A law of time dilation proportionality in Keplerian orbits

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Abstract

We examine the Lorentz factor and the Schwarzschild solution in relation to the estimation and verification of time dilation particularly from available Global Positioning System data. We as a result detect the possible occurrence of a proportionality between time dilation effects of special and general relativity in free-fall motion in Keplerian orbits. This observation is then mathematically proved. The results show that gravitational time dilation during free fall in Keplerian orbits must be exactly double that caused in special relativity due to linear velocity. We propose that a law has to be enunciated in view of the proof provided, and of the experimental and technological verification of time dilation effects during the past six to seven decades. The importance of this finding as a universal phenomenon and in the further development of stable clocks and satellite technology is highlighted.

1. Intoduction

In this paper we study the estimation procedures and experimental verification of time dilation as a result of which we produce proof that, in Keplerian or free falling orbital motions resulting from the inverse square gravitational attraction of a central body, time dilation due to special relativity (SR) must be exactly half that due to gravitational effect. Since time dilation in relativistic effects has been proved in several experiments and technological applications, the conclusion developed in this paper deserves to be enunciated as a Law.

In the special relativity theory time dilation [1.3] is described by the Lorentz transformations. The Schwarzschild solution describes gravitational time dilation in general relativity (GR) [2.3]. These two tools have been of vital and fundamental importance in arriving at the law of proportionality of time dilation and we dwell on them with some details in the text.

Time dilation values predicted by the Lorentz factor and the Schwarzschild gravitational time dilation formula have been experimentally and technologically confirmed. Particle accelerators routinely carry out experimental tests of the time dilation of special relativity since the 1950s. Early SR time dilation tests which confirm Einstein's prediction include the measurement [4.5] of the Doppler shift of the radiation emitted from cathode rays, on the direct observation of the transverse Doppler shift [6], and the accurate time dilation incidence recorded in elementary particles decay [7].

Recent experimental evidence of SR time dilation verification includes the significantly improved test of time dilation in special relativity performed with laser spectroscopy on fast ions at the heavy-ion storage-ring in Heidelberg at $v = 0.0064\,c$ [8] The result confirms the relativistic Doppler formula. More recently even more sensitive measurements of time dilation using the Hiedelberg storage-ring confirmed time dilation with unprecedented accuracy at $v = 0.0065\,c$ [9].

In the case of gravitational time dilation confirmation the classical example is the gravitational red-shift measurement [10] with results within 10% of the predictions of general relativity which was later [11] improved to 1%. Using cesium clocks travelling in commercial airliners [12], the occurrence of SR and GR combined time dilation effects was demonstrated. Recently these results were confirmed [13] in a trial using technologically improved cesium clocks with greater accuracy, within 4% of the predictions of relativity. Other recent work includes the measurement of the frequency shift of radio photons to and from the Cassini spacecraft as they passed by the sun, which agrees with the predictions of general relativity[14].

The Global Positioning System and Glonass can be considered as stable clocks tests in special and general relativity. It has been emphasized [15] that the GPS provides a fascinating menu of applications of special and general relativity. The use of stable clocks in space navigation technologies depends enormously on the relativity predictions of Einstein and on the use of the Lorentz factor and the Schwarzschild solution.

A very recent paper[16] aptly says that relativity has already entered a status of an applied technology in daily life, a point illustrated by the role of the GPS in the success of the German toll system on highways which is a market with a flow of several billion euros annually. In addition the international atomic time is defined by comparing and averaging the times provided by more than 200 atomic clocks distributed worldwide. The average value obtained is more accurate than any individual value. Comparisons between individual clocks and one on board a GPS satellite may need uncertainly less than a few nanoseconds(ns)[16].However relativistic effects much greater than this have to be corrected. Timing errors of one ns will lead to positioning errors of the order 30 m[17].

In this work the data shown below have the corresponding values:

Speed of light c = 299 792 458 m/s

Gravitational constant G = 6.6742 x 10^{11} $m^3 s^{-2} k^{-1}$

Mass of earth M = 6 x 10^{24} kg

Radius of earth R = 6 380 000 m

Data for the GPS satellites have been given by various authors[15.16.17.18.19]. Some selected data are the following:

Altitude = 20 200 km

Orbital radius r =26 580 000 m

Velocity v = 3 888 m/s

SR time dilation = +7 260 ns/day

GR time dilation = -45 570 ns/day compared to earth based stationary clock

Net time dilation = -38 310 ns/day compared with earth based clock

Computing the reported orbital radius against v of the satellite gives a g value of 0.5687. Using the $\frac{GM}{r^2}$ formula the g is 0.5668. The GPS has a semi-synchronous orbital period of 11 h 58 m which is half of the earth average rotational period of 23 h 56 m. The GPS's successful demonstration of time dilation resulting from an array of relativity influences is a clear proof of the validity of Einstein's relativity predictions.

