A Proof of the Erdös-Straus Conjecture

Zhang Tianshu

Emails: chinazhangtianshu@126.com; xinshijizhang@hotmail.com

Zhanjiang city, Guangdong province, China

Abstract

In this article, we classify positive integers step by step, and use the formulation to represent a certain class therein until all classes.

First, divide all integers ≥2 into 8 kinds, and formulate each of 7 kinds therein into a sum of 3 unit fractions.

For the unsolved kind, again divide it into 3 genera, and formulate each of 2 genera therein into a sum of 3 unit fractions.

For the unsolved genus, further divide it into 5 sorts, and formulate each of 3 sorts therein into a sum of 3 unit fractions.

For two unsolved sorts $\frac{4}{49+120c}$ and $\frac{4}{121+120c}$ where c≥0, we use an unit fraction plus a proper fraction to replace each of them, then take out the unit fraction as $\frac{1}{X}$. After that, we take out an unit fraction from the proper fraction and regard the unit fraction as $\frac{1}{Y}$, and finally, prove that the remainder can be identically converted to $\frac{1}{Z}$.

AMS subject classification: 11D72; 11D45; 11P81

Keywords: Erdös-Straus conjecture; Diophantine equation; unit fraction

1. Introduction

The Erdös-Straus conjecture relates to Egyptian fractions. In 1948, Paul Erdös conjectured that for any integer n≥2, there are invariably $\frac{4}{\mathrm{n}}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$, where x, y and z are positive integers; [1].

Later, Ernst G. Straus further conjectured that x, y and z satisfy x≠y, y≠z and z≠x, because there are the convertible formulas $\frac{1}{2\mathrm{r}}+\frac{1}{2r}=\frac{1}{r+1}+\frac{1}{r(r+1)}$ and $\frac{1}{2r+1}+\frac{1}{2r+1}=\frac{1}{r+1}+\frac{1}{(r+1)(2r+1)}$ where r≥1; [2].

Thus, the Erdös conjecture and the Straus conjecture are equivalent from each other, and they are called the Erdös-Straus conjecture collectively.

As a general rule, the Erdös-Straus conjecture states that for every integer n≥2, there are positive integers x, y and z, such that $\frac{4}{\mathrm{n}}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$.

Yet it remains a conjecture that has neither is proved nor disproved; [3].

2. Divide integers≥2 into 8 kinds and formulate 7 kinds therein

First, divide integers≥2 into 8 kinds, i.e. 8k+1with k≥1, and 8k+2, 8k+3, 8k+4, 8k+5, 8k+6, 8k+7, 8k+8, where k≥0, and arrange them as follows:

K\n:	8k+1,	8k+2,	8k+3,	8k+4,	8k+5,	8k+6,	8k+7,	8k+8
0,	①,	2,	3,	4,	5,	6,	7,	8,
1,	9,	10,	11,	12,	13,	14,	15,	16,

2,	17,	18,	19,	20,	21,	22,	23,	24,
3,	25,	26,	27,	28,	29,	30,	31,	32,
…,	…,	…,	…,	…,	…,	…,	…,	…,

Excepting n=8k+1,formulate each of other 7 kinds into $\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$:

(1) For n=8k+2, there are $\frac{4}{8k+2}=\frac{1}{4k+1}+\frac{1}{4k+2}+\frac{1}{(4k+1)(4k+2)}$;

(2) For n=8k+3, there are $\frac{4}{8k+3}=\frac{1}{2k+2}+\frac{1}{(2k+1)(2k+2)}+\frac{1}{(2k+1)(8k+3)}$;

(3) For n=8k+4, there are $\frac{4}{8k+4}=\frac{1}{2k+3}+\frac{1}{(2k+2)(2k+3)}+\frac{1}{(2k+1)(2k+2)}$;

(4) For n=8k+5, there are $\frac{4}{8k+5}=\frac{1}{2k+2}+\frac{1}{(8k+5)(2k+2)}+\frac{1}{(8k+5)(k+1)}$;

(5) For n=8k+6, there are $\frac{4}{8k+6}=\frac{1}{4k+3}+\frac{1}{4k+4}+\frac{1}{(4k+3)(4k+4)}$;

(6) For n=8k+7, there are $\frac{4}{8k+7}=\frac{1}{2k+3}+\frac{1}{(2k+2)(2k+3)}+\frac{1}{(2k+2)(8k+7)}$;

(7) For n=8k+8, there are $\frac{4}{8k+8}=\frac{1}{2k+4}+\frac{1}{(2k+2)(2k+3)}+\frac{1}{(2k+3)(2k+4)}$.

