Wavefunctions as gravitational waves

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Abstract : This paper explores the implications of associating the components of the wavefunction with a physical dimension: force per unit *mass* – which is, of course, the dimension of acceleration (m/s^2) and gravitational fields. The classical electromagnetic field equations for energy densities, the Poynting vector and spin angular momentum are then re-derived by substituting the electromagnetic N/C unit of field strength (mass per unit *charge*) by the new N/kg = m/s^2 dimension.

The results are elegant and insightful. For example, the energy densities are proportional to the square of the absolute value of the wavefunction and, hence, to the probabilities, which establishes a *physical* normalization condition. Also, Schrödinger's wave equation may then, effectively, be interpreted as a diffusion equation for energy, and the wavefunction itself can be interpreted as a propagating gravitational wave. As an added bonus, concepts such as the Compton scattering radius for a particle, spin angular momentum, and the boson-fermion dichotomy, can also be explained more intuitively. Finally, we show the interpretation may lead to a natural explanation of relativistic length contraction.

While the approach offers a physical interpretation of the wavefunction, the author argues that the *core* of the Copenhagen interpretation revolves around the complementarity principle, which remains unchallenged because the interpretation of amplitude waves as traveling fields does *not* explain the particle nature of matter.

Contents :

Introduction

This paper offers a *physical* interpretation of wave mechanics. We do *not* challenge the complementarity principle: the interpretation of the wavefunction that is offered here explains the *wave* nature of matter only. It explains diffraction and interference of amplitudes but it does *not* explain why a particle will hit the detector *as a particle* (*not* as a wave). Hence, the Copenhagen interpretation of the wavefunction remains relevant: we just push its boundaries.

The basic ideas in this paper stem from a simple observation: the *geometric* similarity between the quantum-mechanical wavefunctions and electromagnetic waves is remarkably similar. The components of both waves are orthogonal to the direction of propagation and to each other. Only the relative phase differs : the electric and magnetic field vectors (**E** and **B**) have the same phase. In contrast, the phase of the real and imaginary part of the (elementary) wavefunction ($\psi = a\cdot e^{-i\cdot\theta} = a\cdot\cos\theta - a\cdot\sin\theta$) differ by 90 degrees ($\pi/2$).[1] Pursuing the analogy, we explore the following question: if the oscillating electric and magnetic field vectors of an electromagnetic wave carry the energy that one associates with the wave, can we analyze the real and imaginary part of the wavefunction in a similar way?

We show the answer is positive and remarkably straightforward. If the physical dimension of the electromagnetic field is expressed in newton per coulomb (force per unit charge), then the physical dimension of the components of the wavefunction may be associated with force per unit mass (newton per kg).[2]

Of course, force over some distance is energy. The question then becomes: what is the energy concept here? Kinetic? Potential? Both?

The similarity between the energy of a (one-dimensional) linear oscillator ($E = m\cdot a^2\cdot\omega^2/2$) and Einstein's relativistic energy equation $E = m\cdot c^2$ inspires us to interpret the energy as a *two*-dimensional oscillation of mass. To assist the reader, we construct a two-piston engine metaphor.[3] We then adapt the formula for the electromagnetic energy density to calculate the energy densities for the wave function. The results are elegant and intuitive: the energy densities are proportional to the square of the absolute value of the wavefunction and, hence, to the probabilities. Schrödinger's wave equation may then, effectively, be interpreted as a diffusion equation for energy itself.

As an added bonus, concepts such as the Compton scattering radius for a particle and spin angular, as well as the boson-fermion dichotomy can be explained in a fully intuitive way.[4] Finally, we show the interpretation may lead to a natural explanation of relativistic length contraction.

Of course, such interpretation is also an interpretation of the wavefunction itself, and the immediate reaction of the reader is predictable: the electric and magnetic field vectors are, somehow, to be looked at as *real* vectors. In contrast, the real and imaginary components of the wavefunction are not. However, this objection needs to be phrased more carefully. First, it may be noted that, in a classical

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[1] Of course, an *actual* particle is localized in space and can, therefore, *not* be represented by the elementary wavefunction $\psi = a\cdot e^{-i\cdot\theta} = a\cdot e^{-i[E\cdot t - \mathbf{p}\cdot\mathbf{x}]/\hbar} = a\cdot(\cos\theta - i\cdot a\cdot\sin\theta)$. We must build a wave *packet* for that: a sum of wavefunctions, each with its own amplitude a_k and its own argument $\theta_k = (E_k\cdot t - \mathbf{p}_k\cdot\mathbf{x})/\hbar$. This is dealt with in this paper as part of the discussion on the mathematical and physical interpretation of the normalization condition.

[2] The N/kg dimension immediately, and naturally, reduces to the dimension of acceleration (m/s^2), thereby facilitating a direct interpretation in terms of Newton's force law.

[3] In physics, a two-*spring* metaphor is more common. Hence, the pistons in the author's *perpetuum mobile* may be replaced by springs.

[4] The author re-derives the equation for the Compton scattering radius in section VII of the paper, and discusses the boson-fermion dichotomy in section VIII.

analysis, the magnetic force is a pseudovector itself.[5] Second, a suitable choice of coordinates may make quantum-mechanical rotation matrices irrelevant.[6]

Therefore, we are of the opinion that this little paper may provide some fresh perspective on the question, thereby further exploring Einstein's basic sentiment in regard to quantum mechanics, which may be summarized as follows: there must be some *physical* explanation for the calculated probabilities.[7]

We will, therefore, start with Einstein's relativistic energy equation ($E = mc^2$) and wonder what it could possibly tell us.

I. Energy as a two-dimensional oscillation of mass

The structural similarity between the relativistic energy formula, the formula for the *total* energy of an oscillator, and the *kinetic* energy of a moving body, is striking:

1. $E = mc^2$
2. $E = m\omega^2/2$
3. $E = mv^2/2$

In these formulas, ω, *v* and *c* all describe some velocity.[8] Of course, there is the 1/2 factor in the $E = m\omega^2/2$ formula[9], but that is exactly the point we are going to explore here: can we think of an oscillation in *two* dimensions, so it stores an amount of energy that is equal to $E = 2\cdot m\cdot\omega^2/2 = m\cdot\omega^2$?

That is easy enough. Think, for example, of a V-2 engine with the pistons at a 90-degree angle, as illustrated below. The 90° angle makes it possible to perfectly balance the counterweight and the

[5] The magnetic force can be analyzed as a relativistic effect (see Feynman II-13-6). The dichotomy between the electric force as a polar vector and the magnetic force as an axial vector disappears in the relativistic four-vector representation of electromagnetism.

[6] For example, when using Schrödinger's equation in a central field (think of the electron around a proton), the use of polar coordinates is recommended, as it ensures the symmetry of the Hamiltonian under all rotations (see Feynman III-19-3)

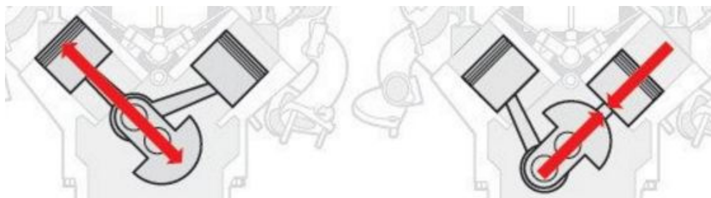
[7] This sentiment is usually summed up in the apocryphal quote: "God does not play dice."The actual quote comes out of one of Einstein's private letters to Cornelius Lanczos, another scientist who had also emigrated to the US. The full quote is as follows: "You are the only person I know who has the same attitude towards physics as I have: belief in the comprehension of reality through something basically simple and unified... It seems hard to sneak a look at God's cards. But that He plays dice and uses 'telepathic' methods... is something that I cannot believe for a single moment." (Helen Dukas and Banesh Hoffman, Albert Einstein, the Human Side: New Glimpses from His Archives, 1979)

[8] Of course, both are different velocities: ω is an *angular* velocity, while *v* is a *linear* velocity: ω is measured in *radians* per second, while *v* is measured in meter per second. However, the definition of a radian implies radians are measured in distance units. Hence, the physical dimensions are, effectively, the same. As for the formula for the total energy of an oscillator, we should actually write: $E = m\cdot a^2\cdot\omega^2/2$. The additional factor (*a*) is the (maximum) amplitude of the oscillator.

[9] We also have a 1/2 factor in the $E = mv^2/2$ formula. Two remarks may be made here. First, it may be noted this is a non-relativistic formula and, more importantly, incorporates kinetic energy only. Using the Lorentz factor (γ), we can write the relativistically correct formula for the kinetic energy as $K.E. = E - E_0 = m_vc^2 - m_0c^2 = m_0\gamma c^2 - m_0c^2 = m_0c^2(\gamma - 1)$. As for the *exclusion* of the potential energy, we may note that we may choose our reference point for the potential energy such that the kinetic and potential energy *mirror* each other. The energy concept that then emerges is the one that is used in the context of the Principle of Least Action: it equals $E = mv^2$. Note 1 provides some comments on that.

pistons, thereby ensuring smooth travel at all times. With permanently closed valves, the air inside the cylinder compresses and decompresses as the pistons move up and down and provides, therefore, a restoring force. As such, it will store potential energy, just like a spring, and the motion of the pistons will also reflect that of a mass on a spring. Hence, we can describe it by a sinusoidal function, with the zero point at the center of each cylinder. We can, therefore, think of the moving pistons as harmonic oscillators, just like mechanical springs.

Figure 1: Oscillations in two dimensions



If we assume there is no friction, we have a *perpetuum mobile* here. The compressed air and the rotating counterweight (which, combined with the crankshaft, acts as a flywheel[10]) store the potential energy. The moving masses of the pistons store the kinetic energy of the system.[11]

At this point, it is probably good to quickly review the relevant math. If the magnitude of the oscillation is equal to a, then the motion of the piston (or the mass on a spring) will be described by $x = a\cdot\cos(\omega\cdot t + \Delta)$.[12] Needless to say, Δ is just a phase factor which defines our $t = 0$ point, and ω is the *natural* angular frequency of our oscillator. Because of the 90° angle between the two cylinders, Δ would be 0 for one oscillator, and $-\pi/2$ for the other. Hence, the motion of one piston is given by $x = a\cdot\cos(\omega\cdot t)$, while the motion of the other is given by $x = a\cdot\cos(\omega\cdot t-\pi/2) = a\cdot\sin(\omega\cdot t)$.

The kinetic and potential energy of *one* oscillator (think of one piston or one spring only) can then be calculated as:

1. $\text{K.E.} = T = m\cdot v^2/2 = (1/2)\cdot m\cdot\omega^2\cdot a^2\cdot\sin^2(\omega\cdot t + \Delta)$
2. $\text{P.E.} = U = k\cdot x^2/2 = (1/2)\cdot k\cdot a^2\cdot\cos^2(\omega\cdot t + \Delta)$

The coefficient k in the potential energy formula characterizes the restoring force: $F = -k\cdot x$. From the dynamics involved, it is obvious that k must be equal to $m\cdot\omega^2$. Hence, the total energy is equal to:

$$E = T + U = (1/2)\cdot m\cdot\omega^2\cdot a^2\cdot[\sin^2(\omega\cdot t + \Delta) + \cos^2(\omega\cdot t + \Delta)] = m\cdot a^2\cdot\omega^2/2$$

To facilitate the calculations, we will briefly assume $k = m\cdot\omega^2$ and a are equal to 1. The motion of our first oscillator is given by the $\cos(\omega\cdot t) = \cos\theta$ function ($\theta = \omega\cdot t$), and its kinetic energy will be equal to $\sin^2\theta$. Hence, the (instantaneous) *change* in kinetic energy at any point in time will be equal to:

$$d(\sin^2\theta)/d\theta = 2\cdot\sin\theta\cdot d(\sin\theta)/d\theta = 2\cdot\sin\theta\cdot\cos\theta$$

Let us look at the second oscillator now. Just think of the second piston going up and down in the V-2 engine. Its motion is given by the sinθ function, which is equal to $\cos(\theta-\pi/2)$. Hence, its kinetic energy is equal to $\sin^2(\theta-\pi/2)$, and how it *changes* – as a function of θ – will be equal to:

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[10] Instead of two cylinders with pistons, one may also think of connecting two springs with a crankshaft.

