A NEW AXIOM FOR ZFC SET THEORY THAT RESULTS IN A PROBLEM

Abstract

This article adds a new axiom to ZFC that assumes there is a set x which is initially the empty set and thereafter the successor function (S) is instantly applied once in-place to x at each time interval ($½^n$ n>0) in seconds. Next, a very simple question is proposed to ZFC. What is x after one second elapses?

By definition, each time S is applied in-place to x, a new element is inserted into x. So, given that S is applied at each time interval ($½^n$ n>0) then an infinite collection of elements is added to x so, x is countable infinite. On the other hand, since x begins as the empty set and only S is applied to x then x cannot be anything other than a finite natural number. Hence, x is finite. Clearly, in-place counting according to the interval timings ($½^n$ n>0) demonstrates a problem in ZFC.

Keywords: ZFC • Axiom of Infinity • Set Theory • Successor Function • Infinity

1. Introduction

This article proposes a new axiom for ZFC that postulates the existence of a set that supports in-place counting by one. Since any child performs this activity, it is certainly a reasonable concept to include with ZFC. As such, this new axiom postulates the existence of a set x which begins as the null set when $t=0$ and then whenever a time interval $1/2^n$ with $n>0$ elapses, the successor function S is instantly applied in place to x. Since only S is applied to x, with x beginning as the empty set, then x is a valid ZFC hereditary set. Hence, x begins as $0=\varnothing$ then $1=\{\varnothing\}$ then $2=\{\varnothing,\{\varnothing\}\}$ and so on. By the law of excluded middle in ZFC, a set is either finite or infinite. So, after 1 second elapses, x is either finite or for this case countable infinite.

Since $S \equiv n \cup \{n\}$ then each time S is performed in place on x, another unique element is inserted into x. More specifically, if $x=n$ before S is applied to x then $x=n \cup \{n\}$ after S is applied to x. So, x contains one additional element after any application of S. Now, if indeed the sequence $\{1/2^n\}$ is actually infinite and given S is performed at each $1/2^n$ with $n>0$ then S is performed a countable infinite collection of times on x. Therefore, a countable infinite collection of elements is inserted into x, thus x is countable infinite.

On the other hand, given that x begins as $\varnothing$ and only S is applied to x thereafter then by the definition of S, x can only be a finite natural number regardless of the circumstances.

As such, it can be argued under ZFC that x is both finite and countable infinite and that will be formally proven below. Therefore, this new axiom in conjunction with ZFC is inconsistent.

2. A New In-Place Counting Axiom for ZFC

Below are standard definitions.

Definition 1.1.

1. $S(n) = n \cup \{n\}$.
2. n *is a natural number iff* $n = \varnothing$ *or* $n = S(m)$ *for some natural number* m.
3. $0 \equiv \varnothing$.
4. $n+1 \equiv S(n)$.

Also, for simplicity, the following partial sums time sequence $\{t_n\}$ is defined.

Definition 1.2. t_n *for* $n \geq 0$ *is defined as:*

i. $t_0 = 0$,

ii. $t_n = \sum_{k=1}^{n} (1/2)^k$ *for* $n > 0$.

Next, the new axiom is introduced. In words it proposes, when $t = t_0$ the set x is the empty set and when $t = t_n$ for any $n > 0$, the successor function S is instantly applied once in-place to the set x. The following definition below spells out the operation $x \Leftarrow S(x)$.

Definition 1.3. $x \Leftarrow S(x)$ *if and only if the successor function is instantly applied in-place to the set* x. *Specifically,* $x \Leftarrow S(x)$ *inserts the set* x *as an element into the existing set* x.

So, the successor function is iteratively applied in-place to x according to the time sequence $\{t_n\}$. The new axiom is now stated.

ZFC Proposed Axiom (ZPA). $\exists x(t = t_0 \to x = \varnothing \wedge \forall n > 0(t = t_n \to x \Leftarrow S(x)))$.

