Constructing Quasi-Exactly Solvable Symmetrized Quartic Anharmonic Oscillators Using a Quotient Polynomial

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Abstract
The quasi-exact solvability of symmetrized quartic anharmonic oscillators has been studied first by Znojil **[2]** and then by Quesne **[3]**. In this work, we examine the solvability of these models using, as basic parameter, the energy-dependent, constant (i.e. position-independent) term of a quotient polynomial. We examine the cases n=0 and n=1, and we show that our results are in agreement with those of Quesne. For n=2, following a different path from that of Znojil, we derive the cubic equation that our parameter satisfies and for the case it has a root at zero, we follow the zero root to obtain an even-parity, ground-state wave function and an odd-parity, third-excited-state wave function. As in the case of the sextic anharmonic oscillator **[6]**, the straightforwardness and transparency of the analysis demonstrates the eligibility of the quotient polynomial as a solvability tool of polynomial oscillators.

Keywords: symmetrized potentials, quartic anharmonic oscillators, symmetrized quartic anharmonic oscillators, polynomial oscillators, quotient polynomial, quasi-exactly solvable potentials, Bethe ansatz

Contents

Introduction

Within the framework of the Bethe ansatz **[7]**, the solvability of one-dimensional, attractive at (least at) long distances, smooth and real polynomial potentials can be examined using a quotient polynomial **[5, 6]**.
In this context, the bound energy eigenfunctions of the potential that can be found in closed form are given by the ansatz

$$\psi(\tilde{x};m,n) = A_n p_n(\tilde{x}) \exp\left(g_{2m}(\tilde{x})\right) \quad (1)$$

with $p_n(\tilde{x})$ being a dimensionless, real monic polynomial of degree $n \geq 0$, and $g_{2m}(\tilde{x})$ an also dimensionless, real polynomial of degree $2m$ with negative leading coefficient, where the tilde indicates a dimensionless quantity **[5]**.
The polynomial $p_n(\tilde{x})$ satisfies the differential equation **[5]**

$$p_n''(\tilde{x}) + 2g_{2m}'(\tilde{x}) p_n'(\tilde{x}) = -q_{2(m-1)}(\tilde{x};n) p_n(\tilde{x}) \quad (2)$$

where $q_{2(m-1)}(\tilde{x};n)$ is the quotient polynomial, a $2(m-1)$-degree polynomial whose coefficients depend on n **[5]**.
In **[5]**, we examine the case where m is a positive integer, and thus both the exponential and the quotient polynomials are of even-degree.
If $m = \dfrac{1}{2}$, then $\deg(q) = -1$, i.e. $q_{-1}(\tilde{x};n)$ is not a polynomial, and the potential is not polynomial either. We'll discuss this case in a separate work.
If $m \geq 1$, then $q_{2(m-1)}(\tilde{x};n)$ is still a polynomial even if m is half-integer.
The potential is given by the general expression **[5]**

$$\tilde{V}(\tilde{x};m,n) = g_{2m}'^2(\tilde{x}) + g_{2m}''(\tilde{x}) - q_{2(m-1)}(\tilde{x};n) + \tilde{E} \quad (3)$$

with $\deg(V) = 2(2m-1)$.

The case m=3/2 – Symmetrized quartic anharmonic oscillators

If $m = \dfrac{3}{2}$, then $\deg(V) = 4$, i.e. the potential is quartic, and $\deg(q) = 1$, i.e. the quotient polynomial is a linear polynomial.
Then, the potential (3) takes the form

$$\tilde{V}\left(\tilde{x};\frac{3}{2},n\right) = g_3'^2(\tilde{x}) + g_3''(\tilde{x}) - q_1(\tilde{x};n) + \tilde{E} \quad (4)$$

We want the potential to be symmetric, i.e. of even parity, and thus the energy eigenfunctions $\psi(\tilde{x};m,n)$ have definite parity **[8]**, i.e. they are of either even or odd parity.
Then, from (1), the exponential part $\exp\left(g_{2m}(\tilde{x})\right)$ must be of even parity, otherwise $\psi(\tilde{x};m,n)$ does not have definite parity, and then the exponential polynomial $g_{2m}(\tilde{x})$ must also be of even parity.

Then, again from (1), $\psi(\tilde{x};m,n)$ and $p_n(\tilde{x})$ have the same parity, i.e. the energy eigenfunctions are of even/odd parity if and only if the polynomials $p_n(\tilde{x})$ are of even/odd parity.
Then, we construct the following symmetrized, and thus of the desired even-parity, exponential polynomial of degree 3,

$$g_3(\tilde{x}) = -\frac{{g_3}^2}{3}|\tilde{x}|^3 + \frac{g_2}{2}\tilde{x}^2 + g_1|\tilde{x}| \quad (5)$$

with $g_3 \neq 0$.
In (5), the factors $\frac{1}{3}$ and $\frac{1}{2}$ are put in for convenience.
The constant term g_0 of the exponential polynomial corresponds to a constant exponential term $\exp(g_0)$, which can be incorporated into the normalization constant A_n of $\psi(\tilde{x};m,n)$, and thus, without loss of generality, we omit the constant term of the exponential polynomial.
Also, the eigenfunction $\psi(\tilde{x};m,n)$ must be square integrable, because it describes a bound state, and thus the exponential part $\exp(g_3(\tilde{x}))$ must vanish at $\pm\infty$, and thus $g_3(\pm\infty) = -\infty$, which is assured by the negativity of the leading coefficient $-\frac{{g_3}^2}{3}$.
Since both the potential and the exponential polynomial are of even parity, from (4) we see that the quotient polynomial is also of even parity*, and since it is linear, it will have the form

$$q_1(\tilde{x};n) = q_1(n)|\tilde{x}| + q_0(n) \quad (6)$$

* Since $g_3(\tilde{x})$ is of even parity, ${g_3}'(\tilde{x})$ is of odd parity, but then ${g_3}'^2(\tilde{x})$ is again of even parity.
Also, ${g_3}''(\tilde{x})$ has the same parity as $g_3(\tilde{x})$, i.e. it is of even parity.
Since from (4), $q_1(\tilde{x};n) = {g_3}'^2(\tilde{x}) + {g_3}''(\tilde{x}) + \tilde{E} - \tilde{V}\left(\tilde{x};\frac{3}{2},n\right)$, we derive that $q_1(\tilde{x};n)$ is of even parity, as sum of even-parity polynomials.

As in the case of the sextic anharmonic oscillator **[6]**, the quotient polynomial $q_1(\tilde{x};n)$ has no intermediate terms, i.e. it is a binomial with a leading and a constant term.
As discussed in **[5]**, if the quotient polynomial has intermediate terms, we cannot find more than one energy eigenfunction of each of the respective potentials in closed form, which is the simplest – and in this sense a trivial – case of quasi-exact solvability.
For $\tilde{x} > 0$, (6) becomes

$$q_1(\tilde{x};n) = q_1(n)\tilde{x} + q_0(n)$$

Then

$$q_1'(\tilde{x};n) = q_1(n)$$

Thus, we have

$$q_1(0^+;n) = q_0(n)$$

$$q_1'(0^+;n) = q_1(n)$$

Similarly, for $\tilde{x} < 0$, (6) becomes

$$q_1(\tilde{x};n) = -q_1(n)\tilde{x} + q_0(n)$$

Then

$$q_1'(\tilde{x};n) = -q_1(n)$$

Thus, we have

$$q_1(0^-;n) = q_0(n)$$

$$q_1'(0^-;n) = -q_1(n)$$

We see that the quotient polynomial is continuous at 0, but its derivative has at 0 a finite jump equal to

$$q_1'(0^+;n) - q_1'(0^-;n) = 2q_1(n) \quad (7)$$

For $\tilde{x} > 0$, the exponential polynomial (5) becomes

$$g_3(\tilde{x}) = -\frac{{g_3}^2}{3}\tilde{x}^3 + \frac{g_2}{2}\tilde{x}^2 + g_1\tilde{x}$$

Then

$$g_3'(\tilde{x}) = -{g_3}^2\tilde{x}^2 + g_2\tilde{x} + g_1$$

$$g_3''(\tilde{x}) = -2{g_3}^2\tilde{x} + g_2$$

Thus, we have

$$g_3'(0^+) = g_1$$

$$g_3''(0^+) = g_2$$

Similarly, for $\tilde{x} < 0$, the exponential polynomial (5) becomes

$$g_3(\tilde{x}) = -\frac{{g_3}^2}{3}(-\tilde{x})^3 + \frac{g_2}{2}\tilde{x}^2 + g_1(-\tilde{x}) = \frac{{g_3}^2}{3}\tilde{x}^3 + \frac{g_2}{2}\tilde{x}^2 - g_1\tilde{x}$$

Then

$$g_3'(\tilde{x}) = {g_3}^2\tilde{x}^2 + g_2\tilde{x} - g_1$$

$g_3''(\tilde{x}) = 2g_3^2\tilde{x} + g_2$

Thus, we have

$g_3'(0^-) = -g_1$

$g_3''(0^-) = g_2$

We observe that $g_3'(\tilde{x})$ has a finite jump at 0, equal to

$g_3'(0^+) - g_3'(0^-) = g_1 - (-g_1) = 2g_1$,

but $g_3''(\tilde{x})$ is continuous at 0, since $g_3''(0^+) = g_3''(0^-)$.

Since $g_3'(0^+) = -g_3'(0^-)$, and $g_3''(0^+) = g_3''(0^-)$, the quantity $g_3'^2(0) + g_3''(0)$ is continuous at 0, and it is equal to $g_1^2 + g_2$, i.e.