By this token, n as above 7 kinds of integers be suitable to the conjecture.

3. Divide the unsolved kind into 3 genera and formulate 2 genera therein

For the unsolved kind n=8k+1 with k≥1, divide it by 3 and get 3 genera:

(1) the remainder is 0 when k=1+3t; (2) the remainder is 2 when k=2+3t;

(3) the remainder is 1 when k=3+3t, where t≥0, and *ut infra*.

k:	1, 2, 3,	4, 5, 6,	7, 8, 9,	10, 11, 12,	13, 14, 15, ...	
8k+1:	9, 17, 25,	33, 41, 49,	57, 65, 73,	81, 89, 97,	105, 113, 121, ...	
The remainder:	0, 2, 1,	0, 2, 1,	0, 2, 1,	0, 2, 1,	0, 2, 1, ...	

Excepting the genus (3), we formulate other 2 genera as follows:

(8) For $\frac{8k+3}{3}$ where the remainder is equal to 0, there are

$$\frac{4}{8k+1}=\frac{1}{\frac{8k+1}{3}}+\frac{1}{8k+2}+\frac{1}{(8k+1)(8k+2)}.$$

Due to k=1+3t and t≥0, there are $\frac{8k+1}{3}=8t+3$, so we confirm that $\frac{8k+1}{3}$ in the preceding equation is an integer.

(9) For $\frac{8k+3}{3}$ where the remainder is equal to 2, there are

$$\frac{4}{8k+1}=\frac{1}{\frac{8k+2}{3}}+\frac{1}{8k+1}+\frac{1}{\frac{(8k+1)(8k+2)}{3}}.$$

Due to k=2+3t and t≥0, there are $\frac{8k+2}{3}=8t+6$, so we confirm that $\frac{8k+2}{3}$ and $\frac{(8k+1)(8k+2)}{3}$ in the preceding equation are two integers.

4. Divide the unsolved genus into 5 sorts and formulate 3 sorts therein

For the unsolved genus $\frac{8k+1}{3}$ where the remainder is equal to 1 when k=3+3t and t≥0, then there are 8k+1=25, 49, 73, 97, 121 etc. So we divide them into 5 sorts: 25+120c, 49+120c, 73+120c, 97+120c and 121+120c where c≥0, and *ut infra*.

C\n:	25+120c,	49+120c,	73+120c,	97+120c,	121+120c,
0,	25,	49,	73,	97,	121,
1,	145,	169,	193,	217,	241,
2,	265,	289,	313,	337,	361,
…,	…,	…,	…,	…,	…,

Excepting n=49+120c and n=121+120c, formulate other 3 sorts, they are:

(10) For n=25+120c, there are $\frac{4}{25+120c}=\frac{1}{25+120c}+\frac{1}{50+240c}+\frac{1}{10+48c}$;

(11) For n=73+120c, there are $\frac{4}{73+120c}=\frac{1}{(73+120c)(10+15c)}+\frac{1}{20+30c}+\frac{1}{(73+120c)(4+6c)}$;

(12) For n=97+120c, there are $\frac{4}{97+120c}=\frac{1}{25+30c}+\frac{1}{(97+120c)(50+60c)}+\frac{1}{(97+120c)(10+12c)}$.

For each of listed above 12 equations which express $\frac{4}{n}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$, please each reader self to make a check respectively.

5. Prove the sort $\frac{4}{49+120c}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$

For a proof of the sort $\frac{4}{49+120c}$, it means that when c is equal to each of positive integers plus 0, there always are $\frac{4}{49+120c}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$.