[11] It is interesting to note that we may look at the energy in the rotating flywheel as *potential* energy because it is energy that is associated with motion, albeit *circular* motion. In physics, one may associate a rotating object with kinetic energy using the rotational equivalent of mass and linear velocity, i.e. rotational inertia (I) and angular velocity ω. The kinetic energy of a rotating object is then given by $\text{K.E.} = (1/2)\cdot I\cdot\omega^2$.

[12] Because of the sideways motion of the connecting rods, the sinusoidal function will describe the linear motion only *approximately,* but you can easily imagine the idealized limit situation.

$$2\cdot\sin(\theta-\pi/2)\cdot\cos(\theta-\pi/2) = = -2\cdot\cos\theta\cdot\sin\theta = -2\cdot\sin\theta\cdot\cos\theta$$

We have our *perpetuum mobile*! While transferring kinetic energy from one piston to the other, the crankshaft will rotate with a constant angular velocity: linear motion becomes circular motion, and vice versa, and the total energy that is stored in the system is $T + U = ma^2\omega^2$.

We have a great *metaphor* here. Somehow, in this beautiful interplay between linear and circular motion, energy is borrowed from one place and then returns to the other, cycle after cycle. We know the wavefunction consist of a sine and a cosine: the cosine is the real component, and the sine is the imaginary component. Could they be equally real? Could each represent *half* of the total energy of our particle? Should we think of the *c* in our $E = mc^2$ formula as an *angular* velocity?

These are sensible questions. Let us explore them.

II. The wavefunction as a two-dimensional oscillation

The elementary wavefunction is written as:

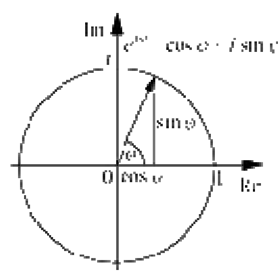
$$\psi = a\cdot e^{-i[\mathrm{E}\cdot t - \mathbf{p}\cdot\mathbf{x}]/\hbar} = a\cdot\cos(\mathbf{p}\cdot\mathbf{x}/\hbar - \mathrm{E}\cdot t/\hbar) + i\cdot a\cdot\sin(\mathbf{p}\cdot\mathbf{x}/\hbar - \mathrm{E}\cdot t/\hbar)$$

When considering a particle at rest ($\mathbf{p} = \mathbf{0}$) this reduces to:

$$\psi = a\cdot e^{-i\cdot\mathrm{E}\cdot t/\hbar} = a\cdot\cos(-\mathrm{E}\cdot t/\hbar) + i\cdot a\cdot\sin(-\mathrm{E}\cdot t/\hbar) = a\cdot\cos(\mathrm{E}\cdot t/\hbar) - i\cdot a\cdot\sin(\mathrm{E}\cdot t/\hbar)$$

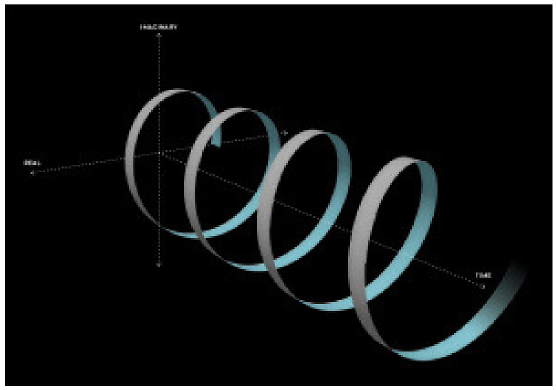
Let us remind ourselves of the geometry involved, which is illustrated below. Note that the argument of the wavefunction rotates *clockwise* with time, while the mathematical convention for measuring the phase angle (φ) is *counter*-clockwise.

Figure 2: Euler's formula



If we assume the momentum **p** is all in the ***x***-direction, then the **p** and **x** vectors will have the same direction, and **p**·**x**/ħ reduces to p·x/ħ. Most illustrations – such as the one below – will either freeze **x** or, else, *t*. Alternatively, one can *google* web animations varying both. The point is: we also have a two-dimensional oscillation here. These two dimensions are perpendicular to the direction of propagation of the wavefunction. For example, if the wavefunction propagates in the *x*-direction, then the oscillations are along the *y*- and *z*-axis, which we may refer to as the real and imaginary axis. Note how the phase difference between the cosine and the sine – the real and imaginary part of our wavefunction – appear to give some spin to the whole. I will come back to this.

Figure 3: Geometric representation of the wavefunction



Hence, *if* we would say these oscillations carry half of the total energy of the particle, then we may refer to the real and imaginary energy of the particle respectively, and the interplay between the real and the imaginary part of the wavefunction may then describe how energy propagates through space over time.

Let us consider, once again, a particle at rest. Hence, p = 0 and the (elementary) wavefunction reduces to $\psi = a \cdot e^{-i \cdot E \cdot t/\hbar}$. Hence, the angular velocity of both oscillations, at some point **x**, is given by $\omega = -E/\hbar$. Now, the energy of our particle includes all of the energy – kinetic, potential and rest energy – and is, therefore, equal to $E = mc^2$.

Can we, somehow, relate this to the $m \cdot a^2 \cdot \omega^2$ energy formula for our V-2 *perpetuum mobile*? Our wavefunction has an amplitude too. Now, if the oscillations of the real and imaginary wavefunction store the energy of our particle, then their amplitude will surely matter. In fact, the energy of an oscillation is, in general, proportional to the *square* of the amplitude: $E \propto a^2$. We may, therefore, think that the a^2 factor in the $E = m \cdot a^2 \cdot \omega^2$ energy will surely be relevant as well.

However, here is a complication: an *actual* particle is localized in space and can, therefore, *not* be represented by the elementary wavefunction. We must build a wave *packet* for that: a sum of wavefunctions, each with their own amplitude a_k, and their own $\omega_i = -E_i/\hbar$. Each of these wavefunctions will *contribute* some energy to the total energy of the wave packet. To calculate the contribution of each wave to the total, both a_i as well as E_i will matter.

What is E_i? E_i varies around some average E, which we can associate with some *average mass* m: m = E/c^2. The Uncertainty Principle kicks in here. The analysis becomes more complicated, but a formula such as the one below might make sense:

$$\mathrm{E} = \sum \mathrm{m}_i \cdot a_i^2 \cdot \omega_i^2 = \sum \frac{\mathrm{E}_i}{c^2} \cdot a_i^2 \cdot \frac{{\mathrm{E}_i}^2}{\hbar^2}$$

We can re-write this as:

$$c^2\hbar^2 = \frac{\sum a_i^2 \cdot \mathrm{E}_i^3}{\mathrm{E}} \Leftrightarrow c^2\hbar^2\mathrm{E} = \sum a_i^2 \cdot \mathrm{E}_i^3$$

What is the meaning of this equation? We may look at it as some sort of *physical* normalization condition when building up the *Fourier sum*. Of course, we should relate this to the *mathematical* normalization condition for the wavefunction. Our intuition tells us that the probabilities must be related to the energy *densities*, but how exactly? We will come back to this question in a moment. Let us first think some more about the enigma: *what is mass?*

Before we do so, let us quickly calculate the value of $c^2ħ^2$: it is about $1×10^{-51}$ $N^2·m^4$. Let us also do a dimensional analysis: the physical dimensions of the E = $m·a^2·ω^2$ equation make sense if we express m in kg, *a* in m, and ω in *rad*/s. We then get: [E] = $kg·m^2/s^2 = (N·s^2/m)·m^2/s^2$ = N·m = J. The dimensions of the left- and right-hand side of the physical normalization condition is $N^3·m^5$.

III. What is mass?

We came up, playfully, with a meaningful interpretation for energy: it is a two-dimensional oscillation of mass. But what is mass? A new *aether* theory is, of course, not an option, but then what *is* it that is oscillating? To understand the physics behind equations, it is always good to do an analysis of the physical dimensions in the equation. Let us start with Einstein's energy equation once again. If we want to look at mass, we should re-write it as $m = E/c^2$:

$$[m] = [E/c^2] = J/(m/s)^2 = N·m·s^2/m^2 = N·s^2/m = kg$$

This is not very helpful. It only reminds us of Newton's definition of a mass: mass is that what gets accelerated by a force. At this point, we may want to think of the physical significance of the *absolute* nature of the speed of light. Einstein's $E = mc^2$ equation implies we can write the ratio between the energy and the mass of *any* particle is always the same, so we can write, for example:

$$\frac{E_{electron}}{m_{electron}} = \frac{E_{proton}}{m_{proton}} = \frac{E_{photon}}{m_{photon}} = \frac{E_{any\ particle}}{m_{any\ particle}} = c^2$$

This reminds us of the $ω^2= C^{-1}/L$ or $ω^2 = k/m$ of harmonic oscillators once again.[13] The key difference is that the $ω^2= C^{-1}/L$ and $ω^2 = k/m$ formulas introduce *two* or more degrees of freedom.[14] In contrast, c^2= E/m for *any* particle, *always*. However, that is exactly the point: we can modulate the resistance, inductance and capacitance of electric circuits, and the stiffness of springs and the masses we put on them, but we live in *one* physical space only: *our* spacetime. Hence, the speed of light *c* emerges here as *the* defining property of spacetime – the resonant frequency, so to speak. We have no further degrees of freedom here.

The Planck-Einstein relation (for photons) and the *de Broglie* equation (for matter-particles) have an interesting feature: both imply that the *energy* of the oscillation is proportional to the frequency, with Planck's constant as the constant of proportionality. Now, for *one-dimensional* oscillations – think of a guitar string, for example – we know the energy will be proportional to the *square* of the frequency.[15] It is a remarkable observation: the two-dimensional matter-wave, or the electromagnetic wave, gives us *two* waves for the price of one, so to speak, each carrying *half* of the *total* energy of the oscillation but, as a result, we get an $E \propto f$ instead of an $E \propto f^2$ proportionality.

However, such reflections do not answer the fundamental question we started out with: what *is* mass? At this point, it is hard to go beyond the circular definition that is implied by Einstein's formula: energy is

[13] The $ω^2= 1/LC$ formula gives us the natural or resonant frequency for a electric circuit consisting of a resistor (*R*), an inductor (*L*), and a capacitor (*C*). Writing the formula as $ω^2= C^{-1}/L$ introduces the concept of *elastance*, which is the equivalent of the mechanical stiffness (k) of a spring.

[14] The resistance in an electric circuit introduces a damping factor. When analyzing a mechanical spring, one may also want to introduce a drag coefficient. Both are usually defined as a fraction of the *inertia*, which is the mass for a spring and the inductance for an electric circuit. Hence, we would write the resistance for a spring as γ*m* and as R = γ*L* respectively.

[15] This is a general result and is reflected in the K.E. = T = $(1/2)·m·ω^2·a^2·\sin^2(ω·t + Δ)$ and the P.E. = U = $k·x^2/2$ = $(1/2)·m·ω^2·a^2·\cos^2(ω·t + Δ)$ formulas for the linear oscillator.

a two-dimensional oscillation of mass, and mass packs energy, and *c* emerges us as the property of spacetime that defines *how* exactly.

When everything is said and done, this does not go beyond stating that mass is some scalar field. Now, a scalar field is, quite simply, some real *number* that we associate with a position in spacetime. The Higgs field is a scalar field but, of course, the theory behind it goes much beyond stating that we should think of mass as some scalar field. The fundamental question is: why and how does energy, or matter, *condense* into elementary particles? That is what the Higgs *mechanism* is about but, as this paper is exploratory only, we cannot even start explaining the basics of it.