$g_3'^2(0) + g_3''(0) = g_1^2 + g_2$ (8)

Then, since the quotient polynomial is continuous at 0, from (4) we derive that the potential $\tilde{V}\left(\tilde{x}; \frac{3}{2}, n\right)$ is continuous at 0.

Thus, imposing the condition that the potential $\tilde{V}\left(\tilde{x}; \frac{3}{2}, n\right)$ should vanish at 0, and using that

$q_1(0^+; n) = q_1(0^-; n) = q_0(n) \equiv q_1(0; n)$,

(4) gives

$g_3'^2(0) + g_3''(0) - q_0(n) + \tilde{E} = 0 \Rightarrow q_0(n) = \tilde{E} + g_3'^2(0) + g_3''(0)$

Using (8), the last equation becomes

$q_0(n) = \tilde{E} + g_1^2 + g_2$ (9)

As in the case of smooth polynomial potentials **[5]**, the constant term of the quotient polynomial is energy-dependent, and in this case, it is equal to the energy of the eigenstate described by the wave function (1) plus the constant $g_1^2 + g_2$.

Let us now find the expression giving the leading coefficient $q_1(n)$ of the quotient polynomial.

In the region $\tilde{x} > 0$, using the expressions of the quotient polynomial and of the derivative of the exponential polynomial, the differential equation (2) is written as, for $m = \frac{3}{2}$,

$p_n''(\tilde{x}) + 2\left(-g_3^2\tilde{x}^2 + g_2\tilde{x} + g_1\right)p_n'(\tilde{x}) = -\left(q_1(n)\tilde{x} + q_0(n)\right)p_n(\tilde{x})$ (10)

The polynomial $p_n(\tilde{x})$ is of degree n.

As in the case of smooth polynomial potentials **[5]**, we can incorporate, without loss of generality, the non-zero leading coefficient of $p_n(\tilde{x})$ into the normalization constant of the energy eigenfunction.

Then, in the positive region, where $|\tilde{x}|^n = \tilde{x}^n$, the leading term of $p_n(\tilde{x})$ is $\tilde{x}^n$, and then the leading term of $p_n'(\tilde{x})$ is $n\tilde{x}^{n-1}$.

Also, we have

$\deg\left(p_n''\right) = n-2$, $\deg\left(\tilde{x}^2 p_n'(\tilde{x})\right) = n+1$, $\deg\left(\tilde{x}p_n'(\tilde{x})\right) = n$, $\deg\left(p_n'(\tilde{x})\right) = n-1$, $\deg(\tilde{x}p_n) = n+1$, and $\deg(p_n(\tilde{x})) = n$.

Then, the highest powers in $\tilde{x}$ in both sides of (10) are of $n+1$ degree, with one term in each side of the equation, with the respective coefficients being $-2ng_3^2$ and $-q_1(n)$.

Since the coefficients of the same degree terms in both sides of (10) must be equal, we obtain

$$q_1(n) = 2ng_3^2 \quad (11)$$

In the region $\tilde{x} < 0$, where $|\tilde{x}| = -\tilde{x}$, the leading term of $p_n(\tilde{x})$ can be $\tilde{x}^n$ or $-\tilde{x}^n$, depending on whether the leading term of $p_n(\tilde{x})$ is in absolute value and on whether n is even/odd.

Then, in the negative region, the leading term of $p_n(\tilde{x})$ is, in general, $\pm\tilde{x}^n$, and the leading term of $p_n'(\tilde{x})$ is, in general, $\pm n\tilde{x}^{n-1}$.

In the region $\tilde{x} < 0$, using the expressions of the quotient polynomial and of the derivative of the exponential polynomial, the differential equation (2) is written as, for $m = \frac{3}{2}$,

$$p_n''(\tilde{x}) + 2\left(g_3^2\tilde{x}^2 + g_2\tilde{x} - g_1\right)p_n'(\tilde{x}) = -\left(-q_1(n)\tilde{x} + q_0(n)\right)p_n(\tilde{x})$$

As we did in the region $\tilde{x} > 0$, equating the coefficients of the two $(n+1)$-degree terms in $\tilde{x}$ in the previous equation, we obtain

$$\pm 2g_3^2 n = -\left(-q_1(n)\right)(\pm 1) \Rightarrow \pm 2g_3^2 n = \pm q_1(n) \Rightarrow q_1(n) = 2ng_3^2$$

We see that the leading coefficient of the quotient polynomial is the same in the positive and negative regions, as expected, since the quotient polynomial is of even parity.

The two continuity conditions

The potential $\tilde{V}\left(\tilde{x};\frac{3}{2},n\right)$ is continuous everywhere, and thus both the energy eigenfunction $\psi\left(\tilde{x};\frac{3}{2},n\right)$ and its derivative must be continuous everywhere **[1]**.

Using (1) for $m=\frac{3}{2}$, the condition that $\psi\left(\tilde{x};\frac{3}{2},n\right)$ is continuous at 0, i.e.

$$\psi\left(0^-;\frac{3}{2},n\right)=\psi\left(0^+;\frac{3}{2},n\right),$$

gives

$$A_n p_n\left(0^-\right)\exp\left(g_3\left(0^-\right)\right)=A_n p_n\left(0^+\right)\exp\left(g_3\left(0^+\right)\right)$$

Using that $A_n\neq 0$ and $g_3\left(0^-\right)=g_3\left(0^+\right)=0$, the previous equation becomes

$$p_n\left(0^-\right)=p_n\left(0^+\right) \quad (12)$$

i.e. the polynomial $p_n\left(\tilde{x}\right)$ is continuous at 0.

Using again (1) for $m=\frac{3}{2}$, the first derivative of $\psi\left(\tilde{x};\frac{3}{2},n\right)$ is

$$\psi'\left(\tilde{x};\frac{3}{2},n\right)=\left(A_n p_n\left(\tilde{x}\right)\exp\left(g_3\left(\tilde{x}\right)\right)\right)'=A_n p_n'\left(\tilde{x}\right)\exp\left(g_3\left(\tilde{x}\right)\right)+A_n p_n\left(\tilde{x}\right)g_3'\left(\tilde{x}\right)\exp\left(g_3\left(\tilde{x}\right)\right)=$$
$$=A_n\left(p_n'\left(\tilde{x}\right)+p_n\left(\tilde{x}\right)g_3'\left(\tilde{x}\right)\right)\exp\left(g_3\left(\tilde{x}\right)\right)$$

That is

$$\psi'\left(\tilde{x};\frac{3}{2},n\right)=A_n\left(p_n'\left(\tilde{x}\right)+p_n\left(\tilde{x}\right)g_3'\left(\tilde{x}\right)\right)\exp\left(g_3\left(\tilde{x}\right)\right)$$

Using the previous relation, the condition that $\psi'\left(\tilde{x};\frac{3}{2},n\right)$ is continuous at 0, i.e.

$$\psi'\left(0^-;\frac{3}{2},n\right)=\psi'\left(0^+;\frac{3}{2},n\right),$$

gives

$$A_n\left(p_n'\left(0^-\right)+p_n\left(0^-\right)g_3'\left(0^-\right)\right)\exp\left(g_3\left(0^-\right)\right)=A_n\left(p_n'\left(0^+\right)+p_n\left(0^+\right)g_3'\left(0^+\right)\right)\exp\left(g_3\left(0^+\right)\right)$$

Using that $A_n\neq 0$ and $g_3\left(0^-\right)=g_3\left(0^+\right)=0$, the previous equation becomes

$$p_n'\left(0^-\right)+p_n\left(0^-\right)g_3'\left(0^-\right)=p_n'\left(0^+\right)+p_n\left(0^+\right)g_3'\left(0^+\right)$$

Using that $g_3'\left(0^-\right)=-g_1$, $g_3'\left(0^+\right)=g_1$, and (12), the previous equation becomes

$$p_n'\left(0^-\right)+p_n\left(0^+\right)\left(-g_1\right)=p_n'\left(0^+\right)+p_n\left(0^+\right)g_1$$

and thus

$$p_n'\left(0^-\right)=p_n'\left(0^+\right)+2g_1 p_n\left(0^+\right) \quad (13)$$

i.e. the first derivative of $p_n(\tilde{x})$ has at 0 a finite jump.
Note that if $g_1 = 0$, i.e. if the linear term of the exponential polynomial vanishes, then the first derivative of $p_n(\tilde{x})$ is continuous at 0.

The recursion relation and the (n+1)-degree equation of $q_0(n)$

Using (11), (10) is written as

$$p_n''(\tilde{x}) + 2\left(-g_3^2\tilde{x}^2 + g_2\tilde{x} + g_1\right)p_n'(\tilde{x}) = -\left(2ng_3^2\tilde{x} + q_0(n)\right)p_n(\tilde{x}) \quad (14)$$

This is the differential equation the polynomials $p_n(\tilde{x})$ satisfy in the region $\tilde{x} > 0$.