This axiom requires timings, so for the below, it is assumed that time started at $t = 0$ and the axiom is applied from there such that when the time is $t = 1$, we evaluate the consequences of the axiom. The abbreviation $ZPA(0,1)$ will be used to represent these conditions. The ZFC claim that time takes on every value of the time sequence $\{t_n\}$ when time elapses from 0 to 1 will also be assumed below.

First, a simple induction argument is proven and the result will be used below.

Lemma 1.4. *For any given* $n > 0$, $n = \{0, \ldots, n-1\}$.

PROOF. The argument proceeds by induction with a base case of 1. By definition 1.1, $1 = S(0) = S(\varnothing) = \varnothing \cup \{\varnothing\} = \{\varnothing\} = \{0\}$. So, the lemma is true at $n = 1$. Now assume the induction hypothesis $n > 0$, $n = \{0, \ldots, n-1\}$. By definition 1.1, $n+1 = S(n)$. Combining the induction hypothesis and definition 1.1, $n+1 = S(n) = \{0, \ldots, n-1\} \cup \{n\}$. Thus, $n+1 = \{0, \ldots, n\}$ and the induction is proven. □

Next, it is shown by an induction argument that the axiom does indeed operate as an in-place successor operation. Hence, the set x is an increasing natural number as time elapses from 0 to 1.

Lemma 1.5. *For any given* n, $t = t_n \to x = n$.

PROOF. The argument proceeds by induction. By ZPA, at $n=0$, $t=t_0 \to x=\varnothing=0$. So, the lemma is true with $n=0$. Now assume $t=t_n \to x=n$ is true for $n \geq 0$. Since by assumption time takes on every value of the sequence $\{t_n\}$ when time elapses from 0 to 1 then we can also assume the condition such that $t=t_n$ is true. Next assume time then elapsed from $t=t_n$ to $t=t_{n+1}$. By ZPA, $x \Leftarrow S(x)$ is applied at $t=t_{n+1}$ and since $x=n$ before S is applied to x then $x=S(n)=n+1$ after the application of S in-place to x. Hence, $t=t_{n+1} \to x=n+1$ and the induction is proven. □

The next lemma shows that x is also a set of natural numbers that expands by one element at a time for each $t=t_n$ as time elapses from 0 to 1. This simply results from ZPA and definition 1.1.

Lemma 1.6 *For any given* $n>0$, $t=t_n \to x=\{0,\ldots,n-1\}$.

PROOF. Apply induction with a base case of 1. At $n=1$ by ZPA and definition 1.1, $t=t_1 \to x=S(\varnothing)=\varnothing \cup \{\varnothing\}=\{\varnothing\}=\{0\}$. Hence, $t=t_1 \to x=\{0\}$. Therefore, the lemma is true at $n=1$. Now assume the induction hypothesis $n>0$, $t=t_n \to x=\{0,\ldots,n-1\}$. Then at $t=t_{n+1}$ by ZPA $x=\{0,\ldots,n-1\} \cup \{\{0,\ldots,n-1\}\}$. From lemma 1.4, $n=\{0,\ldots,n-1\}$ so, $x=\{0,\ldots,n-1\} \cup \{n\}=\{0,\ldots,n\}$. Hence, $t=t_{n+1} \to x=\{0,\ldots,n\}$ which completes the induction. □

Note that lemma 1.6 also reveals a mapping between elements of the sequence $\{t_n\}$ and the elements added to x by ZPA. More specifically, $f:\{t_n\}-\{t_0\} \to x$ where $f(t_{n+1})=n$ for $n \geq 0$. Of course, in ZFC this supposedly demonstrates that x is countable infinite.