What we *can* do, however, is look at the wave *equation* again (Schrödinger's equation), as we can now analyze it as an energy diffusion equation.

IV. Schrödinger's equation as an energy diffusion equation

The interpretation of Schrödinger's equation as a diffusion equation is straightforward. Feynman (*Lectures*, III-16-1) briefly summarizes it as follows:

> "We can think of Schrödinger's equation as describing the diffusion of the probability amplitude from one point to the next. […] But the imaginary coefficient in front of the derivative makes the behavior completely different from the ordinary diffusion such as you would have for a gas spreading out along a thin tube. Ordinary diffusion gives rise to real exponential solutions, whereas the solutions of Schrödinger's equation are complex waves."[16]

Let us review the basic math. For a particle moving in free space – with no external force fields acting on it – there is no potential (U = 0) and, therefore, the Uψ term disappears. Therefore, Schrödinger's equation reduces to:

$$\partial\psi(\mathbf{x}, t)/\partial t = i\cdot(1/2)\cdot(\hbar/m_{eff})\cdot\nabla^2\psi(\mathbf{x}, t)$$

The ubiquitous diffusion equation in physics is:

$$\partial\phi(\mathbf{x}, t)/\partial t = D\cdot\nabla^2\phi(\mathbf{x}, t)$$

The *structural* similarity is obvious. The key difference between both equations is that the wave equation gives us *two* equations for the price of one. Indeed, because ψ is a complex-valued function, with a *real* and an *imaginary* part, we get the following equations[17]:

1. $Re(\partial\psi/\partial t) = -(1/2)\cdot(\hbar/m_{eff})\cdot Im(\nabla^2\psi)$
2. $Im(\partial\psi/\partial t) = (1/2)\cdot(\hbar/m_{eff})\cdot Re(\nabla^2\psi)$

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[16] Feynman further formalizes this in his *Lecture on Superconductivity* (Feynman, III-21-2), in which he refers to Schrödinger's equation as the "equation for continuity of probabilities". The analysis is centered on the *local* conservation of energy, which confirms the interpretation of Schrödinger's equation as an energy diffusion equation.

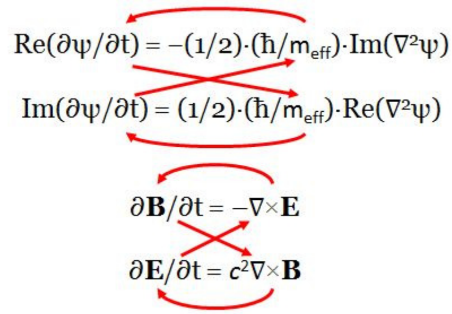
[17] The m_{eff} is the *effective* mass of the particle, which depends on the medium. For example, an electron traveling in a solid (a transistor, for example) will have a different effective mass than in an atom. In free space, we can drop the subscript and just write m_{eff} = m. Note 2 provides some additional comments on the concept. As for the equations, they are easily derived from noting that two complex numbers $a + i\cdot b$ and $c + i\cdot d$ are equal if, and only if, their real and imaginary parts are the same. Now, the $\partial\psi/\partial t = i\cdot(\hbar/m_{eff})\cdot\nabla^2\psi$ equation amounts to writing something like this: $a + i\cdot b = i\cdot(c + i\cdot d)$. Now, remembering that $i^2 = -1$, you can easily figure out that $i\cdot(c + i\cdot d) = i\cdot c + i^2\cdot d = -d + i\cdot c$.

These equations make us think of the equations for an electromagnetic wave in free space (no stationary charges or currents):

1. $\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}$
2. $\partial\mathbf{E}/\partial t = c^2\nabla\times\mathbf{B}$

The above equations effectively describe a *propagation* mechanism in spacetime, as illustrated below.

Figure 4: Propagation mechanisms



The Laplacian operator (∇^2), when operating on a *scalar* quantity, gives us a flux density, i.e. something expressed per square meter ($1/m^2$). In this case, it is operating on ψ(**x**, t), so what is the dimension of our wavefunction ψ(**x**, t)? To answer that question, we should analyze the diffusion constant in Schrödinger's equation, i.e. the $(1/2)\cdot(\hbar/m_{eff})$ factor:

1. As a *mathematical* constant of proportionality, it will *quantify* the relationship between both derivatives (i.e. the time derivative and the Laplacian);
2. As a *physical* constant, it will ensure the *physical dimensions* on both sides of the equation are compatible.

Now, the $\hbar/m_{eff}$ factor is expressed in $(N\cdot m\cdot s)/(N\cdot s^2/m) = m^2/s$. Hence, it does ensure the dimensions on both sides of the equation are, effectively, the same: ∂ψ/∂t is a time derivative and, therefore, its dimension is s^{-1} while, as mentioned above, the dimension of $\nabla^2\psi$ is m^{-2}. However, this does not solve our basic question: what is the dimension of the real and imaginary part of our wavefunction?

At this point, mainstream physicists will say: it does not have a physical dimension, and there is no geometric interpretation of Schrödinger's equation. One may argue, effectively, that its argument, (**p**·**x** – E·*t*)/ħ, is just a number and, therefore, that the real and imaginary part of ψ is also just some number.

To this, we may object that ħ may be looked as a *mathematical* scaling constant only. If we do that, the argument of ψ will, effectively, be expressed in *action* units, i.e. in N·m·s. It then does make sense to also associate a physical dimension with the real and imaginary part of ψ. What could it be?

We may have a closer look at Maxwell's equations for inspiration here. The electric field vector is expressed in *newton* (the unit of force) per unit of *charge* (*coulomb*). Now, there is something interesting here. The physical dimension of the magnetic field is N/C *divided* by m/s.[18] We may write **B** as the following vector cross-product: $\mathbf{B} = (1/c)\cdot\boldsymbol{e}_x\times\mathbf{E}$, with $\boldsymbol{e}_x$ the unit vector pointing in the *x*-direction (i.e. the direction of propagation of the wave). Hence, we may associate the $(1/c)\cdot\boldsymbol{e}_x\times$ *operator*, which amounts to a rotation by 90 degrees, with the s/m dimension. Now, multiplication by *i* also amounts to a

[18] The dimension of **B** is usually written as N/(m·A), using the SI unit for current, i.e. the *ampere* (A). However, 1 C = 1 A·s and, hence, 1 N/(m·A) = 1 (N/C)/(m/s).

rotation by 90° degrees. Hence, we may boldly write: **B** = (1/c)·$\boldsymbol{e_x}$×**E** = (1/c)·i·**E**. This allows us to also geometrically interpret Schrödinger's equation in the way we interpreted it above (see Figure 3).[19]

Still, we have not answered the question as to what the physical dimension of the real and imaginary part of our wavefunction should be. At this point, we may be inspired by the structural similarity between Newton's and Coulomb's force laws:

$$F = k_e \frac{q_1 \cdot q_2}{r^2}$$

$$F = G \frac{m_1 \cdot m_2}{r^2}$$

Hence, if the electric field vector **E** is expressed in force per unit *charge* (N/C), then we may want to think of associating the real part of our wavefunction with a force per unit *mass* (N/kg). We can, of course, do a substitution here, because the mass unit (1 kg) is equivalent to 1 N·s^2/m. Hence, our N/kg dimension becomes:

$$\text{N/kg} = \text{N/(N·s}^2\text{/m)} = \text{m/s}^2$$

What is this: m/s^2? Is *that* the dimension of the $a·cos\theta$ term in the $a·e^{-i\theta} = a·cos\theta - i·a·sin\theta$ wavefunction?

My answer is: why not? Think of it: m/s^2 is the physical dimension of *acceleration*: the increase or decrease in velocity (m/s) per second. It ensures the wavefunction for *any* particle – matter-particles or particles with zero rest mass (photons) – and the associated wave *equation* (which has to be the same for all, as the spacetime we live in is *one*) are mutually consistent.

In this regard, we should think of how we would model a *gravitational* wave. The physical dimension would surely be the same: force per mass unit. It all makes sense: wavefunctions may, perhaps, be interpreted as traveling distortions of spacetime, i.e. as tiny gravitational waves.

V. Energy densities and flows

Pursuing the geometric equivalence between the equations for an electromagnetic wave and Schrödinger's equation, we can now, perhaps, see if there is an equivalent for the energy density. For an electromagnetic wave, we know that the energy density is given by the following formula:

$$u = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{\epsilon_0 \cdot c^2}{2}\mathbf{B}\cdot\mathbf{B}$$

E and **B** are the electric and magnetic field vector respectively. The Poynting vector will give us the directional energy flux, i.e. the energy flow per unit area per unit time. We write:

$$\frac{\partial u}{\partial t} = -\nabla \cdot \boldsymbol{S}$$

[19] Of course, multiplication with i amounts to a *counter*clockwise rotation. Hence, multiplication by $-i$ also amounts to a rotation by 90 degrees, but *clockwise*. Now, to uniquely identify the clockwise and counterclockwise directions, we need to establish the equivalent of the right-hand rule for a proper geometric interpretation of Schrödinger's equation in three-dimensional space: if we look at a clock from the back, then its hand will be moving *counter*clockwise. When writing **B** = (1/c)·i·**E**, we assume we are looking in the *negative* x-direction. If we are looking in the positive x-direction, we should write: **B** = –(1/c)·i·**E**. Of course, Nature does not care about our conventions. Hence, both should give the same results in calculations. We will show in a moment they do.

Needless to say, the $\nabla\cdot$ operator is the divergence and, therefore, gives us the magnitude of a (vector) field's *source* or *sink* at a given point. To be precise, the divergence gives us the volume density of the outward *flux* of a vector field from an infinitesimal volume around a given point. In this case, it gives us the *volume density* of the flux of ***S***.

We can analyze the dimensions of the equation for the energy density as follows:

1. **E** is measured in *newton per coulomb*, so [**E**·**E**] = [E^2] = N^2/C^2.
2. **B** is measured in (N/C)/(m/s), so we get [**B**·**B**] = [B^2] = (N^2/C^2)·(s^2/m^2). However, the dimension of our c^2 factor is (m^2/s^2) and so we are also left with N^2/C^2.
3. The ϵ_0 is the electric constant, aka as the vacuum permittivity. As a *physical* constant, it should ensure the dimensions on both sides of the equation work out, and they do: [ε_0] = $C^2/(N\cdot m^2)$ and, therefore, if we multiply that with N^2/C^2, we find that *u* is expressed in J/m^3.[20]

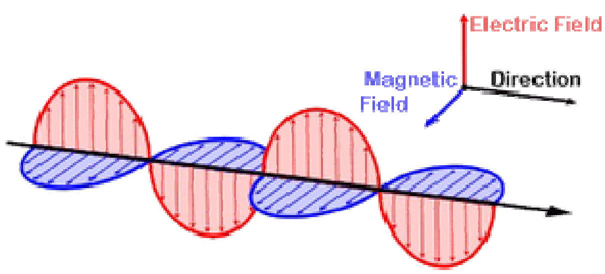
Replacing the *newton per coulomb* unit (N/C) by the *newton per kg* unit (N/kg) in the formulas above should give us the equivalent of the energy density for the wavefunction. We just need to substitute ϵ_0 for an equivalent constant. We may give it a try. If the energy densities can be calculated – which are also mass densities, obviously – then the probabilities should be proportional to them.