In the region $\tilde{x} > 0$, $|\tilde{x}|^k = \tilde{x}^k$, and thus for the terms of degree k, we have the coefficients

$$p_n''(\tilde{x}) \to (k+2)(k+1)p_{k+2}$$

$$2\left(-g_3^2\tilde{x}^2 + g_2\tilde{x} + g_1\right)p_n'(\tilde{x}) \to 2\left(-g_3^2(k-1)p_{k-1} + g_2kp_k + g_1(k+1)p_{k+1}\right)$$

$$-\left(2ng_3^2\tilde{x} + q_0(n)\right)p_n(\tilde{x}) \to -\left(2ng_3^2p_{k-1} + q_0(n)p_k\right)$$

Thus, equating the coefficients of the terms of degree k in $\tilde{x}$ in both sides of (14), we obtain

$$(k+2)(k+1)p_{k+2} + 2\left(-g_3^2(k-1)p_{k-1} + g_2kp_k + g_1(k+1)p_{k+1}\right) = -\left(2ng_3^2p_{k-1} + q_0(n)p_k\right) \Rightarrow$$
$$\Rightarrow (k+2)(k+1)p_{k+2} - 2g_3^2(k-1)p_{k-1} + 2g_2kp_k + 2g_1(k+1)p_{k+1} = -2ng_3^2p_{k-1} - q_0(n)p_k$$

Thus, we obtain the four-term recursion relation

$$(k+2)(k+1)p_{k+2} = -2(k+1)g_1p_{k+1} - \left(q_0(n) + 2kg_2\right)p_k + 2(k-n-1)g_3^2p_{k-1} \quad (15)$$

Observe that if $g_1 = 0$, i.e. if the linear term of the exponential polynomial vanishes, (15) becomes a three-term recursion relation, which is typical of a quasi-exactly solvable system **[4, 6]**.
For $k = 0$, dropping p_{-1}, whose index is negative, we obtain from (15)

$$2p_2 = -2g_1p_1 - q_0(n)p_0 \quad (16)$$

For $k = 1, 2, ..., n-2$, all four terms are present in (15).
For $k = n-1$, dropping p_{n+1}, whose index exceeds the degree of $p_n(\tilde{x})$, we obtain from (15)

$$0 = -2ng_1p_n - \left(q_0(n) + 2(n-1)g_2\right)p_{n-1} - 4g_3^2p_{n-2}$$

In the region $\tilde{x} > 0$, $p_n = 1$, and thus the previous equation becomes

$$-2ng_1 - \left(q_0(n) + 2(n-1)g_2\right)p_{n-1} - 4g_3^2p_{n-2} = 0 \quad (17)$$

For $k = n$, dropping p_{n+1} and p_{n+2}, whose indices exceed the degree of $p_n(\tilde{x})$, we obtain from (15)

$$0 = -\left(q_0(n) + 2ng_2\right)p_n - 2g_3^{\ 2}p_{n-1}$$

Using again that $p_n = 1$, the previous equation becomes

$$-\left(q_0(n) + 2ng_2\right) - 2g_3^{\ 2}p_{n-1} = 0$$

Since $g_3^{\ 2} \neq 0$, we end up to

$$p_{n-1} = -\frac{q_0(n) + 2ng_2}{2g_3^{\ 2}} \quad (18)$$

For $k = n+1$, dropping p_{n+1}, p_{n+2}, and p_{n+3}, whose indices exceed the degree of $p_n(\tilde{x})$, we obtain from (15)

$$0 = 2\left(\underbrace{n+1-n-1}_{0}\right)g_3^{\ 2}\underbrace{p_n}_{1} \Rightarrow 0 = 0$$

i.e. for $k = n+1$ the recursion relation (15) holds identically.
This is expected, since for $k = n+1$, we calculate the leading coefficient of the quotient polynomial, which we have precalculated and incorporated into the initial differential equation (14).
Since for $k = n+1$ the recursion relation (15) holds identically, we'll use it for

$k = 0, 1, .., n$,

as we did in the case of the sextic anharmonic oscillator **[6]**.
From (18), we see that p_{n-1} is a first-degree polynomial in $q_0(n)$.
Then from (17), which is written as

$$-2ng_1 - \left(q_0(n) + 2(n-1)g_2\right)p_{n-1} - 4g_3^{\ 2}p_{n-2} = 0 \Rightarrow$$

$$\underbrace{\Rightarrow}_{g_3^{\ 2}\neq 0} p_{n-2} = -\frac{1}{4g_3^{\ 2}}q_0(n)\,p_{n-1} - \frac{(n-1)g_2}{2g_3^{\ 2}}p_{n-1} - \frac{ng_1}{2g_3^{\ 2}},$$

we see that p_{n-2} is a second-degree polynomial in $q_0(n)$.
Thus, since $p_n = 1$, we have

p_{n-0} is a zero-degree polynomial in $q_0(n)$,

p_{n-1} is a first-degree polynomial in $q_0(n)$,

p_{n-2} is a second-degree polynomial in $q_0(n)$.

Besides, the recursion relation (15) is written as

$$2(k-n-1)g_3^{\ 2}p_{k-1} = \left(q_0(n) + 2kg_2\right)p_k + 2(k+1)g_1p_{k+1} + (k+2)(k+1)p_{k+2} =$$
$$= q_0(n)\,p_k + 2kg_2p_k + 2(k+1)g_1p_{k+1} + (k+2)(k+1)p_{k+2}$$

Since $k \leq n \Rightarrow k-n-1 \leq -1 \neq 0$, and also $g_3^{\ 2} \neq 0$, we end up to

$$p_{k-1} = \frac{1}{2(k-n-1)g_3^{\ 2}} q_0(n) p_k + \frac{kg_2}{(k-n-1)g_3^{\ 2}} p_k + \frac{(k+1)g_1}{(k-n-1)g_3^{\ 2}} p_{k+1} + \frac{(k+2)(k+1)}{2(k-n-1)g_3^{\ 2}} p_{k+2} \quad (19)$$

Then, we can easily show by induction that

p_k is a $(n-k)$-degree polynomial in $q_0(n)$, for $k = n, n-1, ..., 0$

Indeed, for $k = n, n-1, n-2$, it holds.
Then, assuming that

p_{k+2} is a polynomial of degree $n-(k+2) = n-k-2$ in $q_0(n)$.

p_{k+1} is a polynomial of degree $n-(k+1) = n-k-1$ in $q_0(n)$.

p_k is a polynomial of degree $n-k$ in $q_0(n)$,

from (19) we derive that

p_{k-1} is a polynomial of degree $n-k+1 = n-(k-1)$ in $q_0(n)$.

Thus

p_2 is a polynomial of degree $n-2$ in $q_0(n)$,

p_1 is a polynomial of degree $n-1$ in $q_0(n)$, and

p_0 is a polynomial of degree n in $q_0(n)$.

Then, (16), i.e. the recursion relation (15) for $k = 0$, is a polynomial equation of degree $n+1$ in $q_0(n)$, and thus it can have up to $n+1$ real roots.
Then, solving (9) for $\tilde{E}$, i.e.

$\tilde{E} = q_0(n) - (g_1^{\ 2} + g_2)$ (20)

we obtain the energies of the respective symmetrized quartic anharmonic oscillator (4), which thus can be up to $n+1$, with n being the degree of the polynomial $p_n(\tilde{x})$.
Since in the one-dimensional bound states there is no degeneracy **[8]**, each energy corresponds to only one eigenstate, and thus, for every value of n, we can find up to $n+1$ eigenstates of the respective symmetrized quartic anharmonic oscillator (4).
Having calculated the polynomial $p_n(\tilde{x})$ in the region $\tilde{x} > 0$, we find its expression in the region $\tilde{x} < 0$ by using that $p_n(\tilde{x})$ is of either even or odd parity, and it satisfies the two continuity conditions (12) and (13).

Examples

n=0

The polynomial $p_0(\tilde{x})$ is of zero degree, i.e. it is a constant, which, in the region $\tilde{x} > 0$, is 1.

Since $p_0(\tilde{x})$ must have definite parity, it can be only of even parity, and then $p_0(\tilde{x})=1$ in the region $\tilde{x}<0$.
The condition (12) is then satisfied, while the condition (13) gives

$$0=0+2g_1 \Rightarrow g_1=0$$

We see that the linear term of the exponential polynomial vanishes.
Besides, for $n=0$, $k=0$, and the recursion relation (15) gives

$$0=-q_0(0)p_0,$$

where we also used that $g_1=0$.
Since $p_0=1$, we end up to

$$q_0(0)=0$$

Then, from (20), the energy is

$$\tilde{E}=-g_2 \quad (21)$$

Also, for $n=0$, (11) gives that $q_1(0)=0$, and since $q_0(0)=0$, the quotient polynomial is zero, i.e. $q_1(\tilde{x};0)=0$.
Then, from (4) we have

$$\tilde{V}\left(\tilde{x};\frac{3}{2},0\right)=g_3'^2(\tilde{x})+g_3''(\tilde{x})+\tilde{E}$$

Then, using (21) and the expressions the first and second derivatives of $g_3(\tilde{x})$ take in the region $\tilde{x}>0$ for $g_1=0$, the potential in the positive region – let us denote it by $\tilde{V}_+$ – is

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},0\right)=\left(-g_3^2\tilde{x}^2+g_2\tilde{x}\right)^2+\left(-2g_3^2\tilde{x}+g_2\right)-g_2=g_3^4\tilde{x}^4+g_2^2\tilde{x}^2-2g_2g_3^2\tilde{x}^3-2g_3^2\tilde{x}$$

That is

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},0\right)=g_3^4\tilde{x}^4-2g_2g_3^2\tilde{x}^3+g_2^2\tilde{x}^2-2g_3^2\tilde{x}$$

Since the potential is symmetric, we end up to

$$\tilde{V}\left(\tilde{x};\frac{3}{2},0\right)=g_3^4\tilde{x}^4-2g_2g_3^2\left|\tilde{x}\right|^3+g_2^2\tilde{x}^2-2g_3^2\left|\tilde{x}\right| \quad (22)$$

Also, the energy eigenfunction (1) becomes

$$\psi\left(\tilde{x};\frac{3}{2},0\right)=A_0\exp\left(-\frac{g_3^2}{3}\left|\tilde{x}\right|^3+\frac{g_2}{2}\tilde{x}^2\right) \quad (23)$$

Since it has no zeros, (23) is the ground-state wave function.
Therefore, the ground state of the symmetrized quartic anharmonic oscillator (22) is described by the even-parity wave function (23) and has energy given by (21).

