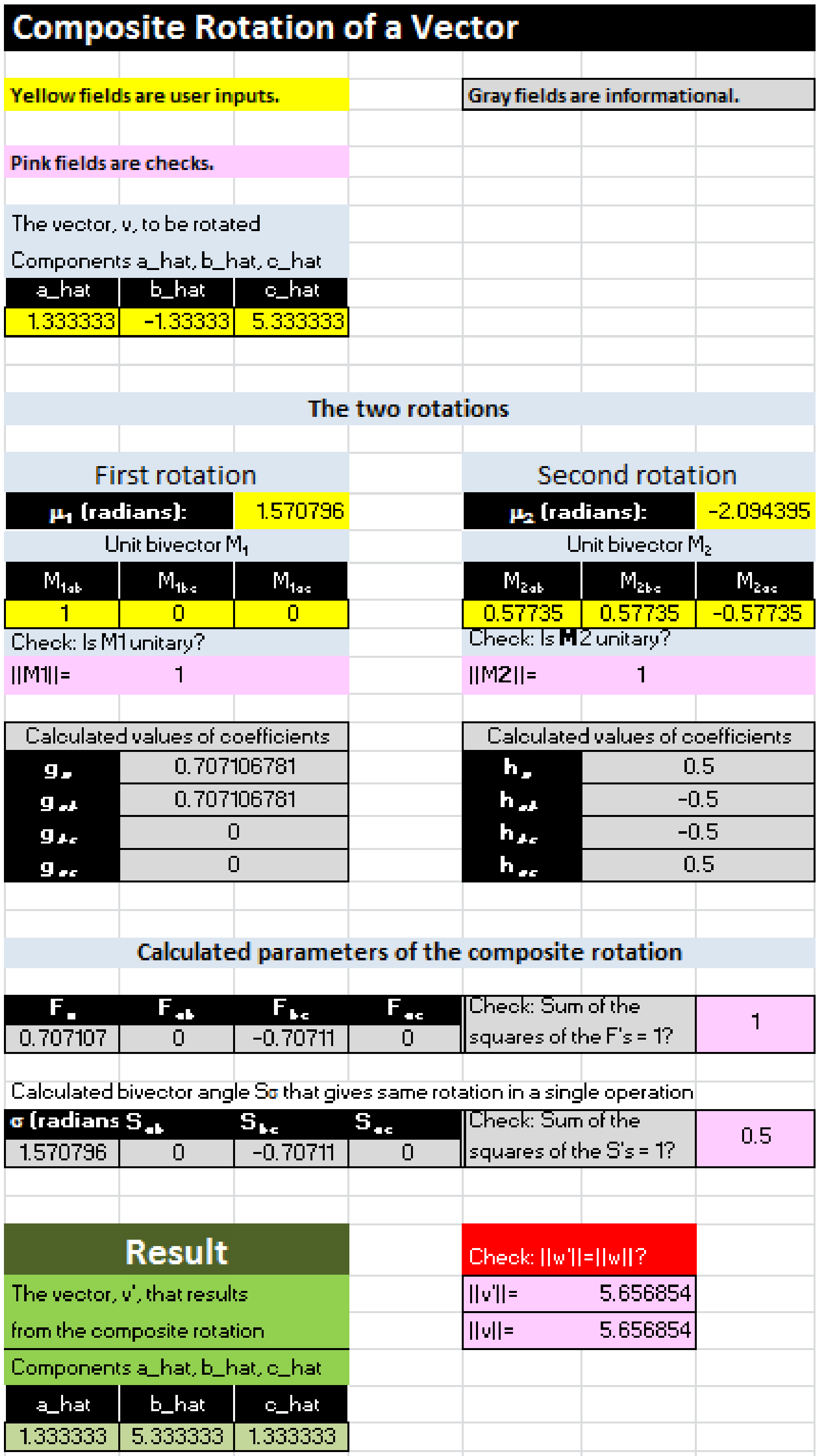

Figure 9: A spreadsheet (Ref. [6]) that uses Eq. (3.2) to calculate $\mathbf{v}''$ via the composite rotation of $\mathbf{v}$.

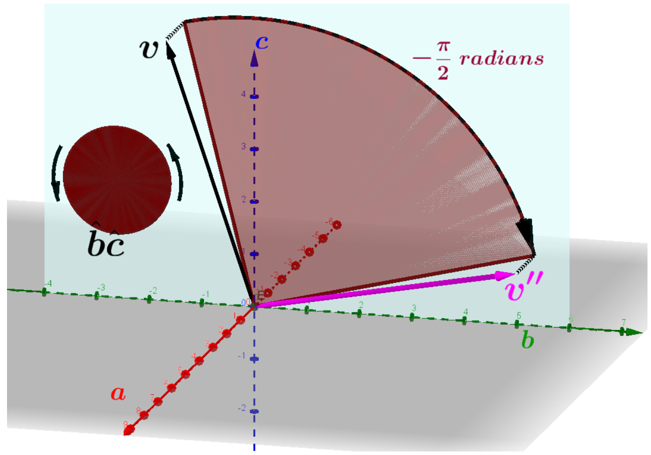

Figure 10: Rotation of $\mathbf{v}$ by $\mathbf{S}\sigma$ to produce $\mathbf{v}''$ in a single operation. Note the significance of the negative sign of the scalar angle: the direction in which $\mathbf{v}'$ rotated is contrary to the orientation of the bivector $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and contrary also to the direction of the rotation from $\hat{\mathbf{b}}$ to $\hat{\mathbf{c}}$.