Let us first see what we get for a photon, assuming the electromagnetic wave represents its wavefunction. Substituting **B** for (1/*c*)·*i*·**E** or for −(1/*c*)·*i*·**E** gives us the following result:

$$u = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{\epsilon_0 \cdot c^2}{2}\mathbf{B}\cdot\mathbf{B} = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{\epsilon_0 \cdot c^2}{2}\frac{i\cdot\mathbf{E}}{c}\frac{i\cdot\mathbf{E}}{c} = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} - \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} = 0$$

Zero. An unexpected result? Perhaps not. We have no stationary charges and no currents: only an electromagnetic wave in free space. Hence, the local energy conservation principle needs to be respected at all points in space and in time. The geometry makes sense of the result: for an electromagnetic wave, the magnitudes of **E** and **B** reach their maximum, minimum and zero point *simultaneously*, as shown below.[21] This is because their *phase* is the same.

Figure 5: Electromagnetic wave: **E** and **B**



Should we expect a similar result for the energy densities that we would associate with the real and imaginary part of the matter-wave? For the matter-wave, we have a phase difference between *a·cos*θ and *a·sin*θ, which gives a different picture of the *propagation* of the wave (see Figure 3).[22] In fact, the geometry of the suggestion suggests some inherent spin, which is interesting. I will come back to this. Let us first guess those densities. Making abstraction of any scaling constants, we may write:

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[20] In fact, when multiplying $C^2/(N\cdot m^2)$ with N^2/C^2, we get N/m^2, but we can multiply this with 1 = m/m to get the desired result. It is significant that an energy density (*joule* per unit *volume*) can also be measured in *newton* (force per unit *area*.

[21] The illustration shows a linearly polarized wave, but the obtained result is general.

[22] The sine and cosine are essentially the same functions, except for the difference in the phase: sinθ = cos(θ−π /2).

$$u = a^2(cos\theta)^2 + a^2(-i \cdot sin\theta)^2 = a^2\ (cos^2\theta + sin^2\theta) = a^2$$

We get what we hoped to get: the absolute square of our amplitude is, effectively, an energy density !

$$|\psi|^2 = |a \cdot e^{-i \cdot E \cdot t/\hbar}|^2 = a^2 = u$$

This is very deep. A photon has no rest mass, so it borrows and returns energy from empty space as it travels through it. In contrast, a matter-wave carries energy and, therefore, has some (*rest*) mass. It is therefore associated with an energy density, and this energy density gives us the probabilities. Of course, we need to fine-tune the analysis to account for the fact that we have a wave packet rather than a single wave, but that should be feasible.

As mentioned, the phase difference between the real and imaginary part of our wavefunction (a cosine and a sine function) appear to give some spin to our particle. We do not have this particularity for a photon. Of course, photons are bosons, i.e. spin-zero particles, while elementary matter-particles are fermions with spin-1/2. Hence, our geometric interpretation of the wavefunction suggests that, after all, there may be some more intuitive explanation of the fundamental dichotomy between bosons and fermions, which puzzled even Feynman:

> "Why is it that particles with half-integral spin are Fermi particles, whereas particles with integral spin are Bose particles? We apologize for the fact that we cannot give you an elementary explanation. An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity. He has shown that the two must necessarily go together, but we have not been able to find a way of reproducing his arguments on an elementary level. It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation. The explanation is deep down in relativistic quantum mechanics. This probably means that we do not have a complete understanding of the fundamental principle involved." (Feynman, *Lectures*, III-4-1)

The *physical* interpretation of the wavefunction, as presented here, may provide some better understanding of 'the fundamental principle involved': *the physical dimension of the oscillation is just very different*. That is all: it is force per unit charge for photons, and force per unit mass for matter-particles. We will examine the question of spin somewhat more carefully in section VII. Let us first examine the matter-wave some more.

VI. Group and phase velocity of the matter-wave

The geometric representation of the matter-wave (see Figure 3) suggests a traveling wave and, yes, of course: the matter-wave effectively *travels* through space and time. But *what is traveling, exactly*? It is the pulse – or the *signal* – only: the *phase* velocity of the wave is just a mathematical concept and, even in our physical interpretation of the wavefunction, the same is true for the *group* velocity of our wave packet. The oscillation is two-dimensional, but perpendicular to the direction of travel of the wave. Hence, nothing actually moves *with* our particle.

Here, we should also reiterate that we did not answer the question as to *what* is oscillating up and down and/or sideways: we only associated a *physical* dimension with the components of the wavefunction – *newton* per *kg* (force per unit mass), to be precise. We were inspired to do so because of the physical dimension of the electric and magnetic field vectors (*newton* per *coulomb*, i.e. force per unit charge) we associate with electromagnetic waves which, for all practical purposes, we currently treat as the wavefunction for a photon. This made it possible to calculate the associated *energy densities* and a *Poynting vector* for energy dissipation. In addition, we showed that Schrödinger's equation itself then becomes a diffusion equation for energy. However, let us now focus some more on the asymmetry which is introduced by the phase difference between the real and the imaginary part of the wavefunction. Look at the mathematical shape of the elementary wavefunction once again:

$$\psi = a \cdot e^{-i[E \cdot t - \mathbf{p} \cdot \mathbf{x}]/\hbar} = a \cdot e^{-i[E \cdot t - \mathbf{p} \cdot \mathbf{x}]/\hbar} = a \cdot cos(\mathbf{p} \cdot \mathbf{x}/\hbar - E \cdot t/\hbar) + i \cdot a \cdot sin(\mathbf{p} \cdot \mathbf{x}/\hbar - E \cdot t/\hbar)$$

The minus sign in the argument of our sine and cosine function defines the direction of travel: an $F(x - v \cdot t)$ wavefunction will always describe some wave that is traveling in the *positive* x-direction (with c the wave velocity), while an $F(x + v \cdot t)$ wavefunction will travel in the *negative* x-direction. For a geometric interpretation of the wavefunction *in three dimensions*, we need to agree on how to define i or, what amounts to the same, a convention on how to define clockwise and counterclockwise directions: if we look at a clock from the back, then its hand will be moving *counter*clockwise. So we need to establish the equivalent of the right-hand rule. However, let us not worry about that now. Let us focus on the interpretation. To ease the analysis, we will assume we are looking at a particle at rest. Hence, $\mathbf{p} = \mathbf{0}$, and the wavefunction reduces to:

$$\psi = a \cdot e^{-i \cdot E \cdot t/\hbar} = a \cdot cos(-E \cdot t/\hbar) + i \cdot a \cdot sin(-E_0 \cdot t/\hbar) = a \cdot cos(E_0 \cdot t/\hbar) - i \cdot a \cdot sin(E_0 \cdot t/\hbar)$$

E_0 is, of course, the *rest* mass of our particle and, now that we are here, we should probably wonder *whose* time *t* we are talking about: is it *our* time, or is the proper time of our particle? Well... In this situation, we are both at rest so it does not matter: t *is*, effectively, the proper time so perhaps we should write it as t_0. It does not matter. You can see what we expect to see: $E_0/\hbar$ pops up as the *natural* frequency of our matter-particle: $(E_0/\hbar) \cdot t = \omega \cdot t$. Remembering the $\omega = 2\pi \cdot f = 2\pi/T$ and $T = 1/f$ formulas, we can associate a period and a frequency with this wave, using the $\omega = 2\pi \cdot f = 2\pi/T$. Noting that $\hbar = h/2\pi$, we find the following:

$$T = 2\pi \cdot (\hbar/E_0) = h/E_0 \Leftrightarrow f = E_0/h = m_0 c^2/h$$

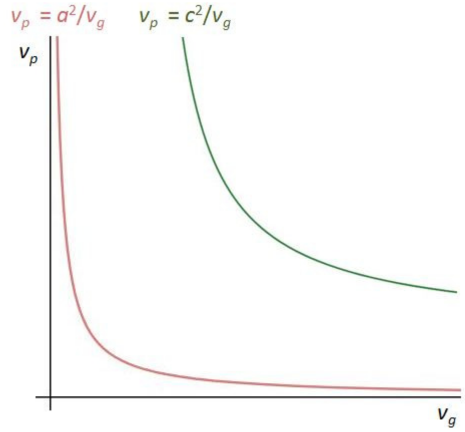
This is interesting, because we can look at the period as a *natural* unit of time for our particle. What about the wavelength? That is tricky because we need to distinguish between group and phase velocity here. The group velocity (v_g) should be zero here, because we assume our particle does not move. In contrast, the phase velocity is given by $v_p = \lambda \cdot f = (2\pi/k) \cdot (\omega/2\pi) = \omega/k$. In fact, we've got something funny here: the wavenumber $k = p/\hbar$ is zero, because we assume the particle is at rest, so $p = 0$. So we have a division by zero here, which is rather strange. What do we get assuming the particle is *not* at rest? We write:

$$v_p = \omega/k = (E/\hbar)/(p/\hbar) = E/p = E/(m \cdot v_g) = (m \cdot c^2)/(m \cdot v_g) = c^2/v_g$$

This is interesting: it establishes a reciprocal relation between the phase and the group velocity, with c as a simple scaling constant. Indeed, the graph below shows the *shape* of the function does *not* change with the value of c, and we may also re-write the relation above as:

$$v_p/c = \beta_p = c/v_p = 1/\beta_g = 1/(c/v_p)$$

Figure 6: Reciprocal relation between phase and group velocity



We can also write the mentioned relationship as $v_p \cdot v_g = c^2$, which reminds us of the relationship between the electric and magnetic constant $(1/\varepsilon_0)\cdot(1/\mu_0) = c^2$. This is interesting in light of the fact we can re-write this as $(c\cdot\varepsilon_0)\cdot(c\cdot\mu_0) = 1$, which shows electricity and magnetism are just two sides of the same coin, so to speak.[23]

Interesting, but how do we interpret the math? What about the implications of the zero value for wavenumber k = p/ħ? We would probably like to think it implies the elementary wavefunction should always be associated with *some* momentum, because the concept of zero momentum clearly leads to weird math: something times *zero* cannot be equal to c^2! Such interpretation is also consistent with the Uncertainty Principle: if $\Delta x \cdot \Delta p \geq \hbar$, then *neither* Δx *nor* Δp can be zero. In other words, the Uncertainty Principle tells us that the idea of a pointlike particle actually *being* at some *specific* point in time and in space does not make sense: it *has* to move. It tells us that our concept of dimensionless points in time and space are *mathematical* notions only. *Actual* particles - including photons - are always a bit spread out, so to speak, and - importantly - they *have to* move.

For a photon, this is self-evident. It has no rest mass, no rest energy, and, therefore, it is going to move at the speed of light itself. We write: $p = m\cdot c = m\cdot c^2/c = E/c$. Using the relationship above, we get:

$$v_p = \omega/k = (E/\hbar)/(p/\hbar) = E/p = c \Rightarrow v_g = c^2/v_p = c^2/c = c$$

This is good: we started out with some reflections on the *matter*-wave, but here we get an interpretation of the electromagnetic wave as a wavefunction for the photon. But let us get back to our matter-wave. In regard to our interpretation of a particle *having to* move, we should remind ourselves, once again, of the fact that an *actual* particle is always localized in space and that it can, therefore, *not* be represented by the elementary wavefunction $\psi = a\cdot e^{-i[E\cdot t - \mathbf{p}\cdot\mathbf{x}]/\hbar}$ or, for a particle at rest, the $\psi = a\cdot e^{-i\cdot E\cdot t/\hbar}$ function. We must build a wave *packet* for that: a sum of wavefunctions, each with their own amplitude a_i, and their own $\omega_i = -E_i/\hbar$. Indeed, in section II, we showed that each of these wavefunctions will *contribute* some energy to the total energy of the wave packet and that, to calculate the contribution of each wave to the total, both a_i as well as E_i matter. This may or may not resolve the apparent paradox. Let us look at the group velocity.