Setting

$g_1 = -b$, $\frac{g_2}{2} = a$, i.e. $g_2 = 2a$, and $g_3{}^2 = 1$,

the energy (21), the potential (22), and the wave function (23) are respectively written as

$$\tilde{E} = -2a$$

$$\tilde{V}\left(\tilde{x};\frac{3}{2},0\right) = \tilde{x}^4 - 4a|\tilde{x}|^3 + 4a^2\tilde{x}^2 - 2|\tilde{x}|$$

$$\psi\left(\tilde{x};\frac{3}{2},0\right) = A_0 \exp\left(-\frac{1}{3}|\tilde{x}|^3 + a\tilde{x}^2\right)$$

in agreement with Quesne **[3]**.

n=1

The polynomial $p_1(\tilde{x})$ is of first degree, i.e. it is a linear polynomial, and thus it has two coefficients, p_0 and p_1.
For $n=1$, $k=0,1$.
For $k=0$, the recursion relation (15) gives

$$0 = -2g_1 p_1 - q_0(1)\, p_0$$

For $k=1$, the recursion relation (15) gives

$$0 = -\left(q_0(1) + 2g_2\right) p_1 - 2g_3{}^2 p_0$$

In the region $\tilde{x} > 0$, the leading coefficient of $p_1(\tilde{x})$ is 1, i.e. $p_1 = 1$, and then the previous two relations become, respectively,

$$-2g_1 - q_0(1)\, p_0 = 0 \quad (24)$$

$$-\left(q_0(1) + 2g_2\right) - 2g_3{}^2 p_0 = 0 \quad (25)$$

(25) is written as

$$-\left(q_0(1) + 2g_2\right) = 2g_3{}^2 p_0$$

Since $g_3{}^2 \neq 0$, the previous equation gives

$$p_0 = -\frac{q_0(1) + 2g_2}{2g_3{}^2} \quad (26)$$

This is the relation we obtain from (18) for $n=1$.
Substituting (26) into (24) yields

$$-2g_1 - q_0(1)\left(-\frac{q_0(1) + 2g_2}{2g_3{}^2}\right) = 0 \Rightarrow -2g_1 + q_0(1)\frac{q_0(1) + 2g_2}{2g_3{}^2} = 0 \Rightarrow$$

$$\Rightarrow -2g_1 + \frac{1}{2g_3{}^2} q_0{}^2(1) + \frac{g_2}{g_3{}^2} q_0(1) = 0$$

Thus, $q_0(1)$ satisfies the quadratic equation

$$q_0^2(1)+2g_2q_0(1)-4g_1g_3^2=0 \quad (27)$$

The discriminant of the trinomial in the left-hand side is

$$4g_2^2+16g_1g_3^2$$

If the discriminant is non-negative, i.e. if

$$4g_2^2+16g_1g_3^2\geq 0$$

or, since $g_3^2>0$,

$$g_1\geq -\frac{g_2^2}{4g_3^2} \quad (28)$$

the equation (27) has real roots, which are

$$q_0(1)=-g_2\pm\sqrt{g_2^2+4g_1g_3^2} \quad (29)$$

Substituting (29) into (26), we obtain

$$p_0=-\frac{g_2\pm\sqrt{g_2^2+4g_1g_3^2}}{2g_3^2}$$

or

$$p_0=\frac{-g_2\mp\sqrt{g_2^2+4g_1g_3^2}}{2g_3^2}$$

Thus, in the region $\tilde{x}>0$, the polynomial $p_1(\tilde{x})$ is

$$p_1(\tilde{x})=\tilde{x}+\frac{-g_2\mp\sqrt{g_2^2+4g_1g_3^2}}{2g_3^2} \quad (30)$$

The linear polynomial (30) must have definite parity.
i. If

$$-g_2\mp\sqrt{g_2^2+4g_1g_3^2}\neq 0\Rightarrow g_2\neq\mp\sqrt{g_2^2+4g_1g_3^2}\Rightarrow |g_2|\neq\sqrt{g_2^2+4g_1g_3^2}\Rightarrow$$
$$\Rightarrow\underbrace{|g_2|^2}_{g_2^2}\neq g_2^2+4g_1g_3^2\Rightarrow 0\neq 4g_1g_3^2\underset{g_3^2\neq 0}{\Rightarrow} g_1\neq 0,$$

i.e. if $g_1\neq 0$, the polynomial (30) can be only of even parity, and this happens if and only if, in the region $\tilde{x}<0$,

$$p_1(\tilde{x})=-\tilde{x}+\frac{-g_2\mp\sqrt{g_2^2+4g_1g_3^2}}{2g_3^2},$$

and then

$$p_1\left(\tilde{x}\right)=\left|\tilde{x}\right|+\frac{-g_2\mp\sqrt{{g_2}^2+4g_1{g_3}^2}}{2{g_3}^2} \quad (31)$$

The condition (12) is then satisfied, while the condition (13) takes the form

$$-1=1+2g_1\frac{-g_2\mp\sqrt{{g_2}^2+4g_1{g_3}^2}}{2{g_3}^2}$$

or

$$g_1\frac{-g_2\mp\sqrt{{g_2}^2+4g_1{g_3}^2}}{{g_3}^2}=-2 \quad (32)$$

As a consequence of (32), $g_1\neq 0$, and $-g_2\mp\sqrt{{g_2}^2+4g_1{g_3}^2}\neq 0$ which again gives $g_1\neq 0$. Thus, the assumption that $-g_2\mp\sqrt{{g_2}^2+4g_1{g_3}^2}\neq 0$ is consistent with (32), as expected.
From (32), we obtain

$$\frac{-g_2\mp\sqrt{{g_2}^2+4g_1{g_3}^2}}{2{g_3}^2}=-\frac{1}{g_1},$$

and the polynomial (31) becomes

$$p_1\left(\tilde{x}\right)=\left|\tilde{x}\right|-\frac{1}{g_1} \quad (33)$$

ii. If $g_1=0$, the equation (27) becomes

$$q_0^{\,2}\left(1\right)+2g_2q_0\left(1\right)=0\Rightarrow q_0\left(1\right)\left(q_0\left(1\right)+2g_2\right)=0\Rightarrow q_0\left(1\right)=0 \text{ or } q_0\left(1\right)=-2g_2$$

Substituting into (26) the two values of $q_0\left(1\right)$, we obtain, respectively,

$$p_0=-\frac{g_2}{{g_3}^2} \text{ or } p_0=0,$$

and thus, in the region $\tilde{x}>0$,

$$p_1\left(\tilde{x}\right)=\tilde{x}-\frac{g_2}{{g_3}^2} \text{ or } p_1\left(\tilde{x}\right)=\tilde{x},$$

respectively.
Besides, for $g_1=0$, the condition (13) takes the form $p_1'\left(0^-\right)=p_1'\left(0^+\right)$, and since in both previous cases $p_1'\left(0^+\right)=1$, we obtain $p_1'\left(0^-\right)=1$, i.e. the leading term of $p_1\left(\tilde{x}\right)$ in the negative region is also $\tilde{x}$, in both cases.
Then, since $p_1\left(\tilde{x}\right)$ must have definite parity, it can be only of odd parity, and thus the accepted solution is the second, i.e. $p_1\left(\tilde{x}\right)=\tilde{x}$, which corresponds to $p_0=0$, and thus $q_0\left(1\right)=-2g_2$.

The linear polynomial $p_1(\tilde{x}) = \tilde{x}$, for $\tilde{x} \in \mathbb{R}$, satisfies both continuity conditions (12) and (13).
Therefore, if $g_1 = 0$, then

$q_0(1) = -2g_2$ (34)

and

$p_1(\tilde{x}) = \tilde{x}$ (35)

To summarize,
If $g_1 \geq -\frac{g_2^2}{4g_3^2}$ and $g_1 \neq 0$, then

$q_0(1) = -g_2 \pm \sqrt{g_2^2 + 4g_1g_3^2}$ and $p_1(\tilde{x}) = |\tilde{x}| - \frac{1}{g_1}$ (even parity), with

$g_1 \frac{-g_2 \mp \sqrt{g_2^2 + 4g_1g_3^2}}{g_3^2} = -2$.

If $g_1 = 0$, then

$q_0(1) = -2g_2$ and $p_1(\tilde{x}) = \tilde{x}$ (odd parity).

Note that in this case, the condition $g_1 \geq -\frac{g_2^2}{4g_3^2}$, i.e. the non-negativity of the discriminant of (27), holds.