The next collection of lemmas will use ZFC's law of excluded middle on infinite sets $\exists n \neg P(n) \vee \forall n P(n)$ (LEMI). As a few examples, LEMI is used Chang and Keisler (1990, p. 582) to prove transfinite induction. It is also used by Kunen (1980, p. 19) (2007, p. 37) to argue that a set which satisfies the axiom of infinity contains every natural number. To apply LEMI, assume $\exists n \neg P(n)$ is true and reach a contradiction or $\neg \exists n \neg P(n)$. Then $\forall n P(n)$ is concluded.

As noted by Feferman (p. 25), the indirect method of LEMI is controversial because $\exists n \neg P(n) \vee \forall n P(n)$ for infinite sets is not based on human intuition. More specifically, finite humans cannot grasp every possible element of an infinite set, if such a concept exists. Therefore, there is no way human intuition can certify that $P(n)$ is true at every possible element of an infinite set in order to demonstrate that $\exists n \neg P(n) \vee \forall n P(n)$ is true. So in ZFC, one must accept LEMI as a fundamental principle of logic completely on blind faith.

It is odd that mathematics is supposed to be the most perfect human science and yet its foundation rests on faith, which of course makes the infinitary part a religion rather than a science. In any event, this "faith" turns out to be false, as this article will prove below.

Since $n>0 \wedge t=t_n \to x=n=\{0,\ldots,n-1\}$, it can be seen that the set x has two distinct ZFC set theoretic personalities. One, it is an increasing finite natural number as time elapses toward 1 second. Second, it is also a growing set of natural numbers that grows one element at a time at each $t=t_n$. However, it will be shown further below that these two different personalities are contradictory.

The next lemma uses lemma 1.6 to demonstrate a LEMI argument from ZFC that shows at $t=1$, x contains all natural numbers.

Lemma 1.7. *At* $t=1$, *for all natural numbers* n, $n \in x$.

PROOF. Apply ILEM and assume there is some natural number n such that $n \notin x$. By lemma 1.6, $t=t_{n+1} \to x=\{0,\ldots,n\}$. Since by assumption time takes on every value of the sequence $\{t_n\}$

when time elapses from 0 to 1 then it is the case that $t=t_{n+1}$ evaluates as true between $t=0$ and $t=1$. So, when $t=t_{n+1}$, n is inserted into x at which point $n \in x$. ZPA does not provide for element deletion once an element is inserted into x. But just to be sure that $n \in x$ at $t=1$, assume there was some $m>n+1$ such that $t=t_m \wedge n \notin x$. By lemma 1.6, $t=t_m \rightarrow x=\{0,\ldots,m-1\}$. Since $n<m-1$ then $n \in x$, which is a contradiction. Thus, no such $m>n+1$ exists where $t=t_m \wedge n \notin x$. Therefore by LEMI, for all $m>n+1$, $t=t_m \rightarrow n \in x$ and so once n was inserted into x at $t=t_{n+1}$, it remained in x. Hence, at $t=1$, $n \in x$ which contradicts the assumption $\exists n(n \notin x)$ and then the lemma follows by LEMI. □

Next, it is shown that x only contains natural numbers.

Lemma 1.8. *At $t=1$, for all $y \in x$, y is a natural number.*

PROOF. Apply LEMI and assume there is some $y \in x$ such that y is not a natural number. Based on ZPA, x begins as the empty set and elements are only added to x by the in-place application of S at some $t=t_n$. Hence, y was added to x by the application of S at some time $t=t_n$. By lemma 1.6, $t=t_n \rightarrow x=\{0,\ldots,n-1\}$. So, x contains only natural numbers at $t=t_n$ hence, y is a natural number, which is a contradiction. The lemma then follows by LEMI. □

By lemma 1.7, x contains every possible natural number. Based on this conclusion, next it is proven that x satisfies the axiom of infinity.

The axiom of infinity is stated for clarity.

Axiom of Infinity (INF). $\exists x(0 \in x \wedge \forall y \in x(S(y) \in x))$.