To calculate a meaningful group velocity, we must assume the $v_g = \partial\omega_i/\partial k_i = \partial(E_i/\hbar)/\partial(p_i/\hbar) = \partial(E_i)/\partial(p_i)$ exists. So we must have some *dispersion relation*. How do we calculate it? We need to calculate ω_i as a

[23] I must thank a physics blogger for re-writing the $1/(\varepsilon_0\cdot\mu_0) = c^2$ equation like this. See: http://reciprocal.systems/phpBB3/viewtopic.php?t=236 (retrieved on 29 September 2017).

function of k_i here, or E_i as a function of p_i. How do we do that? Well... There are a few ways to go about it but one interesting way of doing it is to re-write Schrödinger's equation as we did, i.e. by distinguishing the real and imaginary parts of the $\partial\psi/\partial t = i\cdot[\hbar/(2m)]\cdot\nabla^2\psi$ wave equation and, hence, re-write it as the following *pair* of two equations:

1. $Re(\partial\psi/\partial t) = -[\hbar/(2m_{eff})]\cdot Im(\nabla^2\psi) \Leftrightarrow \omega\cdot\cos(kx - \omega t) = k^2\cdot[\hbar/(2m_{eff})]\cdot\cos(kx - \omega t)$
2. $Im(\partial\psi/\partial t) = [\hbar/(2m_{eff})]\cdot Re(\nabla^2\psi) \Leftrightarrow \omega\cdot\sin(kx - \omega t) = k^2\cdot[\hbar/(2m_{eff})]\cdot\sin(kx - \omega t)$

Both equations imply the following dispersion relation:

$$\omega = \hbar\cdot k^2/(2m_{eff})$$

Of course, we need to think about the subscripts now: we have ω_i, k_i, but... What about m_{eff} or, dropping the subscript, m? Do we write it as m_i? If so, what is it? Well... It is the *equivalent* mass of E_i obviously, and so we get it from the mass-energy equivalence relation: $m_i = E_i/c^2$. It is a fine point, but one most people forget about: they usually just write m. However, if there is uncertainty in the energy, then Einstein's mass-energy relation tells us we must have some uncertainty in the (equivalent) mass too. Here, I should refer back to Section II: E_i varies around some *average* energy E and, therefore, the Uncertainty Principle kicks in.

Let us put this aside for the moment and focus on something else. If you look at the illustrations above, you see we can sort of distinguish (1) a linear velocity – the speed with which those wave crests (or troughs) move – and (2) some kind of circular or tangential velocity – the velocity along the red contour line above. We'll need the formula for a tangential velocity: $v_t = a\cdot\omega$.

Now, if λ is zero, then $v_t = a\cdot\omega = a\cdot E/\hbar$ is just all there is. We may double-check this as follows: the distance traveled in one period will be equal to $2\pi a$, and the period of the oscillation is $T = 2\pi\cdot(\hbar/E)$. Therefore, v_t will, effectively, be equal to $v_t = 2\pi a/(2\pi\hbar/E) = a\cdot E/\hbar$.

However, if λ is non-zero, then the distance traveled in one period will be equal to $2\pi a + \lambda$. The period remains the same: $T = 2\pi\cdot(\hbar/E)$. Hence, we can write:

$$v_t = \frac{\Delta s}{\Delta t} = \frac{2\pi a + \lambda}{2\pi\hbar/E} = \frac{2\pi a}{2\pi\hbar/E} + \frac{\lambda/p}{2\pi\hbar/E} = a\cdot\frac{E}{\hbar} + \frac{E}{p} = v_p + a\cdot\frac{E}{\hbar}$$

Now, in the next section, we will be calculating a formula for *a*. In fact, we will find *a* is just the (reduced) Compton radius (if we are considering an electron, at least). We will equate an angular momentum formula (for an electron) with the actual $+\hbar/2$ or $-\hbar/2$ values of its spin and, remarkably, we do get the Compton radius, which is the scattering radius for an electron. Let us write it out:

$$\frac{a^2\cdot E^2}{2\cdot\hbar\cdot c^2} = \frac{\hbar}{2} \Leftrightarrow a^2 = \frac{\hbar^2\cdot c^2}{E^2} = \frac{\hbar^2}{m^2\cdot c^2} \Leftrightarrow a = \frac{\hbar}{m\cdot c}$$

Substituting the various constants with their numerical values, we find that *a* is equal 3.8616×10^{-13} m. Here, however, we will just want to substitute the formula itself:

$$v_t = v_p + a\cdot\frac{E}{\hbar} = v_p + \frac{\hbar}{m\cdot c}\cdot\frac{E}{\hbar} = v_p + \frac{m\cdot c^2}{m\cdot c} = c + v_p$$

This is fascinating ! Of course, if v_p is equal to *c*, then we find this tangential velocity is equal to:

$$v_t = 2c$$

Hence, the *tangential* velocity is *twice* the *linear* velocity. Of course, the question is: what is the *physical* significance of this? Here, we need to remind ourselves, once again, that wave velocities are, essentially, *mathematical* concepts only: the wave propagates through space, but *nothing else* is really moving. However, the geometric implications are obviously quite interesting and, hence, further exploration might yield other fascinating insights. Some of our initial thoughts on this are further elaborated below.

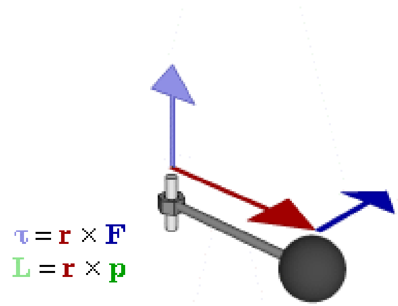
VII. Explaining spin

The elementary wavefunction *vector* – i.e. the vector sum of the real and imaginary component – rotates around the *x*-axis, which gives us the direction of propagation of the wave (see Figure 3). Its *magnitude* remains constant. In contrast, the magnitude of the electromagnetic vector – defined as the vector sum of the electric and magnetic field vectors – oscillates between zero and some maximum (see Figure 5).

We already mentioned that the *rotation* of the wavefunction vector appears to give some *spin* to the particle. Of course, a *circularly* polarized wave would also appear to have spin (think of the **E** and **B** vectors *rotating around* the direction of propagation - as opposed to oscillating up and down or sideways only). In fact, a circularly polarized light does carry angular momentum, as the *equivalent mass* of its energy may be thought of as rotating as well. But so here we are looking at a *matter*-wave.

The basic idea is the following: *if* we look at $\psi = a \cdot e^{-i \cdot E \cdot t/\hbar}$ as some *real* vector – as a two-dimensional oscillation of mass, to be precise – *then* we may associate its rotation around the direction of propagation with some torque. The illustration below reminds of the math here.

Figure 7: Torque and angular momentum vectors



A torque on some mass about a fixed axis gives it *angular momentum*, which we can write as the vector cross-product $\mathbf{L} = \boldsymbol{r} \times \mathbf{p}$ or, perhaps easier for our purposes here as the product of an *angular* velocity (**ω**) and rotational inertia (I), aka as the *moment of inertia* or the *angular mass*. We write:

$$\mathbf{L} = \mathrm{I} \cdot \boldsymbol{\omega}$$

Note we can write **L** and **ω** in **boldface** here because they are (axial) vectors. If we consider their magnitudes only, we write $L = I \cdot \omega$ (no boldface). We can now do some calculations. Let us start with the angular velocity. In our previous posts, we showed that the *period* of the matter-wave is equal to $T = 2\pi \cdot (\hbar/E_0)$. Hence, the angular velocity must be equal to:

$$\omega = 2\pi/[2\pi \cdot (\hbar/E_0)] = E_0/\hbar$$

We also know the distance *r*, so that is the magnitude of ***r*** in the $\mathbf{L} = \boldsymbol{r} \times \mathbf{p}$ vector cross-product: it is just *a*, so that is the *magnitude* of $\psi = a \cdot e^{-i \cdot E \cdot t/\hbar}$. Now, the momentum (**p**) is the product of a *linear* velocity (***v***) - in this case, the *tangential* velocity - and some mass (m): $\mathbf{p} = m \cdot \boldsymbol{v}$. If we switch to *scalar* instead of vector quantities, then the (tangential) velocity is given by $v = r \cdot \omega$. So now we only need to think about what we should use for m or, if we want to work with the *angular* velocity (ω), the *angular* mass (I). Here we need to make some assumption about the mass (or energy) *distribution*. Now, it may or may not sense to assume the energy in the oscillation – and, therefore, the mass – is distributed uniformly. In that case, we may use the formula for the angular mass of a solid cylinder: $I = m \cdot r^2/2$. If we keep the analysis non-relativistic, then $m = m_0$. Of course, the energy-mass equivalence tells us that $m_0 = E_0/c^2$. Hence, this is what we get:

$$L = I \cdot \omega = (m_0 \cdot r^2/2) \cdot (E_0/\hbar) = (1/2) \cdot a^2 \cdot (E_0/c^2) \cdot (E_0/\hbar) = a^2 \cdot E_0^2/(2 \cdot \hbar \cdot c^2)$$

Does it make sense? Maybe. Maybe not. Let us do a dimensional analysis: that won't check our logic, but it makes sure we made no mistakes when mapping mathematical and physical spaces. We have $m^2 \cdot J^2 = m^2 \cdot N^2 \cdot m^2$ in the numerator and $N \cdot m \cdot s \cdot m^2/s^2$ in the denominator. Hence, the dimensions work out: we get N·m·s as the dimension for L, which is, effectively, the physical dimension of angular momentum. It is also the *action* dimension, of course, and that cannot be a coincidence. Also note that the $E = mc^2$ equation allows us to re-write it as:

$$L = a^2 \cdot E_0^2/(2 \cdot \hbar \cdot c^2)$$

Of course, in quantum mechanics, we associate spin with the *magnetic* moment of a *charged* particle, not with its *mass* as such. Is there way to link the formula above to the one we have for the quantum-mechanical angular momentum, which is also measured in N·m·s units, and which can only take on one of two possible values: $J = +\hbar/2$ and $-\hbar/2$? It looks like a long shot, right? How do we go from $(1/2) \cdot a^2 \cdot m_0^2/\hbar$ to $\pm (1/2) \cdot \hbar$? Let us do a numerical example. The energy of an electron is typically 0.510 MeV $\approx 8.1871 \times 10^{-14}$ N·m, and a… What value should we take for a?

We have an obvious *trio* of candidates here: the Bohr radius, the classical electron radius (aka the Thompon scattering length), and the Compton scattering radius.

Let us start with the Bohr radius, so that is about $0.\times 10^{-10}$ N·m. We get $L = a^2 \cdot E_0^2/(2 \cdot \hbar \cdot c^2) = 9.9 \times 10^{-31}$ N·m·s. Now that is about 1.88×10^4 *times* $\hbar/2$. That is a *huge* factor. It cannot be right.

Let us try the classical electron radius, which is about 2.818×10^{-15} m. We get an L that is equal to about 2.81×10^{-39} N·m·s, so now it is a tiny *fraction* of $\hbar/2$! This, too, does not work.

Let us use the Compton scattering length, so that is about 2.42631×10^{-12} m. This gives us an L of 2.08×10^{-13} N·m·s, which is only 20 times ħ. This is not so bad, but it is good enough?

Let us calculate it the other way around: what value should we *take* for a so as to ensure $L = a^2 \cdot E_0^2/(2 \cdot \hbar \cdot c^2) = \hbar/2$? Let us write it out:

$$\frac{a^2 \cdot \mathrm{E_0}^2}{2 \cdot \hbar \cdot c^2} = \frac{\hbar}{2} \Leftrightarrow a^2 = \frac{\hbar^2 \cdot c^2}{\mathrm{E_0}^2} = \frac{\hbar^2}{\mathrm{m_0}^2 \cdot c^2} \Leftrightarrow a = \frac{\hbar}{\mathrm{m_0} \cdot c}$$

In fact, this is the formula for the so-called *reduced* Compton wavelength. This is perfect. We found what we wanted to find. Substituting this value for a (you can calculate it: it is about 3.8616×10^{-13} m), we get what we should find:

$$L = \frac{a^2 \cdot \mathrm{E_0}^2}{2 \cdot \hbar \cdot c^2} = J = \frac{\hbar}{2} = 5.272859 \times 10^{-35} \mathrm{N \cdot m \cdot s}$$

This is a rather spectacular result, and one that would – a priori – support the interpretation of the wavefunction that is being suggested in this paper.