In the first case, i.e. if $g_1 \geq -\frac{g_2^2}{4g_3^2}$ and $g_1 \neq 0$, solving the condition (32) for g_2 yields

$$-g_2 \mp \sqrt{g_2^2 + 4g_1g_3^2} = -\frac{2g_3^2}{g_1} \Rightarrow -g_2 + \frac{2g_3^2}{g_1} = \pm\sqrt{g_2^2 + 4g_1g_3^2} \Rightarrow \left(-g_2 + \frac{2g_3^2}{g_1}\right)^2 = g_2^2 + 4g_1g_3^2 \Rightarrow$$

$$\Rightarrow g_2^2 + \frac{4g_3^4}{g_1^2} - \frac{4g_2g_3^2}{g_1} = g_2^2 + 4g_1g_3^2 \Rightarrow -\frac{4g_2g_3^2}{g_1} = 4g_1g_3^2 - \frac{4g_3^4}{g_1^2} \underbrace{\Rightarrow}_{g_3^2 \neq 0} -\frac{g_2}{g_1} = g_1 - \frac{g_3^2}{g_1^2}$$

Thus

$g_2 = -g_1^2 + \frac{g_3^2}{g_1}$ (36)

Note

For g_2 given by (36), the condition $g_1 \geq -\frac{g_2^2}{4g_3^2}$ holds.

Indeed, using (36), the previous condition becomes

$$g_1 \geq -\frac{\left(-g_1^2+\frac{g_3^2}{g_1}\right)^2}{4g_3^2} = -\frac{g_1^4+\frac{g_3^4}{g_1^2}-2g_1g_3^2}{4g_3^2} = -\frac{g_1^6+g_3^4-2g_1^3g_3^2}{4g_1^2g_3^2}$$

That is

$$g_1 \geq -\frac{g_1^6+g_3^4-2g_1^3g_3^2}{4g_1^2g_3^2}$$

Since $4g_1^2g_3^2>0$, the previous inequality becomes

$$4g_1^3g_3^2 \geq -\left(g_1^6+g_3^4-2g_1^3g_3^2\right) \Rightarrow -4g_1^3g_3^2 \leq g_1^6+g_3^4-2g_1^3g_3^2 \Rightarrow$$

$$\Rightarrow g_1^6+g_3^4+2g_1^3g_3^2 \geq 0 \Rightarrow \left(g_1^3+g_3^2\right)^2 \geq 0$$, which holds.

Besides, from (29) we obtain

$$\mp\sqrt{g_2^2+4g_1g_3^2} = -g_2-q_0(1),$$

and substituting into (32), we obtain

$$g_1\frac{-g_2-g_2-q_0(1)}{g_3^2} = -2 \Rightarrow -2g_2-q_0(1) = -\frac{2g_3^2}{g_1} \Rightarrow q_0(1) = -2g_2+\frac{2g_3^2}{g_1}$$

By means of (36), the last equation becomes

$$q_0(1) = -2\left(-g_1^2+\frac{g_3^2}{g_1}\right)+\frac{2g_3^2}{g_1} = 2g_1^2$$

That is

$$q_0(1) = 2g_1^2 \quad (37)$$

By means of (36) and (37), (20) becomes, for $n=1$,

$$\tilde{E} = 2g_1^2-\left(g_1^2-g_1^2+\frac{g_3^2}{g_1}\right) = 2g_1^2-\frac{g_3^2}{g_1}$$

That is

$$\tilde{E} = 2g_1^2-\frac{g_3^2}{g_1} \quad (38)$$

This is the energy of the eigenstate corresponding to the linear polynomial (33).
Besides, for $n=1$, from (11) we obtain $q_1(1)=2g_3^2$ and thus, using also (37), the quotient polynomial $q_1(\tilde{x};1)$, for the case where $g_1 \geq -\frac{g_2^2}{4g_3^2}$ and $g_1 \neq 0$, is

$$q_1(\tilde{x};1) = 2g_3^2\left|\tilde{x}\right|+2g_1^2 \quad (39)$$

Also, by means of (36), the exponential polynomial (5) becomes

$$g_3\left(\tilde{x}\right)=-\frac{g_3^{\ 2}}{3}\left|\tilde{x}\right|^3+\frac{-g_1^{\ 2}+\dfrac{g_3^{\ 2}}{g_1}}{2}\tilde{x}^2+g_1\left|\tilde{x}\right|=-\frac{g_3^{\ 2}}{3}\left|\tilde{x}\right|^3+\frac{-g_1^{\ 3}+g_3^{\ 2}}{2g_1}\tilde{x}^2+g_1\left|\tilde{x}\right|$$

That is

$$g_3\left(\tilde{x}\right)=-\frac{g_3^{\ 2}}{3}\left|\tilde{x}\right|^3+\frac{-g_1^{\ 3}+g_3^{\ 2}}{2g_1}\tilde{x}^2+g_1\left|\tilde{x}\right|$$

Then, for $\tilde{x}>0$, we have

$$g_3'\left(\tilde{x}\right)=-g_3^{\ 2}\tilde{x}^2+\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}\tilde{x}+g_1$$

$$g_3''\left(\tilde{x}\right)=-2g_3^{\ 2}\tilde{x}+\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}$$

Plugging the previous derivatives, the quotient polynomial (39) for $\tilde{x}>0$, and the energy (38) into the potential (4), we obtain

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},1\right)=\left(-g_3^{\ 2}\tilde{x}^2+\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}\tilde{x}+g_1\right)^2-2g_3^{\ 2}\tilde{x}+\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}-\left(2g_3^{\ 2}\tilde{x}+2g_1^{\ 2}\right)+2g_1^{\ 2}-\frac{g_3^{\ 2}}{g_1}=$$

$$=g_3^{\ 4}\tilde{x}^4+\left(\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}\right)^2\tilde{x}^2+g_1^{\ 2}-2g_3^{\ 2}\left(\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}\right)\tilde{x}^3-2g_1g_3^{\ 2}\tilde{x}^2+2\left(-g_1^{\ 3}+g_3^{\ 2}\right)\tilde{x}-2g_3^{\ 2}\tilde{x}+$$

$$+\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}-2g_3^{\ 2}\tilde{x}-2g_1^{\ 2}+2g_1^{\ 2}-\frac{g_3^{\ 2}}{g_1}=g_3^{\ 4}\tilde{x}^4-2g_3^{\ 2}\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}\tilde{x}^3+\left(\left(\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}\right)^2-2g_1g_3^{\ 2}\right)\tilde{x}^2-$$

$$-2\left(g_1^{\ 3}+g_3^{\ 2}\right)\tilde{x}+g_1^{\ 2}+\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}-\frac{g_3^{\ 2}}{g_1}=$$

$$=g_3^{\ 4}\tilde{x}^4-2g_3^{\ 2}\frac{-g_1^{\ 3}+g_3^{\ 2}}{g_1}\tilde{x}^3+\frac{g_1^{\ 6}+g_3^{\ 4}-2g_1^{\ 3}g_3^{\ 2}-2g_1^{\ 3}g_3^{\ 2}}{g_1^{\ 2}}\tilde{x}^2-2\left(g_1^{\ 3}+g_3^{\ 2}\right)\tilde{x}$$

That is, the potential in the region $\tilde{x}>0$ is

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},1\right)=g_3^{\ 4}\tilde{x}^4+2g_3^{\ 2}\frac{g_1^{\ 3}-g_3^{\ 2}}{g_1}\tilde{x}^3+\frac{g_1^{\ 6}+g_3^{\ 4}-4g_1^{\ 3}g_3^{\ 2}}{g_1^{\ 2}}\tilde{x}^2-2\left(g_1^{\ 3}+g_3^{\ 2}\right)\tilde{x}$$

Since the potential is symmetric, we end up to

$$\tilde{V}\left(\tilde{x};\frac{3}{2},1\right)=g_3^{\ 4}\tilde{x}^4+2g_3^{\ 2}\frac{g_1^{\ 3}-g_3^{\ 2}}{g_1}\left|\tilde{x}\right|^3+\frac{g_1^{\ 6}+g_3^{\ 4}-4g_1^{\ 3}g_3^{\ 2}}{g_1^{\ 2}}\tilde{x}^2-2\left(g_1^{\ 3}+g_3^{\ 2}\right)\left|\tilde{x}\right| \quad (40)$$

Therefore, the symmetrized quartic anharmonic oscillator (40) has an eigenstate of energy (38), which is described by the even-parity wave function

$$\psi\left(\tilde{x};\frac{3}{2},1\right)=A_1\left(\left|\tilde{x}\right|-\frac{1}{g_1}\right)\exp\left(-\frac{g_3^{\ 2}}{3}\left|\tilde{x}\right|^3+\frac{-g_1^{\ 3}+g_3^{\ 2}}{2g_1}\tilde{x}^2+g_1\left|\tilde{x}\right|\right) \quad (41)$$

with $g_1\neq 0$.

Observe that we've expressed the potential, the energy, and the wave function in terms of g_1 and g_3 only, g_2 does not appear.

If $g_1 < 0$, then $|\tilde{x}| - \frac{1}{g_1} > 0$, and then the wave function (41) has no (real) zeros, and thus it describes the ground-state of the symmetrized quartic anharmonic oscillator (40).

If $g_1 > 0$, then the equation $|\tilde{x}| = \frac{1}{g_1}$ has two real roots, at $\tilde{x} = \pm\frac{1}{g_1}$, and then (41) has two zeros, and thus it describes the second-excited state of the symmetrized quartic anharmonic oscillator (40).