After c is given any value, $\frac{4}{49+120c}$ can be substituted by each of infinite more a sum of an unit fraction plus a proper fraction, and that these fractions are different from one another, as listed below:

$$\frac{4}{49+120c}$$

$$=\frac{1}{13+30c}+\frac{3}{(13+30c)(49+120c)}$$

$$=\frac{1}{14+30c}+\frac{7}{(14+30c)(49+120c)}$$

$$=\frac{1}{15+30c}+\frac{11}{(15+30c)(49+120c)}$$

…

$$=\frac{1}{13+\alpha+30c}+\frac{3+4\alpha}{(13+\alpha+30c)(49+120c)}$$, where α≥0 and c≥0

…

As listed above, we can first let $\frac{1}{13+\alpha+30c}=\frac{1}{X}$, then go to prove

$$\frac{3+4\alpha}{(13+\alpha+30c)(49+120c)}=\frac{1}{Y}+\frac{1}{Z}$$ where c≥0 and α≥0, *ut infra*.

***Proof*·** First, we analyse 3+4α on the place of numerator, it is not hard to see, except 3+4α as one numerator, it can also be expressed as the sum of an even number plus an odd number to act as two numerators, i.e. (4α+3), (4α+2)+1, (4α+1)+2, (4α)+3, (4α-1)+4, (4α-2)+5, (4α-3)+6, ...

If there are two addends on the place of numerator, then these two addends are regarded as two matching numerators, and that two matching numerators are denoted by ψ and φ, also there is ψ > φ.

In numerators with the same denominator, largest ψ is denoted as ψ_1. It is obvious that ψ_1 matches with smallest φ, when ψ_1=4α+2, smallest φ=1.

And then, let us think about the denominator (13+α+30c)(49+120c), actually just 13+α+30c is enough, while reserve 49+120c for later.

In the fraction $\frac{3+4\alpha}{13+\alpha+30c}$, let each α be assigned a value for each time, according to the order α= 0, 1, 2, 3,... So the denominator 13+α+30c can be assigned into infinite more consecutive positive integers.

As the value of α goes up, accordingly numerators are getting more and more, and newly- added numerators are getting bigger and bigger.

When α =0, 1, 2, 3 and otherwise, these denominators of 13+α+30c and their numerators 4α+3, ψ and φ are listed below.

13+α+30c,	α,	(4α+3),	(4α+2)+1,	(4α+1)+2,	(4α)+3,	(4α-1)+4,	(4α-2)+5,	(4α-3)+6, ...		
13+30c,	0,	3,	2+1,	1+2						
14+30c,	1,	7,	6+1,	5+2,	4+3,	3+4 ,	2+5,	1+6		
15+30c,	2,	11,	10+1,	9+2,	8+3,	7+4,	6+5,	5+6, ...		
16+30c,	3,	15,	14+1,	13+2,	12+3,	11+4,	10+5,	9+6, ...		
17+30c,	4,	19,	18+1,	17+2,	16+3,	15+4,	14+5,	13+6,...		
...,	...,	...,	...,	...,	...,	...,	...,	...,	...	

As can be seen from the list above, every denominator 13+α+30c corresponds with two special matching numerators ψ_1 and 1, from this, we get the unit fraction $\frac{1}{13+\alpha+30c}$.

For the unit fraction $\frac{1}{13+\alpha+30c}$, multiply its denominator by 49+120c reserved, then we get the unit fraction $\frac{1}{(13+\alpha+30c)(49+120c)}$, and let $\frac{1}{(13+\alpha+30c)(49+120c)}=\frac{1}{Y}$.

After that, let us prove that $\frac{\psi_1}{13+\alpha+30c}$ i.e. $\frac{4\alpha+2}{13+\alpha+30c}$ is an unit fraction.

Since the numerator 4α+2 is an even number, such that the denominator 13+α+30c must be an even numbers. Only in this case, it can reduce the fraction, so α in the denominator 13+α+30c is only an odd number.