Time dilation consideration is heavily relied upon in this paper on account of its experimental verifications in both research and practical applications. Although space contraction has not been experimentally verified, there is little doubt that it will be shown to be true once relativistic velocities are realized for long distances.

This study explores a new perspective of the Lorentz factor for determining time dilation and length contraction. Since the scheme developed (SR scheme 1) works well we will apply it also to the Schwarzschild Solution for gravitational time dilation. We shall then see related approaches for estimating time dilation.

2. Time dilation estimation

The Lorentz Factor may be derived from a valid Pythagorean Theorem based on light signals. Here we will adopt the thought experiment based on a moving train with an observer B standing on the embankment. The train moves with velocity *v*. A light emitter on the floor of the train sends a light signal straight to a mirror fixed on the train's ceiling exactly opposite the emitter, a distance *L* from the emitter. To a train traveller (observer A) the light signal bounces to the mirror and straight back to the floor where the emitter is located For observer B the signal moves obliquely up towards the mirror and obliquely down to the floor. Since the velocity of light is constant in all reference frames observer A finds the light signal to take time $t = \frac{2L}{c}$. For observer B on the embankment the time taken is different and is $t' = \frac{2L'}{c}$.In the meantime the train has moved a distance $t'v$, v being the train's velocity and t', the time observer B sees the light signal

The three distances fit into a right angle triangle to constitute a valid Pythagorean Theorem and are:

$$\frac{t'c}{2}, \frac{t'v}{2}, \frac{tc}{2}$$

So this gives:

$$(\frac{t'c}{2})^2 = (\frac{t'v}{2})^2 + (\frac{tv}{2})^2$$

from which we can derive the multiplier factor

$$\frac{c}{\sqrt{c^2 - v^2}}$$

which in turn can be translated to the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

For the sake of simplification we shall first restrict our discussion to the binomial expansion of the Lorentz factor which is:

$$\frac{v^2}{2c^2}$$

for non-relativistic velocities which Einstein[1] also utilized. The Schwarzschild solution,below, is meant to apply to space-time in the vicinity of a non-rotating massive object:

$$t = \frac{t_0}{\sqrt{1 - \frac{2GM}{Rc^2}}}$$

where t is the time interval measured by a clock which is situated very far away from a central mass.

If the clock is at the earth surface or not very far from it we use the gravitational time dilation formula:

$$t = \frac{t_0}{\sqrt{1 - \frac{2gR}{c^2}}}$$

Employing a binomial expansion of the gravitational time dilation expression, the first approximation to the time expression, ignoring subsequent ones since they are very negligible, is:

$$\frac{gr}{c^2}$$

This formula is a good approximation for estimating time dilation in GRT for r values at the earth surface including earth orbital satellites.

We will then, based on the Lorentz Factor (SR scheme 1), initially develop a linear sequence of ideas and calculations which will lead us eventually to the proportionality of SR and GR time dilation effects. To start with, without making any fundamental change, we propose modifying the formula

$$\frac{v^2}{2c^2} \qquad \text{to} \qquad \frac{dv}{2c^2},$$

where d is the distance covered by any moving object with velocity v in time t as seen by an external observer in an inertial frame of reference. We will now examine how we can paint a descriptive and logical sequence of ideas on $\frac{dv}{2c^2}$, that enables us to arrive at time dilation and distance contraction in a moving body. In other words instead of v^2 and c^2, we shall proceed by adopting a linear sequence of mathematical calculations, bypassing the direct use of squares and square roots (SR scheme 1 below). This approach can be adapted to any kind of motion at uniform velocity such as walking, travelling with clocks to synchronize them, to the motion of an airplane, or the linear motion of a Glonass or GPS satellite. Irrespective of the kind of object and its uniform motion we can proceed as follows.

3. SR scheme 1 and GR scheme 1

The sequence of ideas and their corresponding sequence of calculations are as follows:

1. First we obtain the value of *d*, the distance a massive object (x) moves in time *t* at uniform velocity *v*.
2. We divide this value by 2.
3. We then divide $\frac{d}{2}$ by *c* to give the time that light would theoretically

take to cover distance $\frac{d}{2}$.

4. By multiplying the time that light theoretically takes as obtained in (3) above with v we get the theoretical distance that object x would cover during that time.
5. We now calculate the theoretical time light takes to cover that distance: this actually gives us the amount of time dilation in the inertial reference frame of x.
6. We then calculate the distance x would have theoretically covered in that amount of dilated time to get the length contraction in distance.

We shall use this scheme to calculate SR time dilation and distance contraction (SR scheme 1) for a GPS satellite moving at 3 888 m/s for 11h 58 m to complete one orbital round and we obtain the following data, bearing in mind that the calculated data will be slightly different from the actual observed data owing to some fluctuations in altitude, g value and linear velocity due to a number of factors including orbital eccentricity of the satellites.

