Common description of quantum electromagnetism and relativistic gravitation

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Abstract

A particle at rest is described by the equation $E^2 = (m_0c^2)^2$ and a moving particle by $E^2 = p^2c^2 + m_0^2c^4$ or $E = \sqrt{p^2c^2 + m_0^2c^4} = \gamma m_0 c^2$ with γ reltivistic factor. To describe interactions we add extra dimensions (at least 1 by interaction): $E^2 = p^2c^2 + p_{int}^2c^2 + m_0^2c^4$ giving, in the limit $m_0c \gg p, m_0c \gg p_{int}$, $E = \sqrt{p^2c^2 + p_{int}^2c^2 + m_0^2c^4} \approx m_0c^2 + \frac{p^2}{2m_0} + \frac{p_{int}^2}{2m_0} = m_0c^2 + \frac{p^2}{2m_0} + \frac{k}{r}$ such that the classical energy $\epsilon = E - m_0c^2$ will be given by $\epsilon = \frac{p^2}{2m_0} + \frac{k}{r}$ with $k < 0$ if p_{int} imaginary. Since E and m_0 are constant, classical Kepler problems exhibit an SO(4) symmetry, as we discuss in first part : there are two angular momenta, $\vec{L}$ and $\vec{K}$ since 6 rotations are allowed in 4 dimensions. In the atom, the second angular momenta will be identified with the spin. In this work, we will consider central problems (motion of planets around the sun and electrons around atomic nucleus) and we will approximate the equation $E = \sqrt{p^2c^2 + p_{int}^2c^2 + m_0^2c^4}$ by $E = \sqrt{p^2c^2 + m_0^2c^4} - \frac{k}{r}$ producing 2 postulates: $\mathbf{P_{EM}}$ referring to electromagnetism, $\mathbf{P_{Grav}}$ referring to gravitation : both sould be undertood as approximations. We add a last one, $\mathbf{P_{QM}}$ refering to quantum mechanic.

$\mathbf{P_{EM}}$: $E = \gamma m_0c^2 - k/r$ with $k = Zq^2/(4\pi\epsilon_0)$
$\mathbf{P_{Grav}}$: $E = \gamma m_0c^2 - k/r$ with $k = GM\gamma m_0$ (the central mass M is assumed to be at rest, γ referring then to the mass m_0)
$\mathbf{P_{QM}}$: $\Psi(\vec{x}, t) = \int \hat{\Psi}(\vec{k}, w) e^{iS(\vec{x},t)/\hbar} \mathrm{d}\vec{k}\mathrm{d}w$ defines the wavefunction, with S relativistic action, deduced from $\mathbf{P_{EM}}$. $\mathbf{P_{QM}}$ generalizes the Fourrier transform to the Lagrangian formalism. We will see that it reduces to the usual Fourrier tranform in the free particle case.

Combining $\mathbf{P_{EM}}$ with "Sommerfeld's quantum rules" corresponds to the original quantum theory of Hydrogen, which produces the correct relativistic energy levels of atoms (Sommerfeld's and Dirac's theories of matter produces the same energy levels, and Schrodinger's theory produces the approximation of those energy levels). $\mathbf{P_{QM}}$ implies that Ψ is solution of both Schrodinger's and Klein-Gordon's equations in the non interacting case ($k = 0$ in $\mathbf{P_{EM}}$) while, in the interacting case ($k \neq 0$), it implies "Sommerfeld's quantum rules" : $\mathbf{P_{EM}}$ and $\mathbf{P_{QM}}$ then produce the correct relativistic energy levels of atoms (the same as Dirac's energy levels). We check that the required degeneracy is justified by pure deduction, without any other assumption (Schrodinger's theory only justifies one half of the degeneracy). We observe the connection between $\mathbf{P_{QM}}$, Quantum Field Theories and tunnel effect.
From $\mathbf{P_{Grav}}$ we deduce an equation of motion very similar to general relativity (with accuracy 10^{-6} at the surface of the Sun), our postulate being explicitly an approximation.
First of all, we discuss classical Kepler problems (Newtonian motion of the Earth around the Sun), explain the link between Kelpler's law of periods (1619) and Plank's law (1900) and observe the links between all historical models of atoms (Bohr, Sommerfeld, Pauli, Schrodinger, Dirac, Fock).