After α is assigned to odd numbers 1, 3, 5 and otherwise, and the fraction $\frac{4\alpha+2}{13+\alpha+30c}$ after the values assignment divided by 2, then the fraction $\frac{4\alpha+2}{13+\alpha+30c}$ is turned into the fraction $\frac{3+4t}{k+15c}$ identically, where c≥0, t≥0 and k≥7.

The point above is that 3+4t and k+15c after the values assignment make up a fraction, they are on the same order of taking values of t and k, according to the order from small to large, i.e. $\frac{3+4t}{k+15c}=\frac{3}{7+15c}$, $\frac{7}{8+15c}$, $\frac{11}{9+15c}$, ...

Such being the case, let the numerator and denominator of the fraction

$\frac{3+4t}{k+15c}$ divided by 3+4t, then we get a temporary indeterminate unit fraction, and its denominator is $\frac{k+15c}{3+4t}$, and its numerator is 1.

Thus, we are necessary to prove that the denominator $\frac{k+15c}{3+4t}$ is able to become a positive integer in the case where t≥0, k≥7 and c≥0.

In the fraction $\frac{k+15c}{3+4t}$, due to k≥7, the numerator k+15c after the values assignment are infinite more consecutive positive integers, while the denominator 3+4t = 3, 7, 11 and otherwise positive odd numbers.

The key above is that each value of 3+4t after the values assignment can seek its integral multiples within infinite more consecutive positive integers of k+15c, in the case where t≥0, k≥7 and c≥ 0.

As is known to all, there is a positive integer that contains the odd factor 2n+1 within 2n+1 consecutive positive integers, where n=1, 2, 3, ...

Like that, there is a positive integer that contains the odd factor 3+4t within 3+4t consecutive positive integers of k+15c, no matter which odd number that 3+4t is equal to, where t≥0, k≥7 and c≥0. It is obvious that a fraction that consists of such a positive integer as the numerator and 3+4t as the denominator is an improper fraction.

Undoubtedly, every such improper fraction that is found in this way, via the reduction, it is surely a positive integer.

That is to say, $\frac{k+15c}{3+4t}$ as the denominator of the aforesaid temporary indeterminate unit fraction can become a positive integer, and the positive integer is represented by μ, and thus in this case the fraction $\frac{3+4t}{k+15c}$ is exactly $\frac{1}{\mu}$.

For the unit fraction $\frac{1}{\mu}$, multiply its denominator by 49+120c reserved, then we get the unit fraction $\frac{1}{\mu(49+120c)}$, and let $\frac{1}{\mu(49+120c)}=\frac{1}{Z}$.

If 3+4α serve as one numerator such that $\frac{3+4\alpha}{(13+\alpha+30c)(49+120c)}=\frac{1}{Y}$, then we can multiply the denominator Y by 2 to make a sum of two identical unit fractions, then again, convert them into the sum of two each other's -distinct unit fractions by the formula $\frac{1}{2r}+\frac{1}{2r}=\frac{1}{r+1}+\frac{1}{r(r+1)}$.

Thus it can be seen, the fraction $\frac{3+4\alpha}{(13+\alpha+30c)(49+120c)}$ is surely able to be expressed into a sum of two each other's -distinct unit fractions in the case where c≥0 and α≥0. To sum up, there are $\frac{4}{49+120c}=\frac{1}{13+\alpha+30c}+\frac{1}{(13+\alpha+30c)(49+120c)}+\frac{1}{\mu(49+120c)}$ where α≥0, μ is an integer and $\mu=\frac{k+15c}{3+4t}$, t≥0, k≥7 and c≥0.

In other words, we have proved $\frac{4}{49+120c}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$.

6. Prove the sort $\frac{4}{121+120c}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$

The proof in this section is exactly similar to that in the section 5. Namely, for a proof of the sort $\frac{4}{121+120c}$, it means that when c is equal to each of positive integers plus 0, there always are $\frac{4}{121+120c}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$.