Of course, if we can calculate some radius, then we should, perhaps, also try to calculate other dimensions. Note 3 to this paper explores this possibility.[24]

[24] The analysis is rather primitive because the author limits it to one-dimensional space only. However, the results are interesting.

VIII. The boson-fermion dichotomy

Let us do some more thinking on the boson-fermion dichotomy. Again, we should remind ourselves that an *actual* particle is localized in space and that it can, therefore, *not* be represented by the elementary wavefunction $\psi = a \cdot e^{-i[E \cdot t - \mathbf{p} \cdot \mathbf{x}]/\hbar}$ or, for a particle at rest, the $\psi = a \cdot e^{-i \cdot E \cdot t/\hbar}$ function. We must build a wave *packet* for that: a sum of wavefunctions, each with their own amplitude a_i, and their own $\omega_i = -E_i/\hbar$. Each of these wavefunctions will *contribute* some energy to the total energy of the wave packet. Now, we can have another wild but logical theory about this.

Think of the apparent right-handedness of the elementary wavefunction: surely, *Nature* can't be bothered about our convention of measuring phase angles clockwise or counterclockwise. Also, the angular momentum can be positive or negative: $J = +\hbar/2$ or $-\hbar/2$. Hence, we would probably like to think that an actual particle - think of an electron, or whatever other particle you'd think of - may consist of right-handed as well as left-handed elementary waves. To be precise, we may think they *either* consist of (elementary) right-handed waves or, *else*, of (elementary) left-handed waves. An elementary right-handed wave would be written as:

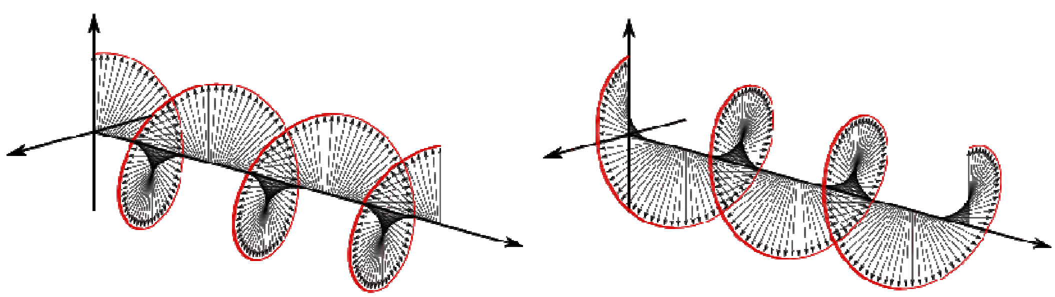
$$\psi(\theta_i) = a_i \cdot (\cos\theta_i + i \cdot \sin\theta_i)$$

In contrast, an elementary left-handed wave would be written as:

$$\psi(\theta_i) = a_i \cdot (\cos\theta_i - i \cdot \sin\theta_i)$$

Both are illustrated below.

Figure 8: Left- and right-handed matter-wave



How does that work out with the $E_0 \cdot t$ argument of our wavefunction? Position is position, and direction is direction, but time? Time has only one direction, but *Nature* surely does not care how we *count* time: counting like 1, 2, 3, etcetera or like −1, −2, −3, etcetera is just the same. If we count like 1, 2, 3, etcetera, then we write our wavefunction like:

$$\psi = a \cdot \cos(E_0 \cdot t/\hbar) - i \cdot a \cdot \sin(E_0 \cdot t/\hbar)$$

If we count time like −1, −2, −3, etcetera then we write it as:

$$\psi = a \cdot \cos(-E_0 \cdot t/\hbar) - i \cdot a \cdot \sin(-E_0 \cdot t/\hbar) = a \cdot \cos(E_0 \cdot t/\hbar) + i \cdot a \cdot \sin(E_0 \cdot t/\hbar)$$

Hence, it is just like the left- or right-handed circular polarization of an electromagnetic wave: we can have both for the matter-wave too! This, then, should explain why we can have *either* positive *or* negative quantum-mechanical spin ($+\hbar/2$ or $-\hbar/2$). It is the usual thing: we have two *mathematical* possibilities here, and so we *must* have two *physical* situations that correspond to it.

It is only natural. If we have left- and right-handed photons - or, generalizing, left- and right-handed bosons - then we should also have left- and right-handed fermions (electrons, protons, etcetera). Back to the dichotomy. The textbook analysis of the dichotomy between bosons and fermions may be epitomized by Richard Feynman's *Lecture* on it (Feynman, III-4), which is confusing and – I would dare to say – even inconsistent: how are photons or electrons supposed to *know* that they need to interfere with a positive or a negative sign? They are not supposed to *know* anything: *knowledge* is part of our *interpretation* of whatever it is that is going on there.

Hence, it is probably best to keep it simple, and think of the dichotomy in terms of the different *physical* dimensions of the oscillation: newton per kg versus newton per coulomb. And then, of course, we should also note that matter-particles have a rest mass and, therefore, actually *carry* charge. Photons do not. But both are two-dimensional oscillations, and the point is: the so-called *vacuum* - and the *rest mass* of our particle (which is zero for the photon and non-zero for everything else) - give us the natural frequency for both oscillations, which is beautifully summed up in that remarkable equation for the group and phase velocity of the wavefunction, which applies to photons as well as matter-particles:

$$(v_{phase}\cdot c)\cdot(v_{group}\cdot c) = 1 \Leftrightarrow v_p\cdot v_g = c^2$$

The final question then is: why are photons spin-zero particles? Well... We should first remind ourselves of the fact that they do have spin when circularly polarized.[25] Here we may think of the rotation of the equivalent mass of their energy. However, if they are linearly polarized, then there is no spin. Even for circularly polarized waves, the spin angular momentum of photons is a weird concept. If photons have no (rest) mass, then they cannot carry any *charge*. They should, therefore, not have any *magnetic* moment. Indeed, what I wrote above shows an explanation of quantum-mechanical spin requires both mass *as well as* charge.[26]

IX. An explanation of relativistic length contraction?

As for the wave velocity, and its direction of propagation, we know that the (phase) velocity of any wave F(kx – ωt) is given by v_p = ω/k = (E/ħ)/(p/ħ) = E/p. Of course, the momentum might also be in the negative x-direction, in which case k would be equal to –p and, therefore, we would get a negative phase velocity: v_p = ω/k = –E/p.

E/ħ = ω gives the frequency in *time* (expressed in *radians* per second), while p/ħ = k gives us the wavenumber, or the frequency in *space* (expressed in *radians* per meter). Of course, we may write: *f* = ω/2π and λ = 2π/k, which gives us the two *de Broglie* relations:

1. E = ħ·ω = h·*f*
2. p = ħ·k = h/λ

The frequency in time is easy to interpret. The wavefunction of a particle with more energy, or more mass, will have a *higher density in time* than a particle with less energy.

In contrast, the second *de Broglie* relation is somewhat harder to interpret. According to the p = h/ λ relation, the wavelength is *inversely* proportional to the momentum: λ = h/p. The velocity of a photon,

[25] A circularly polarized electromagnetic wave may be analyzed as consisting of two perpendicular electromagnetic plane waves of equal amplitude and 90° difference in phase.

[26] Of course, the reader will now wonder: what about neutrons? How to explain neutron spin? Neutrons are neutral. That is correct, but neutrons are not elementary: they consist of (charged) quarks. Hence, neutron spin can (or should) be explained by the spin of the underlying quarks.

or a (theoretical) particle with zero rest mass ($m_0 = 0$), is *c* and, therefore, we find that $p = m_v \cdot v = m_c \cdot c = m \cdot c$ (all of the energy is kinetic). Hence, we can write: $p \cdot c = m \cdot c^2 = E$, which we may also write as: $E/p = c$. Hence, for a particle with zero rest mass, the wavelength can be written as:

$$\lambda = h/p = hc/E = h/mc$$

However, this is a limiting situation – applicable to photons only. Real-life *matter*-particles should have *some* mass[27] and, therefore, their velocity will never be *c*.[28]

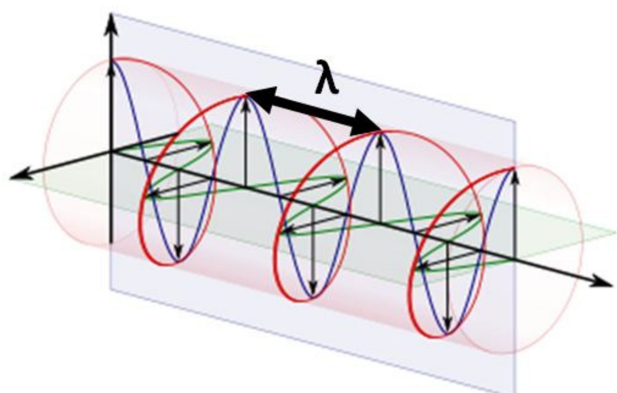
Hence, if p goes to zero, then the wavelength becomes infinitely long: if $p \to 0$ then $\lambda \to \infty$. How should we interpret this inverse proportionality between λ and p? To answer this question, let us first see what this wavelength λ actually represents.

If we look at the $\psi = a \cdot cos(p \cdot x/\hbar - E \cdot t/\hbar) - i \cdot a \cdot sin(p \cdot x/\hbar - E \cdot t/\hbar)$ once more, and if we write $p \cdot x/\hbar$ as Δ, then we can look at $p \cdot x/\hbar$ as a *phase factor*, and so we will be interested to know for what x this phase factor $\Delta = p \cdot x/\hbar$ will be equal to 2π. So we write:

$$\Delta = p \cdot x/\hbar = 2\pi \Leftrightarrow x = 2\pi \cdot \hbar/p = h/p = \lambda$$

So *now* we get a meaningful interpretation for that wavelength. It is the distance between the crests (or the troughs) of the wave, so to speak, as illustrated below. Of course, this two-dimensional wave has no real crests or troughs: we measure crests and troughs against the y-axis here. Hence, our definition depend on the frame of reference.

Figure 9: The *de Broglie* wavelength



Now we know what λ actually represent for our one-dimensional elementary wavefunction. Now, the time that is needed for one cycle is equal to $T = 1/f = 2\pi \cdot (\hbar/E)$. Hence, we can now calculate the wave velocity:

$$v = \lambda/T = (h/p)/[2\pi \cdot (\hbar/E)] = E/p$$

[27] Even neutrinos have some (rest) mass. This was first confirmed by the US-Japan Super-Kamiokande collaboration in 1998. Neutrinos oscillate between three so-called flavors: electron neutrinos, muon neutrinos and *tau* neutrinos. Recent data suggests that the *sum* of their masses is less than a millionth of the rest mass of an electron. Hence, they propagate at speeds that are very near to the speed of light.

[28] Using the Lorentz factor (γ), we can write the relativistically correct formula for the kinetic energy as $KE = E - E_0 = m_v c^2 - m_0 c^2 = m_0 \gamma c^2 - m_0 c^2 = m_0 c^2(\gamma - 1)$. As *v* approaches *c*, γ approaches infinity and, therefore, the kinetic energy would become infinite as well.

Unsurprisingly, we just get the phase velocity that we had calculated already: $v = v_p = E/p$. The question remains: what if p is zero? What if we are looking at some particle at rest? It is an intriguing question: we get an infinitely long wavelength, and an infinite wave velocity.