1. Distance covered in one orbital cycle=43 080 s x 3 888 m=167 495 040 m.
2. Divide by 2 = 83 747 520 m.
3. Divide by c = 0.279351657 s.
4. Multiply by v = 0.27935 x 3 888 = 1 086 m.
5. Time light would take to cover that distance: 1 086 divided by c = 0.000003623 s which gives 3 623 ns per cycle or 7 246 ns per day of time dilation due to linear motion of a GPS satellite.
6. Therefore distance contraction in the satellite frame of reference=0.000003623 x 3 888 = 0.014086 m or 28.171mm per day.

The sequence of steps and the corroboration of the data obtained here on time dilation with reported observed time dilation data from atomic clocks in the GPS augur promisingly for the scheme developed above. It also gives weight to Einstein's other prediction of special relativity, that dealing with distance contraction, yet to be actually measured in real life situation. In this connection the last step in the set of procedures is very likely to be also correct, especially since up to step 5 everything has been experimentally verified. It is only a matter of time and of technological know-how before it can be experimentally measured.

The arguments proposed in the scheme indicate that physical adjustments in time and space, far from being counter-intuitive, can in fact be intuitive. But the main purpose of the scheme is not for determining time dilation and related relativistic parameters, rather it is meant to indicate whether analysis of the scheme developed above provides new insights of the coherence of Einstein's relativity theories that would add to a better understanding of motion and of the universe generally.

We shall now proceed to utilize the above sequence of ideas to see whether each step can be made applicable in GR time dilation (GR scheme 1). To do this we shall apply it to the $\frac{gr}{c^2}$ expansion of the Schwarzschild solution, and then illustrating the ideas being argued by applying them to GPS. Before doing so, however, we will provide some explanatory notes regarding *g* and *r* in the context of this application. . Next, in $\frac{gr}{c^2}$, we will assume that *gr* can be equated to *dv* we saw in the modified Lorentz Factor mentioned earlier. In the present calculation we shall consider *r* to mean the distance *d*, and *g* is arbitrarily interpreted as being a uniform 'potential' velocity since an object at rest on the earth surface would not per se be accelerating, in other words there is no real rate of change of velocity, but would be sensitive to a gravitational field in proportion to the *g* felt. This is a mere assumption and it is realised that acceleration may be fundamentally a difficult phenomenon to circumscribe. A subsequent paper will deal with some concepts of the fundamental nature of acceleration in relation to gravity.

One can also argue that a body at rest in a gravitational field, that is, one which does not actually change its *r* value with respect to the central body exerting the gravitation field, accumulates time dilation values uniformly with time just as a body in uniform motion in SR does. Therefore the scheme in GR, given for the sake of comparison, comes to the following for an object at rest at sea level:

1. We assume *r* to be equivalent to *d*, that is, 6 380 000 m.
2. Point 2 in the scheme is not applicable in GR.
3. Calculate the time light would theoretically take to cover that distance = 0.021281389 s.
4. Calculate how far the object would have, if it were moving at a constant velocity of 9.81 m/s, theoretically covered in that time=0.208770428 m.
5. How long light would theoretically take to cover the distance=0.6963831918 ns. This is the time dilation value, per second, due to GR for an object at sea level. Total for a day=60 000 ns.

Applying the procedure to GPS we obtain the following for a satellite at a radius of approximately 26 580 000 m completing two orbital cycles per day each cycle taking 43 080 s. The gravitational attraction at that point is 0.5687 m/s.

1. The value of d is 26 580 000 m.
2. Not applicable in GR.
3. Time light would theoretically take to cover that distance= 0.088661336 s.
4. The distance that the GPS satellite would have theoretically covered in that time if it were moving at a constant velocity of 0.5687 m/s=0.0886613 m x 0.5687 m/s=0.050421702 m.
5. The time that light would take to cover this distance=0.1681886943 ns/s. Accumulated time dilation for a day=14 491 ns. Daily difference between a clock at sea surface and in the GPS=60 000 ns-14 491 ns=45 509 ns, which means that a clock in the GPS runs faster than an earth surface clock by 45 509 ns per day due to Earth's gravitational field.

The net calculated time dilation therefore is 45 509 minus 7 246 or 38 263ns, compared to an earth based stationary clock. Although apparently an unorthodox approach, the scheme has a fundamental value for it indicates, the occurrence of a proportionality. The SR time dilation(SR scheme 1) for a day of 7 246 ns, if multiplied by 2, gives 14 492 ns practically identical with 14 491 ns as obtained for GR time dilation. The above schemes seem to indicate that in terms of actual estimation the most pertinent difference between schemes SR 1 and GR 1 is that the latter produces twice the time dilation value of SR.