Theorem 1.9. *At $t=1$, the set x satisfies INF.*

PROOF. From lemma 1.7, $0 \in x$. Now assume $\exists y \in x(S(y) \notin x)$. By lemma 1.8, y is a natural number. Hence, $S(y)$ is a natural number by definition 1.1 and then by lemma 1.7 $S(y) \in x$, which is a contradiction. Hence, $\exists y \in x(S(y) \notin x)$ is false and then by LEMI $\forall y \in x(S(y) \in x)$. Thus, the set x satisfies INF. □

But, now there is a problem. At any $t=t_n$, the set x is not only a growing set of natural numbers but, it is also itself a finite natural number as shown by lemma 1.5. Moreover, by the law of excluded middle in ZFC at $t=1$, x is either finite or infinite.

Since ZFC restricts us to the two choices that x is either finite or infinite at $t=1$ then the next theorem assumes x is infinite and that assumption results in a contradiction with the definition of S. So at $t=1$, x is finite.

Theorem 1.10. *At $t=1$, the set x is finite.*

PROOF. At $t=1$, assume x is infinite. We know based on ZPA that x begins as the finite set $\varnothing$ at $t=0$. After that by ZPA, the set x is only changed by a step-by-step application of S. Since x began as a finite set and is only changed by S then in order for x to become infinite some application of S caused x to change from some finite set y to an infinite set. However, by the definition of S, $S(y)=y \cup \{y\}$ and since y is finite then so is $y \cup \{y\}$, which is a contradiction. Hence, x cannot be infinite. Therefore, x is finite at $t=1$. □

Theorem 1.10 simply proves the obvious that the successor function applied to finite sets can only produce finite sets and that state of truth cannot be changed regardless of the speed at which the successor function is applied. Therefore, the conclusion of theorem 1.9, which is based on ILEM, that x is infinite results in a contradiction with the definition of S.

Hence, ZFC's LEMI is not a valid principle of logic since it allowed us to conclude a false statement.

Combining theorems 1.9 and 1.10, at $t = 1$, x is both infinite and finite under ZFC, which is a contradiction. Given the fact that human intuition would agree that in-place incremental counting is an obviously consistent concept then combining the new axiom with ZFC could not invoke a contradiction unless ZFC is inconsistent. So, ZFC is inconsistent is the only valid conclusion as the next theorem shows.

Theorem 1.11. *Under ZFC at $t = 1$, the set x is both infinite and finite. Therefore, ZFC is inconsistent.*

PROOF. The theorem follows directly from the conjunction of theorems 1.9 and 1.10 and the fact that in-place counting is consistent with human intuition. □

3. Conclusions

Surprisingly, ZFC does not have an axiom that supports in-place counting even though that concept is the first exposure most children have with math. As such, this article included such an axiom, ZPA with ZFC. It was shown above that any set that satisfies $ZPA(0,1)$ could only be a finite natural number. It was also shown, using ILEM, that any set that satisfies $ZPA(0,1)$ also satisfies INF. Therefore, INF does not uniquely describe infinite sets and since ZFC claims a set that satisfies INF is necessarily infinite then ZFC is inconsistent.

As such, ZFC never described a concept of actual infinity and therefore, Cantor's infinity has never been anything other than a baseless concept. Moreover, ILEM was eliminated as a valid principle of logic above since it allowed us to conclude the false statement $\forall n(n \in x)$. So, the indirect methods of ILEM have been eliminated as viable proof techniques.

Consequently, without ILEM, any subsequent argument that claims a set of all natural numbers exists is required to directly prove element by element that the set actually contains every possible natural number. However, by using the set x above, with x beginning as the empty set, each time one claims a natural number was verified as being in the set, apply S in-place to x. Then assume the argument completed and all natural numbers were certified as being in the set. But at the argument's completion, x must be a finite natural number regardless of the circumstances, hence all natural numbers were not certified as being in the set, which is a contradiction. In short, ZPA provides a tool which proves there cannot be a set of every possible natural number because such a claim contradicts the assumption that the natural numbers are unbounded.

References

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