Now, re-writing the $v = E/p$ as $v = m \cdot c^2/m \cdot v_g = c/\beta_g$, in which β_g is the relative *classical* velocity[29] of our particle $\beta_g = v_g/c$) tells us that the *phase* velocities will effectively be superluminal ($\beta_g < 1$ so $1/\beta_g > 1$), but what if β_g approaches zero? The conclusion seems unavoidable: for a particle at rest, we only have a frequency *in time*, as the wavefunction reduces to:

$$\psi = a \cdot e^{-i \cdot E \cdot t/\hbar} = a \cdot cos(E \cdot t/\hbar) - i \cdot a \cdot sin(E \cdot t/\hbar)$$

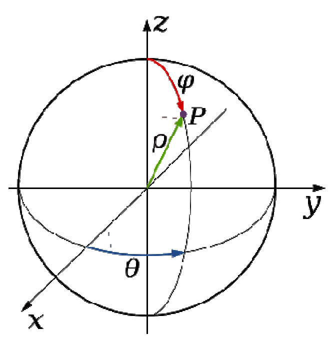
How should we interpret this?

We argued that the oscillations of the wavefunction pack energy. Because the energy of our particle is *finite*, the wave train cannot be infinitely long. Now, if we would assume the wave train consists of a more or less precise *number* of oscillations, then the *string* of oscillations will be shorter as λ *decreases*: **the particle *string* will still consist of the same number of oscillations, *with each of these oscillations packing the same energy in a smaller space*.** Hence, the physical interpretation of the wavefunction that is offered here may explain relativistic length contraction.

X. Concluding remarks

There are, of course, other ways to look at the matter – literally. For example, we can imagine two-dimensional oscillations as *circular* rather than linear oscillations. Think of a tiny ball, whose center of mass stays where it is, as depicted below. Any rotation – around any axis – will be some combination of a rotation around the two other axes. Hence, we may want to think of a two-dimensional oscillation as an oscillation of a polar and azimuthal angle.

Figure 10: Two-dimensional *circular* movement



The point of this paper is not to make any definite statements. That would be foolish. Its objective is just to challenge the simplistic mainstream viewpoint on the *reality* of the wavefunction. Stating that it is a mathematical construct only without *physical significance* amounts to saying it has no meaning at all. That is, clearly, a non-sustainable proposition.

The interpretation that is offered here looks at amplitude waves as traveling fields. Their physical dimension may be expressed in force per mass unit, as opposed to electromagnetic waves, whose amplitudes are expressed in force per (electric) *charge* unit. Also, the amplitudes of matter-waves

[29] Because our particle will be represented by a wave *packet*, i.e. a superimposition of elementary waves with different E and p, the classical velocity of the particle becomes the *group* velocity of the wave, which is why we denote it by v_g.

incorporate a phase factor, but this may actually explain the rather enigmatic dichotomy between fermions and bosons and is, therefore, an added bonus.

The interpretation that is offered here has some advantages over other explanations, as it explains the *how* of diffraction and interference. However, while it offers a great explanation of the wave nature of matter, it does *not* explain its particle nature: while we think of the energy as being spread out, we will still *observe* electrons and photons as pointlike particles once they hit the detector. Why is it that a detector can sort of ‘hook’ the whole blob of energy, so to speak?

The interpretation of the wavefunction that is offered here does *not* explain this. Hence, the *complementarity principle* of the Copenhagen interpretation of the wavefunction surely remains relevant.

Note 1: The *de Broglie* relations and energy

The 1/2 factor in Schrödinger's equation is related to the concept of the *effective* mass (m_{eff}). It is easy to make the wrong calculations. For example, when playing with the famous *de Broglie* relations – aka as the matter-wave equations – one may be tempted to *derive* the following energy concept:

1. $E = h\cdot f$ and $p = h/\lambda$. Therefore, $f = E/h$ and $\lambda = p/h$.
2. $v = f\cdot\lambda = (E/h)\cdot(p/h) = E/p$
3. $p = m\cdot v$. Therefore, $E = v\cdot p = m\cdot v^2$

$E = m\cdot v^2$? This *resembles* the $E = mc^2$ equation and, therefore, one may be enthused by the discovery, especially because the $m\cdot v^2$ also pops up when working with the Least Action Principle in *classical* mechanics, which states that the path that is followed by a particle will minimize the following integral:

$$S = \int_{t_1}^{t_2} (\mathrm{KE} - \mathrm{PE})dt$$

Now, we can choose any reference point for the potential energy but, to reflect the energy conservation law, we can select a reference point that ensures the *sum* of the kinetic and the potential energy is zero *throughout* the time interval. If the force field is uniform, then the integrand will, effectively, be equal to $\mathrm{KE} - \mathrm{PE} = m\cdot v^2$.[30]

However, that is *classical* mechanics and, therefore, not so relevant in the context of the *de Broglie* equations. The apparent paradox is to be solved by distinguishing between the *group* and the *phase* velocity of the matter wave, but the analysis is less straightforward than one might expect. Consider the following.

The $p = m\cdot v$ is the relativistically correct formula for the momentum of an object if $m = m_v$, so that is the same mass concept as used in Einstein's $E = mc^2$ mass-energy equivalence relation. Of course, v here is, obviously, the *group* velocity (v_g), so that is the classical velocity of our particle. Hence, we can write:

$$p = m\cdot v_g = (E/c^2)\cdot v_g \Leftrightarrow v_g = p/m = p\cdot c^2/E$$

This is just another way of writing the formula we derived in our paper: $v_g = c^2/v_p$ or $v_p = c^2/v_g$. Let us substitute in the formula for the wavelength:

$$\lambda = v_p/f = v_p\cdot T = v_p\cdot(h/E) = (c^2/v_g)\cdot(h/E) = h/(m\cdot v_g) = h/p$$

This gives us the second *de Broglie* relation: $\lambda = h/p$. It is interesting to think about it. The $f = E/h$ relation is intuitive: higher energy, higher frequency. In contrast, the $\lambda = h/p$ relation tells us we get an infinitely long wavelength for a stationary particle. As the $E = m\cdot v^2$ is only correct if $v = c$, the $\lambda = h/p$ relation may describe a photon, or a theoretical massless fermion only. For particles with a non-zero rest mass, the relation may only convey an idea or, at the very least, requires a better definition of the velocity variable.

[30] We detailed the mathematical framework and detailed calculations in the following online article: https://readingfeynman.org/2017/09/15/the-principle-of-least-action-re-visited.

Note 2: The concept of the effective mass

The effective mass – as used in Schrödinger's equation – is a rather enigmatic concept. To make sure we are making the right analysis here, we should start by noting you will usually see Schrödinger's equation written as:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_{eff}}\nabla^2\psi + U\psi$$

This formulation includes a term with the potential energy (U). In free space (no potential), this term disappears, and the equation can be re-written as:

$$\partial\psi(\mathbf{x}, t)/\partial t = i\cdot(1/2)\cdot(\hbar/m_{eff})\cdot\nabla^2\psi(\mathbf{x}, t)$$

We just moved the $i\cdot\hbar$ coefficient to the other side, noting that $1/i = -i$. Now, in one-dimensional space, and assuming ψ is just the elementary wavefunction (so we substitute $a\cdot e^{-i\cdot[E\cdot t - p\cdot x]/\hbar}$ for ψ), this implies the following:

$$-a\cdot i\cdot(E/\hbar)\cdot e^{-i\cdot[E\cdot t - p\cdot x]/\hbar} = -i\cdot(\hbar/2m_{eff})\cdot a\cdot(p^2/\hbar^2)\cdot e^{-i\cdot[E\cdot t - p\cdot x]/\hbar}$$

$$\Leftrightarrow E = p^2/(2m_{eff}) \Leftrightarrow m_{eff} = m\cdot(v/c)^2/2 = m\cdot\beta^2/2$$

It is an ugly formula: it *resembles* the kinetic energy formula (K.E. = $m\cdot v^2/2$) but it is, in fact, something completely different. The $\beta^2/2$ factor ensures the *effective* mass is always a fraction of the mass itself. To get rid of the ugly 1/2 factor, we may re-define m_{eff} as *two* times the old m_{eff} (hence, $m_{eff}^{NEW} = 2\cdot m_{eff}^{OLD}$), as a result of which the formula will look somewhat better:

$$m_{eff} = m\cdot(v/c)^2 = m\cdot\beta^2$$

We know β varies between 0 and 1 and, therefore, m_{eff} will vary between 0 and m. Feynman drops the subscript, and just writes m_{eff} as m in his textbook (see Feynman, III-19). On the other hand, the electron mass as used is also the electron mass that is used to calculate the size of an atom (see Feynman, III-2-4). As such, the two mass concepts are, effectively, mutually compatible. It is confusing because the same mass is often defined as the mass of a *stationary* electron (see, for example, the article on it in the online Wikipedia encyclopedia[31]).

In the context of the derivation of the electron orbitals, we do have the potential energy term – which is the equivalent of a *source* term in a diffusion equation – and that may explain why the above-mentioned $m_{eff} = m\cdot(v/c)^2 = m\cdot\beta^2$ formula does not apply.

[31] https://en.wikipedia.org/wiki/Electron_rest_mass (retrieved on 29 September 2017).

Note 3: Energy densities and particle dimensions

If we can calculate some radius with this model, then we should also try to calculate other dimensions. The preliminary analysis that is offered here is rather primitive because we limit ourselves to one-dimensional space only. However, the results are encouraging.

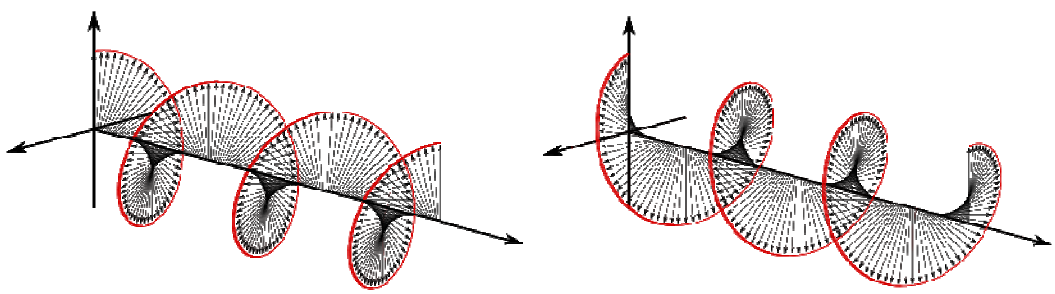
The equation for the *elementary* wavefunction is the usual one:

$$\psi = a \cdot e^{-i[E \cdot t - \mathbf{p} \cdot \mathbf{x}]/\hbar} = a \cdot e^{-i[E \cdot t - \mathbf{p} \cdot \mathbf{x}]/\hbar} = a \cdot cos(\mathbf{p} \cdot \mathbf{x}/\hbar - E \cdot t/\hbar) + i \cdot a \cdot sin(\mathbf{p} \cdot \mathbf{x}/\hbar - E \cdot t/\hbar)$$

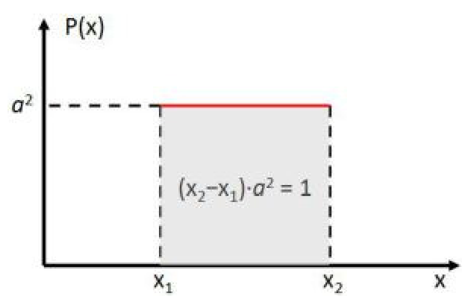
In one-dimensional space (think of a particle traveling along some *line*), the vectors (**p** and **x**) become scalars, and so we simply write:

$$\psi = a \cdot e^{-i[E \cdot t - p \cdot x]/\hbar} = a \cdot e^{-i[E \cdot t - p \cdot x]/\hbar} = a \cdot cos(p \cdot x/\hbar - E \cdot t/\hbar) + i \cdot a \cdot sin(p \cdot x/\hbar - E \cdot t/\hbar)$$

Let us assume our particle is an electron and, as mentioned, we reduced its motion to a *one-dimensional* motion only: we are thinking of it as traveling along the *x*-axis. We can then use the *y*- and *z*-axes as *mathematical* axes only: they will show us how the magnitude and direction of the real and imaginary component of ψ. As mentioned in the paper, the wavefunction can be right-handed *or* left-handed, as shown below.