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I – New results in classical physics

We first defines the quantities in $\mathbf{P_{EM}}$ and $\mathbf{P_{Grav}}$: E [J] is the energy, m_0 [kg] the mass of the orbiting particle/planet, c [m/s] the speed of light, q [C] the unit proton charge ($-q$ is the electron charge), Z [-] the number of protons in an atom (we will fix $Z = 1$ for simplicity, corresponding to Hydrogen), ϵ_0 [F/m] the vacuum permittivity , G [m^3/kg/s^2] Newton's constant, M [kg] the mass of the central object, $\gamma = 1/\sqrt{1 - v^2/c^2}$ the relativistic factor.

In this part, the potential k/r can refer to the electromagnetic ($k = q^2/(4\pi\epsilon_0) = \alpha\hbar c$) or classical gravitationnal potential ($k = GMm_0$). $\alpha \approx 1/137$ [-] is the fine structure constant, and $\hbar$ [J.s] is the reduced Planck constant. Equations (1) to (4) are well known results of classical physics and can be found in any student book. With $\epsilon = E - m_0c^2, \gamma m_0c^2 \approx m_0c^2 + 1/2m_0v^2 = m_0c^2 + p^2/(2m_0)$, the classical energy $\epsilon \leq 0$ is given by :

$$\epsilon = \frac{p^2}{2m_0} - \frac{k}{r} = \frac{(\vec{p}.\vec{r})^2}{2m_0r^2} + \frac{(\vec{p} \wedge \vec{r})^2}{2m_0r^2} - \frac{k}{r} = \frac{p_r^2}{2m_0} + \frac{L^2}{2m_0r^2} - \frac{k}{r} = \frac{m_0\dot{r}^2}{2} + \frac{L^2}{2m_0r^2} - \frac{k}{r} \quad (1)$$

With $L = m_0r^2\frac{d\phi}{dt} \Leftrightarrow \dot{r} = \frac{dr}{dt} = \frac{Ldr}{m_0r^2d\phi}$, and fixing $u = 1/r \Leftrightarrow \frac{du}{d\phi} = -\frac{dr/d\phi}{r^2}$ the previous equation can be rewritten and derivated according to :

$$\epsilon = \frac{L^2}{2m_0}(\frac{du}{d\phi})^2 + \frac{L^2}{2m_0}u^2 - ku \Leftrightarrow 0 = \frac{du}{d\phi}(\frac{L^2}{m_0}\frac{d^2u}{d\phi^2} + \frac{L^2}{m_0}u - k) \Leftrightarrow 0 = \frac{d^2u}{d\phi^2} + u = \frac{m_0k}{L^2} \quad (2)$$

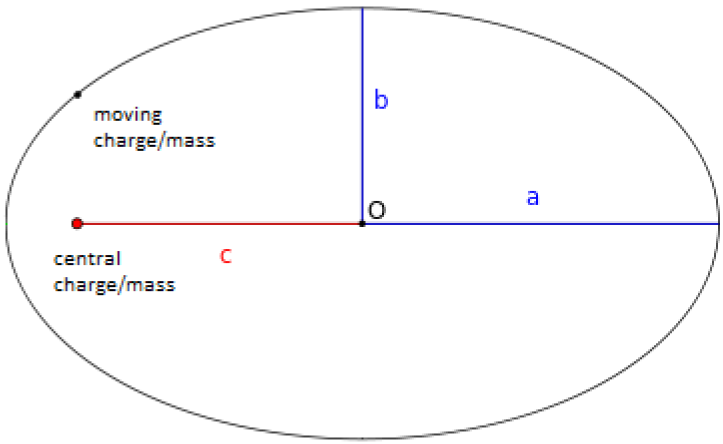
The solution, with u proportionnal to the potential k/r, takes the form

$$u = 1/r = \frac{m_0k}{L^2}(1 + ecos(\phi - \phi_0)) \; with \; e = \sqrt{1 + \frac{2\epsilon L^2}{k^2m_0}} \quad (3)$$

e (the eccentricity) and ϕ_0 are constants of integration, and $r(\phi)$ is given by