Setting

$g_1 = -b$, $\frac{g_2}{2} = a$, i.e. $g_2 = 2a$, and $g_3{}^2 = 1$,

the energy (38), the potential (40), and the wave function (41) are respectively written as

$$\tilde{E} = 2(-b)^2 - \frac{1}{-b} = 2b^2 + \frac{1}{b} = \frac{2b^3+1}{b}$$

$$\tilde{V}\left(\tilde{x};\frac{3}{2},1\right) = \tilde{x}^4 + 2\frac{(-b)^3-1}{-b}|\tilde{x}|^3 + \frac{(-b)^6+1-4(-b)^3}{(-b)^2}\tilde{x}^2 - 2\left((-b)^3+1\right)|\tilde{x}| =$$

$$= \tilde{x}^4 + 2\frac{-b^3-1}{-b}|\tilde{x}|^3 + \frac{b^6+1+4b^3}{b^2}\tilde{x}^2 - 2\left(-b^3+1\right)|\tilde{x}| =$$

$$= \tilde{x}^4 + 2\frac{b^3+1}{b}|\tilde{x}|^3 + \frac{b^6+4b^3+1}{b^2}\tilde{x}^2 + 2\left(b^3-1\right)|\tilde{x}|$$

$$\psi\left(\tilde{x};\frac{3}{2},1\right) = A_1\left(|\tilde{x}| - \frac{1}{-b}\right)\exp\left(-\frac{1}{3}|\tilde{x}|^3 + \frac{-(-b)^3+1}{2(-b)}\tilde{x}^2 + (-b)|\tilde{x}|\right) =$$

$$= A_1\left(|\tilde{x}| + \frac{1}{b}\right)\exp\left(-\frac{1}{3}|\tilde{x}|^3 + \frac{b^3+1}{-2b}\tilde{x}^2 - b|\tilde{x}|\right) = A_1\left(|\tilde{x}| + \frac{1}{b}\right)\exp\left(-\frac{1}{3}|\tilde{x}|^3 - \frac{b^3+1}{2b}\tilde{x}^2 - b|\tilde{x}|\right)$$

in agreement with Quesne **[3]**.

In the second case, i.e. if $g_1 = 0$, then $q_0(1) = -2g_2$, and then (20) gives, for $n = 1$,

$\tilde{E} = -3g_2$ (42)

The leading coefficient of the quotient polynomial is the same as in the previous case, since n remains 1, and thus in this case the quotient polynomial is

$q_1(\tilde{x};1) = 2g_3{}^2|\tilde{x}| - 2g_2$ (43)

Also, for $g_1 = 0$, the exponential polynomial (5) becomes

$$g_3(\tilde{x}) = -\frac{g_3{}^2}{3}|\tilde{x}|^3 + \frac{g_2}{2}\tilde{x}^2$$

Then, in the region $\tilde{x} > 0$, we have

$$g_3'(\tilde{x}) = -g_3{}^2\tilde{x}^2 + g_2\tilde{x}$$

$$g_3''(\tilde{x}) = -2g_3{}^2\tilde{x} + g_2$$

Plugging the previous derivatives, the quotient polynomial (43) for $\tilde{x} > 0$, and the energy (42) into the potential (4), we obtain

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},1\right) = \left(-g_3{}^2\tilde{x}^2 + g_2\tilde{x}\right)^2 - 2g_3{}^2\tilde{x} + g_2 - \left(2g_3{}^2\tilde{x} - 2g_2\right) - 3g_2 =$$
$$= g_3{}^4\tilde{x}^4 + g_2{}^2\tilde{x}^2 - 2g_2g_3{}^2\tilde{x}^3 - 4g_3{}^2\tilde{x}$$

That is, the potential in the region $\tilde{x} > 0$ is

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},1\right) = g_3{}^4\tilde{x}^4 - 2g_2g_3{}^2\tilde{x}^3 + g_2{}^2\tilde{x}^2 - 4g_3{}^2\tilde{x}$$

Since the potential is symmetric, we end up to

$$\tilde{V}\left(\tilde{x};\frac{3}{2},1\right) = g_3{}^4\tilde{x}^4 - 2g_2g_3{}^2\left|\tilde{x}\right|^3 + g_2{}^2\tilde{x}^2 - 4g_3{}^2\left|\tilde{x}\right| \quad (44)$$

Therefore, the symmetrized quartic anharmonic oscillator (44) has an eigenstate of energy (42), which is described by the odd-parity wave function

$$\psi\left(\tilde{x};\frac{3}{2},1\right) = A_1\tilde{x}\exp\left(-\frac{g_3{}^2}{3}\left|\tilde{x}\right|^3 + \frac{g_2}{2}\tilde{x}^2\right) \quad (45)$$

The wave function (45) has one (real) zero, at zero, and thus it describes the first-excited state of the symmetrized quartic anharmonic oscillator (44).
Setting

$g_1 = -b$, $\frac{g_2}{2} = a$, i.e. $g_2 = 2a$, and $g_3{}^2 = 1$,

the energy (42), the potential (44), and the wave function (45) are respectively written as

$$\tilde{E} = -6a$$

$$\tilde{V}\left(\tilde{x};\frac{3}{2},1\right) = \tilde{x}^4 - 4a\left|\tilde{x}\right|^3 + 4a^2\tilde{x}^2 - 4\left|\tilde{x}\right|$$

$$\psi\left(\tilde{x};\frac{3}{2},1\right) = A_1\tilde{x}\exp\left(-\frac{1}{3}\left|\tilde{x}\right|^3 + a\tilde{x}^2\right)$$

in agreement with Quesne **[3]**.
Regarding the case $n = 1$, we observe that the parameter g_1, i.e. the coefficient of the linear term of the exponential polynomial, determines the degree of the excitation of the one known eigenstate of the respective symmetrized quartic anharmonic oscillator.

If the parameter g_1 is negative, the one known eigenstate is the ground state, if it is zero, the one known eigenstate is the first-excited state, while if it is positive, the one known eigenstate is the second-excited state.
We remind that, in the case $n=0$, the parameter g_1 is zero and the one eigenstate we found was the ground state of the respective symmetrized quartic anharmonic oscillator.
Note also that the symmetrized quartic anharmonic oscillators (22) and (44) differ only in the coefficients of their linear terms. This happens because, in both cases, $g_1=0$, and the respective exponential polynomials have the same form.

n=2

The polynomial $p_2(\tilde{x})$ is a second-degree polynomial, and thus it has three coefficients, p_0,p_1, and p_2.
For $n=2$, $k=0,1,2$ in the recursion relation (15).
For $k=0$, the recursion relation (15) gives

$$2p_2=-2g_1p_1-q_0(2)p_0$$

Since $p_2=1$ in the region $\tilde{x}>0$, the previous equation becomes

$$2=-2g_1p_1-q_0(2)p_0 \quad (46)$$

in agreement with (16) for $n=2$.
For $k=1$, using again that $p_2=1$ in the region $\tilde{x}>0$, we obtain from (15)

$$0=-4g_1-\left(q_0(2)+2g_2\right)p_1-4g_3^{\ 2}p_0 \quad (47)$$

For $k=2$, using again that $p_2=1$ in the region $\tilde{x}>0$, we obtain from (15)

$$0=-\left(q_0(2)+4g_2\right)-2g_3^{\ 2}p_1$$

Since $g_3^{\ 2}\neq 0$, the previous equation gives

$$p_1=-\frac{q_0(2)+4g_2}{2g_3^{\ 2}} \quad (48)$$

in agreement with (18) for $n=2$.
Substituting (48) into (47) yields

$$0=-4g_1-\left(q_0(2)+2g_2\right)\left(-\frac{q_0(2)+4g_2}{2g_3^{\ 2}}\right)-4g_3^{\ 2}p_0=$$

$$=-4g_1+\left(q_0(2)+2g_2\right)\frac{q_0(2)+4g_2}{2g_3^{\ 2}}-4g_3^{\ 2}p_0$$

Thus

$$4g_3^{\ 2}p_0=-4g_1+\frac{\left(q_0(2)+2g_2\right)\left(q_0(2)+4g_2\right)}{2g_3^{\ 2}}$$

Since $g_3^{\ 2}\neq 0$, we finally obtain

$$p_0 = -\frac{g_1}{g_3^2} + \frac{(q_0(2)+2g_2)(q_0(2)+4g_2)}{8g_3^4} \quad (49)$$

By means of (48) and (49), (46) becomes

$$2 = -2g_1\left(-\frac{q_0(2)+4g_2}{2g_3^2}\right) - q_0(2)\left(-\frac{g_1}{g_3^2} + \frac{(q_0(2)+2g_2)(q_0(2)+4g_2)}{8g_3^4}\right) =$$

$$= \frac{g_1q_0(2)+4g_1g_2}{g_3^2} + \frac{g_1q_0(2)}{g_3^2} - \frac{q_0(2)(q_0(2)+2g_2)(q_0(2)+4g_2)}{8g_3^4} =$$

$$= \frac{2g_1q_0(2)+4g_1g_2}{g_3^2} - \frac{q_0(2)(q_0^2(2)+6g_2q_0(2)+8g_2^2)}{8g_3^4} =$$

$$= -\frac{q_0^3(2)+6g_2q_0^2(2)+8g_2^2q_0(2)}{8g_3^4} + \frac{2g_1q_0(2)+4g_1g_2}{g_3^2}$$

That is

$$-\frac{q_0^3(2)+6g_2q_0^2(2)+8g_2^2q_0(2)}{8g_3^4} + \frac{2g_1q_0(2)+4g_1g_2}{g_3^2} = 2 \Rightarrow$$

$$\Rightarrow -\left(q_0^3(2)+6g_2q_0^2(2)+8g_2^2q_0(2)\right) + 8g_3^2\left(2g_1q_0(2)+4g_1g_2\right) = 16g_3^4 \Rightarrow$$

$$\Rightarrow q_0^3(2)+6g_2q_0^2(2)+8g_2^2q_0(2) - 8g_3^2\left(2g_1q_0(2)+4g_1g_2\right) = -16g_3^4 \Rightarrow$$

$$\Rightarrow q_0^3(2)+6g_2q_0^2(2)+8g_2^2q_0(2) - 16g_1g_3^2q_0(2) - 32g_1g_2g_3^2 = -16g_3^4$$