These wavefunctions come with *constant* probabilities $|\psi|^2 = a^2$, so we need to define a space outside of which ψ = 0. This is obvious: oscillations pack energy, and the energy of our particle is finite. Hence, each particle - be it a photon or an electron - will pack a *finite* number of oscillations. It will, therefore, occupy a finite amount of space. Mathematically, this corresponds to the normalization condition: all probabilities have to add up to one, as illustrated below.

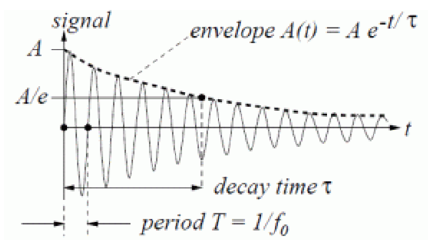


Now, the oscillations of the elementary wavefunction have the same (maximum) amplitude: *a*. [Terminology is a bit confusing here because we use the term amplitude to refer to two very different things here: we may say *a* is the (maximum) amplitude of the (probability) amplitude ψ.] So how many oscillations do we have? If this is a particle in a box, then what is the *size* of the box?

In our one-dimensional model, this amounts to asking how we can calculate the *length* of an electron. The question is interesting: we know the frequency (whose order of magnitude is 10^{15} Hz and 10^{20} Hz for the photon and the electron respectively) gives us the number of oscillations *per second*. But how many oscillations do we have in *one* photon, or in *one* electron?

Let us first think about photons, because we have more clues here. Photons are emitted by atomic oscillators: atoms going from one state (energy level) to another. We know how to calculate to calculate the Q of these atomic oscillators (see, for example, Feynman I-32-3): it is of the order of 10^8, which means the wave train will last about 10^{-8} seconds (to be precise, that is the time it takes for the radiation to die out by a factor 1/*e*). Now, the frequency of sodium light, for example, is 0.5×10^{15} oscillations *per second*, and the decay time is about 3.2×10^{-8} seconds, so that makes for $(0.5\times10^{15})\cdot(3.2\times10^{-8}) = 16$ million oscillations. Now, the wavelength is 600 *nano*meter (600×10^{-9}) m), so that gives us a wave train with a length of $(600\times10^{-9})\cdot(16\times10^{6}) = 9.6$ m.

These oscillations may or may not have the same amplitude and, hence, each of these oscillations may pack a different amount of energies. However, if the *total* energy of our sodium light photon (i.e. about 2 eV ≈ 3.3×10^{-19} J) is to be packed in those oscillations, then each oscillation would pack about 2×10^{-26} J, *on average*, that is. We may speculate on how we might imagine the actual wave *pulse* that atoms emit when going from one energy state to another, but we will not do that here. However, the following illustration of the decay of a transient signal may be useful.



Oscillation with exponential decay in time.
τ = how long to die to 1/e, about 37%

The calculation above is interesting, but gives us a paradox: if a photon is a pointlike particle, how can we say its length is like 10 *meter* or more? Fortunately, relativity theory saves us here. We need to distinguish the reference frame of the photon – riding along the wave as it is being emitted, so to speak – and our stationary reference frame, which is that of the emitting atom. Now, because the photon travels at the speed of light, relativistic length contraction will make it *look* like a pointlike particle.

What about the electron? Can we use similar assumptions? For the photon, we can use the decay time to calculate the effective *number* of oscillations. What can we use for an electron? We will need to make some assumption about the phase velocity or, what amounts to the same, the group velocity of the particle. What formulas can we use?

If our particle is at rest, then p = 0 and the p·x/ħ term in our wavefunction vanishes, so the wavefunction reduces to:

$$\psi = a\cdot e^{-i\cdot \mathrm{E}\cdot t/\hbar} = a\cdot cos(\mathrm{E}\cdot t/\hbar) - i\cdot a\cdot sin(\mathrm{E}\cdot t/\hbar)$$

Hence, our wave does not travel. It has the same amplitude at every point in space *at any point in time*. Both the phase and group velocity become meaningless concepts. Of course, the *amplitude* varies – because of the sine and cosine – but the probability remains the same: $|\psi|^2 = a^2$. How can we calculate the size of our box. We may think of the formula we wrote down in our paper (see section II):

$$\mathrm{E} = \sum \mathrm{m}_i \cdot a_i^2 \cdot \omega_i^2 = \sum \frac{\mathrm{E}_i}{c^2} \cdot a_i^2 \cdot \frac{\mathrm{E}_i{}^2}{\hbar^2}$$

This is a *physical* normalization condition: the energy contributions of the waves that make up a wave packet need to add up to the total energy of our wave. Of course, for our elementary wavefunction here, the subscripts vanish and so the formula reduces to $E = (E/c^2)\cdot a^2\cdot(E^2/\hbar^2)$, out of which we get our formula for the Compton scattering radius: $a = \hbar/mc$. Now how do we *pack* that energy in our cylinder? Assuming that energy is distributed uniformly, we are tempted to write something like $E = a^2\cdot l$ or, looking at the geometry of the situation, to think of the formula for the *volume* of a cylinder:

$$E = \pi\cdot a^2\cdot l \Leftrightarrow l = E/(\pi\cdot a^2)$$

Using the value we got for the Compton scattering radius ($a = 3.8616\times10^{-13}$ m), we find an l that is equal to $(8.19\times10^{-14})/(\pi\cdot14.9\times10^{-26}) =\approx 0.175\times10^{12}$... *Meter?* Yes. We get the following formula:

$$l = \frac{\mathrm{m}^3\cdot c^4}{\pi\cdot\hbar^2} = \frac{\mathrm{E}^3}{\pi\cdot\hbar^2\cdot c^2}$$

0.175×10^{12} m is 175 *million kilo*meter. That is - literally - astronomic. It corresponds to 583 light-seconds, or 9.7 light-*minutes.* So that is about 1.17 times the (average) distance between the Sun and the Earth. Of course, that space is quite large to look for an electron. It just underlines the need to properly build a wave *packet* by making use of the Uncertainty Principle: paradoxically, the uncertainty in the energy will, effectively, reduce the uncertainty in position.

We may wonder if we could possibly get less astronomic proportions *without* uncertainty. What if we *impose* that l should equal a? We get the following condition:

$$\frac{l}{a} = \frac{E}{\pi\cdot a^3} = \frac{\mathrm{m}\cdot c^2}{\pi\cdot\frac{\hbar^3}{\mathrm{m}^3c^3}} = \frac{\mathrm{m}^4\cdot c^5\cdot}{\pi\cdot\hbar^3} = 1 \Leftrightarrow \mathrm{m} = \sqrt[4]{\frac{\pi\cdot\hbar^3}{c^5}}$$

We find that m would have to be equal to $\mathrm{m} \approx 1.11\times10^{-36}$ kg. That is tiny. In fact, it is equivalent to an energy of about 0.623 eV (623 meV. This corresponds to light with a wavelength of about 2 µm (*micro*-meter). That is light in the infrared spectrum. Note the proportionality of the l/a ratio with m^4.

However, these manipulations do not tell us much. Should we make a guess at the equivalent of the electricity constant to see whether we get another, perhaps more meaningful, result?

Let us think about the scaling constant: the probabilities will, obviously, not be *identical* to the energy densities, but *proportional*. Hence, we need to find the constant of proportionality, i.e. the equivalent of the electric constant ε_0 for the energy density formula for the wavefunction. How should we go about this? For inspiration, we may look once again at the structural similarity between Newton's and Coulomb's force laws:

$$F = k_e\frac{q_1\cdot q_2}{r^2}$$

$$F = G\frac{m_1\cdot m_2}{r^2}$$

This is what inspired us to analyze the wavefunction as an energy propagation mechanism. Indeed, we associated the components of the wavefunction with a physical dimension (N/kg, i.e. force per unit *mass*) because we noted the electric and magnetic field vectors were associated with a similar physical dimension (N/C, i.e. force per unit *charge*). Of course, we duly noted that the mass unit (1 kg) is equivalent to 1 $\mathrm{N\cdot s^2/m}$ and, hence, that our N/kg dimension is actually the dimension of acceleration:

$$\mathrm{N/kg} = \mathrm{N/(N\cdot s^2/m)} = \mathrm{m/s^2}$$

This, in turn, inspired us to analyze the wavefunction as a gravitational wave. Should we push the comparison further? Coulomb's constant $k_e = 1/4\pi\varepsilon_0$ serves two purposes as a constant of proportionality:

1. As a *mathematical* constant of proportionality, they give us, effectively a constant of proportionality.
2. As a *physical* constant, it will ensure the *physical dimensions* on both sides of the equation are compatible.

The physical dimension of Coulomb's constant is $\text{N}\cdot\text{m}^2/\text{C}^2$. Likewise, the physical dimension of *G* is equal to $\text{N}\cdot\text{m}^2/\text{kg}^2$. We also know that k_e is equal to $1/4\pi\varepsilon_0$. The $1/4\pi$ factor is, obviously, a geometric factor. Hence, If we denote the equivalent of ε_0 as g_0, we may, perhaps, write the following:

$$g_0 \approx \frac{1}{4\pi G} = \frac{1}{4\pi \cdot 6.674 \times 10^{-11}}$$

We may then guess the following for the energy density:

$$u = \frac{g_0}{2} a^2 (cos\theta)^2 + \frac{g_0}{2} a^2 (-i \cdot sin\theta)^2 = \frac{g_0}{2} a^2\ (cos^2\theta + sin^2\theta) = \frac{g_0}{2} a^2 = \frac{a^2}{8\pi G}$$

In the mentioned article, we calculated a *length* using the energy density. We got a nonsensical result (about 0.175×10^{12} m), but that result may be explained because we did not do any thinking about the proportionality coefficient. Would we get a better result with the energy density formula above? Let us see.

Assuming that energy is distributed uniformly, we may use, once again, the formula for the *volume* of a cylinder. However, this time we will not use the simple $E = \pi\cdot a^2\cdot l$ *formula. We will multiply the* $\pi\cdot a^2$ area (or surface) with the energy density *u*. [This formula may or may not make sense, but the dimensions work out: $m^2\cdot(\text{N/m}^2)\cdot m$ *gives us N·m, so we do get the energy dimension out of it.] So let us do a revised calculation:*

$$\text{E} = \text{m}_\text{e} \cdot c^2 = \pi \cdot a^2 \cdot u \cdot l$$

$$\Leftrightarrow l = \frac{\text{m}_\text{e} \cdot c^2}{\pi \cdot a^2 \cdot u} = \frac{\text{m}_\text{e} \cdot c^2}{\pi \cdot a^2 \cdot \frac{a^2}{8\pi G}} = \frac{8 \cdot G \cdot \text{m}_\text{e} \cdot c^2}{\left(\frac{\hbar}{\text{m}_\text{e} \cdot c}\right)^4} = \frac{8 \cdot G \cdot {\text{m}_\text{e}}^5 \cdot c^6}{\hbar^4} =$$

The *numerical* result we get is even more astronomical:

$$l \approx 1.96581\times10^{27} \text{ m.}$$

This value corresponds to 0.2 billion light years. Our reasoning here clearly needs more tuning.

References

This paper discusses general principles in physics only. Hence, references can be limited to references to physics textbooks only. For ease of reading, any reference to additional material has been limited to a more popular undergrad textbook that can be consulted online: Feynman's Lectures on Physics (http://www.feynmanlectures.caltech.edu). References are per volume, per chapter and per section. For example, Feynman III-19-3 refers to Volume III, Chapter 19, Section 3.

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