$$r = \frac{L^2}{m_0k + \sqrt{m_0^2k^2 + 2m_0\epsilon L^2}cos(\phi - \phi_0)} = \frac{L^2/(m_0k)}{1 + ecos(\phi - \phi_0)} = \frac{l}{1 + ecos(\phi - \phi_0)} \quad (4)$$

l is called "semi latus rectum". It is well known that the solution is an ellipse, described by 3 parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}$ where $\mathsf{a}^2 = \mathsf{b}^2 + \mathsf{c}^2$, $e = \mathsf{c}/\mathsf{a}$ and $l = \mathsf{b}^2/\mathsf{a}$).



$$\mathsf{a} = \frac{m_0k}{2m_0|\epsilon|} = \frac{J}{\sqrt{2m_0|\epsilon|}}; \; \mathsf{b} = \frac{L}{\sqrt{2m_0|\epsilon|}}; \; \mathsf{c} = \frac{\sqrt{m_0^2k^2/(2m_0|\epsilon|) - L^2}}{\sqrt{2m_0|\epsilon|}} = \frac{K}{\sqrt{2m_0|\epsilon|}} \quad (5)$$

Here K, which is proportionnal to the eccentricity e, is the norm of the well known Laplace-Runge-Lenz-Pauli vector, the second converved angular momentum of Kepler problems :

$$\vec{K} = \frac{1}{\sqrt{-2m_0\epsilon}}(\vec{p}\wedge\vec{L} - m_0 k\frac{\vec{r}}{r}) \quad , \quad \epsilon \ classical\ energy < 0 \tag{6}$$

since (with $\dot{\vec{L}} = 0$ and $\vec{r}\wedge\vec{v}\wedge\vec{r} = r^2\vec{v} - (\vec{v}.\vec{r})\vec{r}$, and $\vec{v}.\vec{r} = \dot{x}x + \dot{y}y + \dot{z}z = \dot{r}r$ which can be easily checked from the right to the left) :

$$\frac{d(\vec{p}\wedge\vec{L})}{dt} = \dot{\vec{p}}\wedge\vec{L} = \frac{k\vec{r}}{r^3}\wedge(m_0\vec{v}\wedge\vec{r}) = \frac{m_0 k}{r^3}(r^2\vec{v}-(\vec{v}.\vec{r})\vec{r}) = \frac{m_0 k}{r^3}(r^2\vec{v}-(\dot{r}r)\vec{r}) = \frac{d(m_0 k\vec{r}/r)}{dt} \tag{7}$$

Equation (7) shows that $\vec{K}$ is conserved, which is well known. We can now give new results (equations (9), (10) and (11)), the conserved Runge-Lenz vector can be rewritten :

$$\vec{K} = \frac{1}{\sqrt{-2m_0\epsilon}}(m_0^2\vec{v}\wedge\vec{r}\wedge\vec{v} - m_0 k\frac{\vec{r}}{r}) = \frac{1}{\sqrt{-2m_0\epsilon}}[(m_0^2v^2 - m_0\frac{k}{r})\vec{r} - m_0^2(\vec{v}.\vec{r})\vec{v}] \tag{8}$$

$$\vec{K} = (\frac{m_0^2v^2 - m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})\vec{r} - (\frac{\vec{p}.\vec{r}}{\sqrt{-2m_0\epsilon}})\vec{p} = (p_w)\vec{r} - (w)\vec{p} \tag{9}$$

where we can check that

$$\dot{w} = \frac{\dot{\vec{p}}.\vec{r} + \vec{p}.\dot{\vec{r}}}{\sqrt{-2m_0\epsilon}} = \frac{m_0v^2 - \frac{k}{r}}{\sqrt{-2m_0\epsilon}} = \frac{p_w}{m_0} \tag{10}$$

We can then define 6 rotations and a total angular momentum J such that

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} \qquad \vec{K} = \begin{bmatrix} xp_w - wp_x \\ yp_w - wp_y \\ zp_w - wp_z \end{bmatrix} = \vec{r}p_w - w\vec{p} \tag{11}$$

$$\vec{J} = \vec{L} + \vec{K} \ ; \ J^2 = K^2 + L^2 \ (\vec{K}.\vec{L} = 0 \ can\ be\ easily\ checked\ from\ equations\ (6)\ or\ (11)) \tag{12}$$