Thus, $q_0(2)$ satisfies the following cubic equation

$$q_0^3(2)+6g_2q_0^2(2)+8\left(g_2^2-2g_1g_3^2\right)q_0(2)+16g_3^2\left(g_3^2-2g_1g_2\right)=0 \quad (50)$$

As an odd-degree equation with real coefficients, (50) has at least one real root.
If $g_3^2-2g_1g_2=0$, i.e. if

$$g_3^2 = 2g_1g_2 \quad (51)$$

the equation (50) is written as

$$q_0(2)\left(q_0^2(2)+6g_2q_0(2)+8g_2\left(g_2-4g_1^2\right)\right)=0,$$

and thus

$$q_0(2)=0 \quad (52)$$

or

$$q_0^2(2)+6g_2q_0(2)+8g_2\left(g_2-4g_1^2\right)=0$$

We'll examine the root at zero.
Since g_3^2 is non-zero, from (51) we derive that both g_1 and g_2 are non-zero, and thus all three parameters are non-zero.
Substituting (52) into (48) and (49), we obtain, respectively,

$$p_1 = -\frac{2g_2}{g_3^{\,2}} = -\frac{2g_2}{2g_1g_2} = -\frac{1}{g_1}$$

$$p_0 = -\frac{g_1}{g_3^{\,2}} + \frac{g_2^{\,2}}{g_3^{\,4}} = \frac{g_2^{\,2} - g_1 g_3^{\,2}}{g_3^{\,4}} = \frac{g_2^{\,2} - 2g_1^{\,2}g_2}{\left(2g_1g_2\right)^2} = \frac{g_2\left(g_2 - 2g_1^{\,2}\right)}{4g_1^{\,2}g_2^{\,2}} = \frac{g_2 - 2g_1^{\,2}}{4g_1^{\,2}g_2}$$

where we used (51).
Thus, in the region $\tilde{x} > 0$,

$$p_2\left(\tilde{x}\right) = \tilde{x}^2 - \frac{1}{g_1}\tilde{x} + \frac{g_2 - 2g_1^{\,2}}{4g_1^{\,2}g_2}$$

The polynomial $p_2\left(\tilde{x}\right)$ must have definite parity, and thus

i. If $p_2\left(\tilde{x}\right)$ is of even parity, then

$$p_2\left(\tilde{x}\right) = \tilde{x}^2 - \frac{1}{g_1}\left|\tilde{x}\right| + \frac{g_2 - 2g_1^{\,2}}{4g_1^{\,2}g_2} \quad (53)$$

ii. If $p_2\left(\tilde{x}\right)$ is of odd parity, then p_0 must vanish, i.e.

$$g_2 = 2g_1^{\,2} \quad (54)$$

and

$$p_2\left(\tilde{x}\right) = \begin{cases} \tilde{x}^2 - \dfrac{1}{g_1}\tilde{x}, \ \tilde{x} > 0 \\ -\tilde{x}^2 - \dfrac{1}{g_1}\tilde{x}, \ \tilde{x} < 0 \end{cases} \quad (55)$$

In this case, substituting (54) into (51) yields

$$g_3^{\,2} = 4g_1^{\,3} \quad (56)$$

Note that (54) and (56) give g_2 and $g_3^{\,2}$ in terms of g_1.

Also, from (54), we see that g_2 is positive, while from (56), we see that $g_1^{\,3}$ is also positive, which means that g_1 is positive too.

Thus, for the odd-parity $p_2\left(\tilde{x}\right)$, both g_1 and g_2 are positive.

The polynomials (53) and (55) must satisfy the two continuity conditions (12) and (13).
The condition (12) is satisfied by both polynomials (53) and (55).
The first derivative of the even-parity polynomial (53) is

$$p_2'\left(\tilde{x}\right) = \begin{cases} 2\tilde{x} - \dfrac{1}{g_1}, \ \tilde{x} > 0 \\ 2\tilde{x} + \dfrac{1}{g_1}, \ \tilde{x} < 0 \end{cases}$$

Also, from (53), we have

$$p_2\left(0^+\right)=\frac{g_2-2g_1^{\ 2}}{4g_1^{\ 2}g_2}$$

Thus, for the even-parity $p_2\left(\tilde{x}\right)$, the continuity condition (13) takes the form

$$\frac{1}{g_1}=-\frac{1}{g_1}+2g_1\left(\frac{g_2-2g_1^{\ 2}}{4g_1^{\ 2}g_2}\right)$$

Solving for g_2 yields, after a little algebra,

$$g_2=-\frac{2}{3}g_1^{\ 2} \quad (57)$$

Substituting (57) into (51) yields

$$g_3^{\ 2}=-\frac{4}{3}g_1^{\ 3} \quad (58)$$

Thus, for the even-parity $p_2\left(\tilde{x}\right)$, g_2 and $g_3^{\ 2}$ are also given in terms of g_1.
From (57), we see that g_2 is negative, and from (58) we see that $g_1^{\ 3}$ is negative, which means that g_1 is also negative.
Thus, for the even-parity $p_2\left(\tilde{x}\right)$, both g_1 and g_2 are negative.
Besides, using (57), the even-parity polynomial (53) becomes

$$p_2\left(\tilde{x}\right)=\tilde{x}^2-\frac{1}{g_1}\left|\tilde{x}\right|+\frac{-\frac{2}{3}g_1^{\ 2}-2g_1^{\ 2}}{4g_1^{\ 2}\left(-\frac{2}{3}g_1^{\ 2}\right)}=\tilde{x}^2-\frac{1}{g_1}\left|\tilde{x}\right|+\frac{-2g_1^{\ 2}-6g_1^{\ 2}}{-8g_1^{\ 4}}=\tilde{x}^2-\frac{1}{g_1}\left|\tilde{x}\right|+\frac{1}{g_1^{\ 2}}$$

That is

$$p_2\left(\tilde{x}\right)=\tilde{x}^2-\frac{1}{g_1}\left|\tilde{x}\right|+\frac{1}{g_1^{\ 2}} \quad (59)$$

with $g_1<0$.
The first derivative of the odd-parity polynomial (55) is

$$p_2'\left(\tilde{x}\right)=2\left|\tilde{x}\right|-\frac{1}{g_1}$$

Also, from (55), $p_2\left(0^+\right)=0$, and the continuity condition (13), for the odd-parity $p_2\left(\tilde{x}\right)$, is written as

$$-\frac{1}{g_1}=-\frac{1}{g_1}\Rightarrow 0=0$$

i.e. it holds.
Thus, for $q_0\left(2\right)=0$, we obtain the even-parity polynomial (59) with the conditions (57) and (58), and the odd-parity polynomial (55) with the conditions (54) and (56).

The condition (12) ensures that the polynomials $p_n\left(\tilde{x}\right)$ are continuous at $\tilde{x}=0$, and thus they are defined at $\tilde{x}=0$, with their values at zero being equal to $p_n\left(0^-\right)$ or $p_n\left(0^+\right)$, i.e.

$$p_n\left(0\right)\equiv p_n\left(0^-\right)=p_n\left(0^+\right)$$

Thus, we can include the point $\tilde{x}=0$ in the domain of the two polynomials $p_2\left(\tilde{x}\right)$. Then, the odd-parity polynomial (55) is written as

$$p_2\left(\tilde{x}\right)=\begin{cases}\tilde{x}^2-\dfrac{1}{g_1}\tilde{x},\ \tilde{x}\geq 0\\ -\tilde{x}^2-\dfrac{1}{g_1}\tilde{x},\ \tilde{x}\leq 0\end{cases}$$

or

$$p_2\left(\tilde{x}\right)=\operatorname{sgn}\left(\tilde{x}\right)\tilde{x}^2-\frac{1}{g_1}\tilde{x}\ \ (60)$$

where $\operatorname{sgn}\left(\tilde{x}\right)$ is the sign function, i.e.

$$\operatorname{sgn}\left(\tilde{x}\right)=\begin{cases}1,\ \tilde{x}>0\\ 0,\ \tilde{x}=0\\ -1,\ \tilde{x}<0\end{cases}$$

Now, for the two previous polynomials $p_2\left(\tilde{x}\right)$, we'll calculate the respective symmetrized quartic anharmonic oscillators, the energies, and the eigenfunctions.
For the even-parity polynomial (59), substituting the conditions (57) and (58) into the exponential polynomial (5), we obtain

$$g_3\left(\tilde{x}\right)=-\frac{\left(-\dfrac{4}{3}g_1{}^3\right)}{3}\left|\tilde{x}\right|^3+\frac{\left(-\dfrac{2}{3}g_1{}^2\right)}{2}\tilde{x}^2+g_1\left|\tilde{x}\right|=\frac{4}{9}g_1{}^3\left|\tilde{x}\right|^3-\frac{1}{3}g_1{}^2\tilde{x}^2+g_1\left|\tilde{x}\right|$$

That is

$$g_3\left(\tilde{x}\right)=\frac{4}{9}g_1{}^3\left|\tilde{x}\right|^3-\frac{1}{3}g_1{}^2\tilde{x}^2+g_1\left|\tilde{x}\right|$$

Thus, we obtain the even-parity wave function

$$\psi\left(\tilde{x};\frac{3}{2},2\right)=A_2\left(\tilde{x}^2-\frac{1}{g_1}\left|\tilde{x}\right|+\frac{1}{g_1{}^2}\right)\exp\left(\frac{4}{9}g_1{}^3\left|\tilde{x}\right|^3-\frac{1}{3}g_1{}^2\tilde{x}^2+g_1\left|\tilde{x}\right|\right)\ \ (61)$$

The negativity of g_1 ensures the square integrability of the wave function.