4. GR to SR proportionality

Einstein's SR and GR incorporate a physical space-time and it certainly makes sense that in SR it is different from that of GR. We have earlier developed a scheme to show that the application of Lorentz factor and of the Schwarzschild solution to time dilation determination appears to have some common features. We shall now go even further and get close to a proof of a mathematically and therefore structurally proportionate basis for the space time description in SR and GR.

The obvious differences in those formulae, $\frac{v^2}{2c^2}$ for SR and $\frac{gr}{c^2}$ for GR, appear to fit well with the contextual differences between SR, which is apparently outside any gravitational influence, and GR where gravity has a central role. We shall now see that another system of formulae can be applied to GR and SR time dilation calculation, which will confirm that there must be a proportionality between them. In the case of an object at rest on earth surface, we calculated gravitational time dilation of 0.69638 ns per s or 60 000 ns per day (GR scheme 1) using $\frac{gr}{c^2}$.

For the same time period a GPS satellite in orbit with r=26 580 000 m and with g=0.5687 m/s/s we obtained time dilation of 0.16818869 ns/s or 14 491 ns per day, again using

$\frac{gr}{c^2}$. However since for a massive body in Keplerian orbital motion,

$$v^2 = g \text{ x } r$$

then using GPS satellite data we get:

$$v = \sqrt{0.5687\text{X}26580000 m/s}$$

=3 887.9 m/s, very near to the reported velocity of 3 888 m/s

Calculating GR time dilation the two formulae shown below will give identical values:

$$\frac{gr}{c^2} = \frac{v^2}{c^2}$$

Since the SR velocity time dilation formula $\frac{v^2}{2c^2}$ gives estimates exactly those of GR formula $\frac{v^2}{c^2}$ the calculated time dilation in GR (GR scheme 2) should be exactly twice that of SR, as shown below for a GPS satellite:

Calculated GPS GR time dilation using $\frac{v^2}{c^2}$ =3 887.9 m/s x 3 887.9 m/s divided by *c*2=

14 490.86 ns per day

Calculated GPS SR time dilation (SR scheme 1) =7 246 ns per day

Reported GPS SR time dilation=7 260 ns

Therefore calculated gravitational time dilation for an object in a circular orbital free fall is exactly twice that of SR time dilation for the same orbital velocity. Consequently if the observed data are slightly different that would indicate some slight flaws or inaccuracies in the technological computation of data.

We shall first again base our SR time dilation estimation on the use of the binomial expansion of the Lorentz Factor which, as shown earlier, gives us 7 246 ns per day for a linear velocity of 3 888 m/s for a GPS satellite in orbit. For an object at rest on the earth surface, because it has no linear orbital motion, there is no estimation required. Now let us suppose an object at rest on the earth surface had in fact, if the earth were suddenly to contract in size to a tiny black-hole, been in an orbital motion due to free fall. We would then get a circular velocity of:

$$\sqrt{9.81X6380000} = 7\,911.345161 \text{ m/s}$$

Now using the formula $\frac{v^2}{2c^2}$ we would have had an SR time dilation value of

29 959.69, very nearly half the GR time dilation value of 60 000 ns we obtained earlier for a stationary clock at sea level.

5. Conclusion

We have succeeded in demonstrating a proportionality between SR and GR based on both theoretical reasoning and by verification of actual GPS time dilation data. The results produced in this paper warrant a law of proportionality of Einsteinian time dilation effects in Keplerian orbital motion which states:

"In Keplerian orbital free-fall due to the inverse square gravitational attraction of a central body, SR time dilation effects due to linear velocity and GR time dilation have a proportionality of 1:2. This is irrespective of variation in free fall linear velocity and g value due to orbital eccentricity."

Clearly SR and GR time dilation effects are partially entangled in Keplerian orbits. Therefore an interesting observation arising from this Law is that all Keplerian orbital inertial frames of reference in the universe would have a common nature-given uniform property in their space/time base.Irrespective of their relative motions, as long as they are in free fall motion in a Keplerian orbits, they would all be having the same proportionality in terms of their SR to GR time dilation. So we can have an equation with a constant as follows applicable only to time dilation in Keplerian orbital free fall:

$$td_{GR} = ktd_{SR}$$

where k is a constant 2

td is quantity of actual time dilation

GR is general relativity

SR is special relativity

It also foresees an application of the result reported in this study towards making on board navigational satellite's clocks more stable. Based on the proportionality law it may be envisaged to automatically and simultaneously monitor SR and GR time dilation effects with nano-precision, computed from linear velocity or from a combined g and r value.

Finally, in retrospect and with an appropriate methodology , it could have been possible to arrive at the same basic conclusion from an application of the virial theorem, which connects the average kinetic and potential energies for systems in which the potential is a power of the radius.

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