After c is given any value, $\frac{4}{121+120c}$ can be substituted by each of infinite more a sum of an unit fraction plus a proper fraction, and that these fractions are different from one another, as listed below.

$$\frac{4}{121+120c}$$

$$=\frac{1}{31+30c}+\frac{3}{(31+30c)(121+120c)}$$

$$=\frac{1}{32+30c}+\frac{7}{(32+30c)(121+120c)}$$

$$=\frac{1}{33+30c}+\frac{11}{(33+30c)(121+120c)}$$

…

$$=\frac{1}{31+\alpha+30c}+\frac{3+4\alpha}{(31+\alpha+30c)(121+120c)}$$, where $\alpha \geq 0$ and $c \geq 0$.

…

As listed above, we can first let $\frac{1}{31+\alpha+30c}=\frac{1}{X}$, then go to prove

$$\frac{3+4\alpha}{(31+\alpha+30c)(121+120c)}=\frac{1}{Y}+\frac{1}{Z}$$ where c≥0 and α≥0, *ut infra*.

Proof· First, we analyse 3+4α on the place of numerator, it is not hard to see, except 3+4α as one numerator, it can also be expressed as the sum of an even number and an odd number to act as two numerators, i.e. (4α+3), (4α+2)+1, (4α+1)+2, (4α)+3, (4α-1)+4, (4α-2)+5, (4α-3)+6, ...

If there are two addends on the place of numerator, then these two addends are regarded as two matching numerators, and that two matching numerators are denoted by ψ and φ, also there is ψ > φ.

In numerators with the same denominator, largest ψ is denoted as ψ_1. It is obvious that ψ_1 matches with smallest φ, when ψ_1=4α+2, smallest φ=1.

And then, let us think about the denominator (31+α+30c)(121+120c), actually just 31+α+30c is enough, while reserve 121+120c for later.

In the fraction $\frac{3+4\alpha}{31+\alpha+30c}$, let each α be assigned a value for each time, according to the order α= 0, 1, 2, 3, ...So the denominator 31+α+30c can be assigned into infinite more consecutive positive integers.

As the value of α goes up, accordingly, numerators are getting more and more, and newly- added numerators are getting bigger and bigger.

When α =0, 1, 2, 3 and otherwise, these denominators of 31+α+30c and their numerators 4α+3, ψ and φ are listed below.

31+α+30c,	α,	(4α+3),	(4α+2)+1,	(4α+1)+2,	(4α)+3,	(4α-1)+4,	(4α-2)+5,	(4α-3)+6, ...
31+30c,	0,	3,	2+1,	1+2				

32+30c,	1,	7,	6+1,	5+2,	4+3,	3+4 ,	2+5,	1+6	
33+30c,	2,	11,	10+1,	9+2,	8+3,	7+4,	6+5,	5+6, ...	
34+30c,	3,	15,	14+1,	13+2,	12+3,	11+4,	10+5,	9+6, ...	
35+30c,	4,	19,	18+1,	17+2,	16+3,	15+4,	14+5,	13+6,...	
...,	...,	...,	...,	...,	...,	...,	...,	...,	...

As can be seen from the list above, every denominator of 31+α+30c corresponds with two special matching numerators ψ_1 and 1, from this, we get the unit fraction $\frac{1}{31+\alpha+30c}$.

For the unit fraction $\frac{1}{31+\alpha+30c}$, multiply its denominator by 121+120c reserved, then we get the unit fraction $\frac{1}{(31+\alpha+30c)(121+120c)}$, and let $\frac{1}{(31+\alpha+30c)(121+120c)}=\frac{1}{Y}$.

After that, let us prove that $\frac{\psi_1}{31+\alpha+30c}$ i.e. $\frac{4\alpha+2}{31+\alpha+30c}$ is an unit fraction.

Since the numerator 4α+2 is an even number, such that the denominator 31+α+30c must be an even numbers. Only in this case, it can reduce the fraction, so α in the denominator 31+α+30c is only an odd number.

After α is assigned to odd numbers 1, 3, 5 and otherwise, and the fraction $\frac{4\alpha+2}{31+\alpha+30c}$ after the values assignment divided by 2, then the fraction $\frac{4\alpha+2}{31+\alpha+30c}$ is turned into the fraction $\frac{3+4t}{m+15c}$ identically, where c≥0, t≥0 and m≥16.