6 Appendix: Identifying the Bivector Angle $\mathbf{S}\sigma$ through which the Vector v can be Rotated to Produce $\mathbf{v}''$ in a Single Operation

Let $\mathbf{v}$ be an arbitrary vector. We want to identify the bivector angle $\mathbf{S}\sigma$ through which the initial vector, $\mathbf{v}$, can be rotated to produce the same vector $\mathbf{v}''$ that results from the rotation of $\mathbf{v}$ through the composite rotation by $\mathbf{M}_1\mu_1$, then by $\mathbf{M}_2\mu_2$:

$$\left[e^{-\mathbf{M}_2\mu_2/2}\right]\left[e^{-\mathbf{M}_1\mu_1/2}\right][\mathbf{v}]\left[e^{\mathbf{M}_1\mu_1/2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right] = \mathbf{v}'' = \left[e^{-\mathbf{S}\sigma/2}\right][\mathbf{v}]\left[e^{\mathbf{S}\sigma/2}\right]. \quad (6.1)$$

We want Eq. (6.1) to be true for all vectors $\mathbf{v}$. Therefore, $e^{\mathbf{S}\sigma/2}$ must be equal to $\left[e^{\mathbf{M}_1\mu_1/2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right]$, and $e^{-\mathbf{S}\sigma/2}$ must be equal to $\left[e^{-\mathbf{M}_1\mu_1/2}\right]\left[e^{\mathbf{M}_2\mu_2/2}\right]$. The second of those conditions is the same as saying that the representations of the $\mathbf{S}\sigma$ rotation and the composite rotation must be equal. We'll write that condition using the $\mathcal{F}_o$'s defined in Eq. (3.2), with $\mathbf{S}$ expressed in terms of the unit bivectors $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$:

$$\cos\frac{\sigma}{2} - \underbrace{\left(\hat{\mathbf{a}}\hat{\mathbf{b}}S_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}S_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}S_{ac}\right)}_{\mathbf{S}}\sin\frac{\sigma}{2} = \mathcal{F}_o - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\right).$$

Now, we want to identify σ and the coefficients of $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$. First, we note that both sides of the previous equation are multivectors. According to the postulates of GA, two multivectors $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal if and only if for every grade k, $\langle\mathcal{A}_1\rangle_k = \langle\mathcal{A}_2\rangle_k$. Equating the scalar parts, we see that $\cos\frac{\sigma}{2} = \mathcal{F}_o$. Equating the bivector parts gives $\left(\hat{\mathbf{a}}\hat{\mathbf{b}}S_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}S_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}S_{ac}\right)\sin\frac{\sigma}{2} = \hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} +$

$\hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}$. Comparing like terms, $S_{ab} = \mathcal{F}_{ab}/\sin\frac{\sigma}{2}$, $S_{bc} = \mathcal{F}_{bc}/\sin\frac{\sigma}{2}$, and $S_{ac} = \mathcal{F}_{ac}/\sin\frac{\sigma}{2}$.

Why is it correct to identify the $\mathcal{S}$'s by comparing like terms? In simple terms, because the unit bivectors $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$ $\hat{\mathbf{a}}\hat{\mathbf{b}}$ are orthogonal. Two linear combinations of those bivectors are equal if and only if the coefficients match, term for term.

Next, we need to find $\sin\frac{\sigma}{2}$. Although we could do so via $\sin\frac{\sigma}{2} = \sqrt{1-\cos^2\frac{\sigma}{2}}$, for the purposes of this discussion we will use the fact that $\mathbf{S}$ is, by definition, a unit bivector. Therefore, $||\mathbf{S}|| = 1$, leading to

$$\begin{aligned} \|\sin\frac{\sigma}{2}\| &= \|\hat{\mathbf{a}}\hat{\mathbf{b}}\mathcal{F}_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\mathcal{F}_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\mathcal{F}_{ac}\| \\ &= \sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}\,. \end{aligned}$$

Now, the question is whether we want to use $\sin\frac{\sigma}{2} = +\sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}$, or $\sin\frac{\sigma}{2} = -\sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}$. The truth is that we can use either: if we use $-\sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}$ instead of $+\sqrt{\mathcal{F}_{ab}^2 + \mathcal{F}_{bc}^2 + \mathcal{F}_{ac}^2}$, then the sign of $\mathbf{S}$ changes as well, leaving the product $\mathbf{S}\sin\frac{\sigma}{2}$ unaltered.

The choice having been made, we can find the scalar angle σ from the values of $\sin\frac{\sigma}{2}$ and $\cos\frac{\sigma}{2}$, thereby determining the bivector angle $\mathbf{S}\sigma$.

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[6] J. A. Smith, 2017, "Composite rotation in GA" (an Excel spreadsheet), https://drive.google.com/file/d/0B2C4TqxB32RRaktDZktjcExPeUE/view?usp=sharing .