Since g_1 is negative, $\tilde{x}^2-\frac{1}{g_1}\left|\tilde{x}\right|+\frac{1}{g_1{}^2}\geq\frac{1}{g_1{}^2}>0$, and thus the wave function (61) has no (real) zeros, and then it describes the ground state of the symmetrized quartic anharmonic oscillator that we'll calculate now.

Using (11), the leading coefficient of the quotient polynomial for $n=2$ is $q_1(2)=4g_3^{\,2}$, which by means of (58) becomes $q_1(2)=-\frac{16}{3}g_1^{\,3}$.

Then, the quotient polynomial $q_1(\tilde{x};2)$ for $q_0(2)=0$ is

$$q_1(\tilde{x};2)=-\frac{16}{3}g_1^{\,3}|\tilde{x}| \quad (62)$$

Also, substituting $q_0(2)=0$ and (57) into (20), we obtain the ground-state energy, which is

$$\tilde{E}=-\frac{g_1^{\,2}}{3} \quad (63)$$

Observe that the ground-state energy is negative.

Using the exponential polynomial we found, in the region $\tilde{x}>0$ we have

$$g_3'(\tilde{x})=\frac{4}{3}g_1^{\,3}\tilde{x}^2-\frac{2}{3}g_1^{\,2}\tilde{x}+g_1$$

$$g_3''(\tilde{x})=\frac{8}{3}g_1^{\,3}\tilde{x}-\frac{2}{3}g_1^{\,2}$$

Plugging the previous derivatives, the quotient polynomial (62) for $\tilde{x}>0$, and the energy (63) into the potential (4), we obtain

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},2\right)=\left(\frac{4}{3}g_1^{\,3}\tilde{x}^2-\frac{2}{3}g_1^{\,2}\tilde{x}+g_1\right)^2+\frac{8}{3}g_1^{\,3}\tilde{x}-\frac{2}{3}g_1^{\,2}-\left(-\frac{16}{3}g_1^{\,3}\tilde{x}\right)-\frac{g_1^{\,2}}{3}=$$

$$=\frac{16}{9}g_1^{\,6}\tilde{x}^4+\frac{4}{9}g_1^{\,4}\tilde{x}^2+g_1^{\,2}-\frac{16}{9}g_1^{\,5}\tilde{x}^3+\frac{8}{3}g_1^{\,4}\tilde{x}^2-\frac{4}{3}g_1^{\,3}\tilde{x}+\frac{8}{3}g_1^{\,3}\tilde{x}-\frac{2}{3}g_1^{\,2}+\frac{16}{3}g_1^{\,3}\tilde{x}-\frac{g_1^{\,2}}{3}=$$

$$=\frac{16}{9}g_1^{\,6}\tilde{x}^4-\frac{16}{9}g_1^{\,5}\tilde{x}^3+\left(\frac{4}{9}+\frac{8}{3}\right)g_1^{\,4}\tilde{x}^2+\left(-\frac{4}{3}+\frac{8}{3}+\frac{16}{3}\right)g_1^{\,3}\tilde{x}+g_1^{\,2}-\frac{2}{3}g_1^{\,2}-\frac{g_1^{\,2}}{3}=$$

$$=\frac{16}{9}g_1^{\,6}\tilde{x}^4-\frac{16}{9}g_1^{\,5}\tilde{x}^3+\frac{28}{9}g_1^{\,4}\tilde{x}^2+\frac{20}{3}g_1^{\,3}\tilde{x}$$

That is

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},2\right)=\frac{16}{9}g_1^{\,6}\tilde{x}^4-\frac{16}{9}g_1^{\,5}\tilde{x}^3+\frac{28}{9}g_1^{\,4}\tilde{x}^2+\frac{20}{3}g_1^{\,3}\tilde{x}$$

Since the potential is symmetric, we end up to

$$\tilde{V}\left(\tilde{x};\frac{3}{2},2\right)=\frac{4}{3}g_1^{\,3}\left(\frac{4}{3}g_1^{\,3}\tilde{x}^4-\frac{4}{3}g_1^{\,2}|\tilde{x}|^3+\frac{7}{3}g_1\tilde{x}^2+5|\tilde{x}|\right) \quad (64)$$

with $g_1<0$.

Therefore, the ground state of the symmetrized quartic anharmonic oscillator (64) is described by the even-parity wave function (61) and has energy given by (63).

For the odd-parity polynomial (60), substituting the conditions (54) and (56) into the exponential polynomial (5), we obtain

$$g_3(\tilde{x}) = -\frac{4}{3}g_1^{\ 3}|\tilde{x}|^3 + g_1^{\ 2}\tilde{x}^2 + g_1|\tilde{x}|$$

Then, we obtain the odd-parity wave function

$$\psi\left(\tilde{x};\frac{3}{2},2\right) = A_2\left(\operatorname{sgn}(\tilde{x})\tilde{x}^2 - \frac{1}{g_1}\tilde{x}\right)\exp\left(-\frac{4}{3}g_1^{\ 3}|\tilde{x}|^3 + g_1^{\ 2}\tilde{x}^2 + g_1|\tilde{x}|\right)$$

Using that $\operatorname{sgn}(\tilde{x})\tilde{x} = |\tilde{x}|$, we write the wave function as

$$\psi\left(\tilde{x};\frac{3}{2},2\right) = A_2\tilde{x}\left(|\tilde{x}| - \frac{1}{g_1}\right)\exp\left(-\frac{4}{3}g_1^{\ 3}|\tilde{x}|^3 + g_1^{\ 2}\tilde{x}^2 + g_1|\tilde{x}|\right) \quad (65)$$

In this case, the square integrability of the wave function is ensured by the positivity of g_1.

Since g_1 is positive, the wave function (65) has three zeros, at 0, and at $|\tilde{x}| = \frac{1}{g_1}$, i.e. at $\tilde{x} = \pm\frac{1}{g_1}$, and thus it describes the third-excited state of the symmetrized quartic anharmonic oscillator that we'll calculate now.

Using (56), the leading coefficient $q_1(2) = 4g_3^{\ 2}$ of the quotient polynomial for $n = 2$ becomes $q_1(2) = 16g_1^{\ 3}$, and then, the quotient polynomial for this case is

$$q_1(\tilde{x};2) = 16g_1^{\ 3}|\tilde{x}| \quad (66)$$

Also, substituting $q_0(2) = 0$ and (54) into (20), we obtain the third-excited-state energy, which is

$$\tilde{E} = -3g_1^{\ 2} \quad (67)$$

Observe that the third-excited-state energy is negative.

Using the exponential polynomial we found, in the region $\tilde{x} > 0$ we have

$$g_3'(\tilde{x}) = -4g_1^{\ 3}\tilde{x}^2 + 2g_1^{\ 2}\tilde{x} + g_1$$

$$g_3''(\tilde{x}) = -8g_1^{\ 3}\tilde{x} + 2g_1^{\ 2}$$

Plugging the previous derivatives, the quotient polynomial (66) for $\tilde{x} > 0$, and the energy (67) into the potential (4), we obtain

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},2\right) = \left(-4g_1^{\ 3}\tilde{x}^2 + 2g_1^{\ 2}\tilde{x} + g_1\right)^2 - 8g_1^{\ 3}\tilde{x} + 2g_1^{\ 2} - 16g_1^{\ 3}\tilde{x} - 3g_1^{\ 2} =$$
$$= 16g_1^{\ 6}\tilde{x}^4 + 4g_1^{\ 4}\tilde{x}^2 + g_1^{\ 2} - 16g_1^{\ 5}\tilde{x}^3 - 8g_1^{\ 4}\tilde{x}^2 + 4g_1^{\ 3}\tilde{x} - 8g_1^{\ 3}\tilde{x} + 2g_1^{\ 2} - 16g_1^{\ 3}\tilde{x} - 3g_1^{\ 2} =$$
$$= 16g_1^{\ 6}\tilde{x}^4 - 16g_1^{\ 5}\tilde{x}^3 - 4g_1^{\ 4}\tilde{x}^2 - 20g_1^{\ 3}\tilde{x} = 4g_1^{\ 3}\left(4g_1^{\ 3}\tilde{x}^4 - 4g_1^{\ 2}\tilde{x}^3 - g_1\tilde{x}^2 - 5\tilde{x}\right)$$

That is

$$\tilde{V}_+\left(\tilde{x};\frac{3}{2},2\right) = 4g_1^{\ 3}\left(4g_1^{\ 3}\tilde{x}^4 - 4g_1^{\ 2}\tilde{x}^3 - g_1\tilde{x}^2 - 5\tilde{x}\right)$$

Since the potential is symmetric, we end up to

$$\tilde{V}\left(\tilde{x};\frac{3}{2},2\right)=4g_1^{3}\left(4g_1^{3}\tilde{x}^4-4g_1^{2}\left|\tilde{x}\right|^3-g_1\tilde{x}^2-5\left|\tilde{x}\right|\right) \quad (68)$$

with $g_1>0$.

Therefore, the third-excited state of the symmetrized quartic anharmonic oscillator (68) is described by the odd-parity wave function (65) and has energy given by (67).

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