The point above is that 3+4t and m+15c after the values assignment make up a fraction, they are on the same order of taking values of t and m, according to the order from small to large, i.e. $\frac{3+4t}{m+15c}=\frac{3}{16+15c}$, $\frac{7}{17+15c}$, $\frac{11}{18+15c}$, ...

Such being the case, let the numerator and denominator of the fraction $\frac{3+4t}{m+15c}$ divided by 3+4t, then we get a temporary indeterminate unit fraction, and its denominator is $\frac{m+15c}{3+4t}$, and its numerator is 1.

Thus, we are necessary to prove that the denominator $\frac{m+15c}{3+4t}$ is able to become a positive integer in the case where t≥0, m≥16 and c≥0.

In the fraction $\frac{m+15c}{3+4t}$, due to m≥16, the numerator m+15c after the values assignment are infinite more consecutive positive integers, while the denominator 3+4t=3, 7, 11 and otherwise positive odd numbers.

The key above is that each value of 3+4t after the values assignment can seek its integral multiples within infinite more consecutive positive integers of m+15c in the case where t≥0, m≥16 and c≥ 0.

As is known to all, there is a positive integer that contains the odd factor 2n+1 within 2n+1 consecutive positive integers, where n=1, 2, 3, ...

Like that, there is a positive integer that contains the odd factor 3+4t within 3+4t consecutive positive integers of m+15c, no matter which odd number that 3+4t is equal to, where t≥0, m≥16 and c≥0. It is obvious that a fraction that consists of such a positive integer as the numerator and 3+4t as the denominator is an improper fraction.

Undoubtedly, every such improper fraction that is found in this way, via the reduction, it is surely a positive integer.

That is to say, $\frac{m+15c}{3+4t}$ as the denominator of the aforesaid temporary indeterminate unit fraction can become a positive integer, and the positive integer is represented by λ, and thus in this case, the fraction $\frac{3+4t}{m+15c}$ is exactly $\frac{1}{\lambda}$.

For the unit fraction $\frac{1}{\lambda}$, multiply its denominator by 121+120c reserved, then we get the unit fraction $\frac{1}{\lambda(121+120c)}$, and let $\frac{1}{\lambda(121+120c)}=\frac{1}{Z}$.

If 3+4α serve as one numerator such that $\frac{3+4\alpha}{(31+\alpha+30c)(121+120c)}=\frac{1}{Y}$, then we can multiply the denominator Y by 2 to make a sum of two identical unit fractions, then again, convert them into the sum of two each other's -distinct unit fractions by the formula $\frac{1}{2\mathrm{r}}+\frac{1}{2r}=\frac{1}{r+1}+\frac{1}{r(r+1)}$.

Thus it can be seen, the fraction $\frac{3+4\alpha}{(31+\alpha+30c)(121+120c)}$ is surely able to be expressed into a sum of two each other's -distinct unit fractions in the case where c≥0 and α≥0. To sum up, there are $\frac{4}{121+120c}=\frac{1}{31+\alpha+30c}+\frac{1}{(31+\alpha+30c)(121+120c)}+\frac{1}{\lambda(121+120c)}$ where λ is an integer and $\lambda=\frac{m+15c}{3+4t}$, t≥0, m≥16, and c≥0.

In other words, we have proved $\frac{4}{121+120c}=\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$.

The proof was thus brought to a close. As a consequence, the Erdös-Straus conjecture is tenable.

References

[1] J. W. Sander, *On* $\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ *and Iwaniec' Half Dimensional Sieve*; Journal of Number Theory, Volume 46, Issue 2, February 1994, Pages 123-136; https://www.sciencedirect.com/science/article/pii/S0022314X84710080?via%3Dihub

[2] Wolfram Math world, the web's most extensive mathematics resource; https://mathworld.wolfram.com/Erdos-StrausConjecture.html

[3] Konstantine Zelator, *An ancient Egyptian problem: the diophantine equation* $\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$, *n>or=2*; arXiv: 0912.2458;