The 6 independant previous rotations defines the SO(4) symmetry. The two angular momenta have opposite parity. In 1926, Pauli [Pauli, 1926] used this definition of equation (12) for J to deduce the non relativistic energy levels of Hydrogen, without solving for the wave function. In 1935, Fock [Fock, 1935], studying Schrodinger's Hydrogen [Schrodinger, 1926] in momentum space, observed that, with an $1/r$ potential (SO(3) symmetry) he could describe an intrinsic SO(4) in the model. Concerning SO(4), we now give a new result involving p_w and $\vec{J}$:

$$r^2P^2 = r^2[p_x^2 + p_y^2 + p_z^2 + p_w^2] = r^2[p^2 + (\frac{m_0^2v^2 - m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})^2] \tag{13}$$

$$r^2P^2 = r^2[p^2 + (\frac{m_0^2v^2 - 2m_0\frac{k}{r} + m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})^2] = r^2[p^2 + (-\sqrt{-2m_0\epsilon} + \frac{m_0k/r}{\sqrt{-2m_0\epsilon}})^2] \tag{14}$$

$$r^2P^2 = r^2[p^2 - 2m_0\epsilon - 2m_0k/r + J^2/r^2) = J^2 \tag{15}$$

This shows that J combines both SO(3) and SO(4) symmetries. From this (and the definition of angular momentum L) we easily deduce

$$J^2 = r^2(p^2 + p_w^2) \ ; \ L^2 = r^2(p^2 - p_r^2) \ ; \ K^2 = r^2(p_w^2 + p_r^2) \tag{16}$$

We now recall Kepler's third law of periods for Planetary motion (left hand side of equation (17) below), and observe that it can be rewritten in a new form (using equation (5)):

$$T^2 = \frac{4\pi^2}{GM}\mathsf{a}^3 = \frac{4\pi^2}{GM}(\mathsf{a})(\mathsf{a}^2) = \frac{4\pi^2}{GM}(\frac{m_0^2GM}{2m_0|\epsilon|})(\frac{J^2}{2m_0|\epsilon|}) \Leftrightarrow |\epsilon|T = \pi J \ (new\ form) \tag{17}$$

Equations (15), (16) and (17) are new results for classical physics. Evidently, with h as Planck constant (and $\hbar = h/(2\pi)$), fixing J equal to one unit of angular momentum, $J = \hbar$, and introducing the frequency $\nu = 1/T$ gives $|\epsilon| = h\nu/2$. In particular, when the motion around the central object is a circle (corresponding to Bohr's model of Hydrogen [Bohr, 1913]), the electromagnetic interaction energy $V(r) = -k/r = E_i$ is constant, and $|E_i| = 2|\epsilon| = h\nu$: we recognize here Planck's law [Planck, 1900] for the electromagnetic field, hidden in Kepler's third law [Kepler, 1619]. Considering non circular orbits, and calling $< E_i >$ the time average value of the potential, we can write, (using equation (1) and the definition of L below equation (1)) :

$$T < E_i >= T(\frac{1}{T}\int_0^T \frac{-k}{r}\mathrm{d}t) = \int_0^T \epsilon - \frac{m_0\dot{r}^2}{2} - \frac{L^2}{2m_0r^2}\mathrm{d}t = \epsilon T - \int_0^T \frac{m_0\dot{r}^2}{2} + \frac{L}{2}\dot{\phi}\mathrm{d}t \tag{18}$$

With equation (17) and a simple change of the integration variables we have

$$T < E_i >= -\pi J - \int_{r(t=0)}^{r(T)} \frac{p_r}{2}\mathrm{d}r - \int_{\phi(t=0)}^{\phi(T)} \frac{L}{2}\mathrm{d}\phi = -\pi J - \oint \frac{p_r}{2}\mathrm{d}r - \pi L = -\pi J - [\pi(J-L)] - \pi L = -2\pi J \tag{19}$$

The term $[\pi(J-L)]$ as result of the integral over $\mathrm{d}r$ is justified in part II. The analogy between Kepler's third law and Planck's law remains valid for non circular orbits, while equation (19) looks like to a saturation of Heisenberg's relation $\Delta t\Delta E \approx h$ for $J = \hbar$. This is our last new result for classical physics.

II – Sommerfeld′s model of atoms :

We reproduce here Sommerfeld's book [Sommerfeld, 1916], nothing is new except equation (31) and maybe equations (24) and (25). Starting with $\mathbf{P_{EM}}$: $E = \gamma m_0c^2 - k/r$ which, in polar coordinates (cylindrical coordinates with $z = 0$), becomes

$$(E + \frac{k}{r})^2 = p^2c^2 + m_0^2c^4 \Leftrightarrow E^2 - m_0^2c^4 = p^2c^2 - \frac{2Ek}{r} - \frac{k^2}{r^2} = p_r^2c^2 + \frac{L^2c^2 - k^2}{r^2} - \frac{2Ek}{r} \tag{20}$$

$$E^2 - m_0^2c^4 = (\gamma m_0\dot{r})^2c^2 + \frac{L'^2c^2}{r^2} - \frac{2Ek}{r} \tag{21}$$

$L'^2 = L^2 - (k/c)^2 = L^2 - (\alpha\hbar)^2$ will be important. With $L = \vec{p} \wedge \vec{r} = \gamma m_0 r^2\frac{d\phi}{dt} \Leftrightarrow \dot{r} = \frac{dr}{dt} = \frac{Ldr}{\gamma m_0 r^2 d\phi}$, and fixing $u = 1/r \Leftrightarrow \frac{du}{d\phi} = -\frac{dr/d\phi}{r^2}$ the previous equation can be rewritten and derivated according to :

$$E^2 - m_0^2c^4 = L^2c^2(\frac{du}{d\phi})^2 + L'^2c^2u^2 - 2Eku \tag{22}$$

$$0 = \frac{du}{d\phi}(2L^2c^2\frac{d^2u}{d\phi^2} + 2L'^2c^2u - 2Ek) \Leftrightarrow \frac{d^2u}{d\phi^2} + u(1 - (\frac{k}{Lc})^2) = \frac{Ek}{L^2c^2} \tag{23}$$

The solution takes the form $u = 1/r = \frac{Ek}{L^2c^2}(1 + ecos(\Gamma\phi - \phi_0))$ with $e = \sqrt{1 + \frac{(E^2 - m_0^2c^4)L'^2c^2}{E^2k^2}}$ and $\Gamma^2 = 1 - (\frac{k}{Lc})^2$. The Γ factor produces a shift of the perihelion, as illustrated by Sommerfeld :

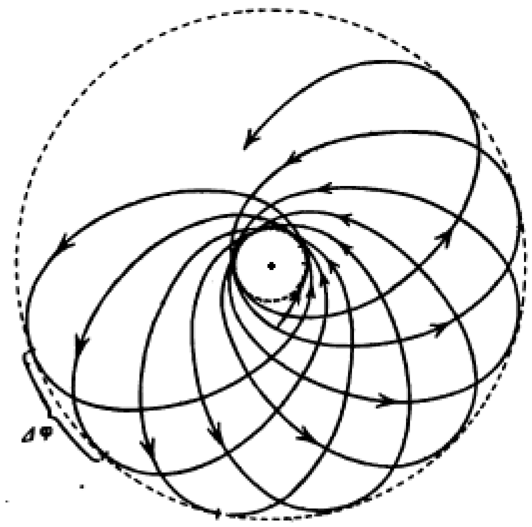


Figure 1 : Perihelion's shift for 8 loops (from [Sommerfeld, 1916])

The 3 parameters of the ellipse are now given by (with $E < m_0c^2$) :

$$\mathsf{a} = \frac{Ek}{m_0^2c^4 - E^2} = \frac{Jc}{\sqrt{m_0^2c^4 - E^2}} \;;\; \mathsf{b} = \frac{c\sqrt{L^2 - (k/c)^2}}{\sqrt{m_0^2c^4 - E^2}} = \frac{L'c}{\sqrt{m_0c^2 - E^2}} \tag{24}$$

$$\mathsf{c} = \frac{\sqrt{(Ek)^2/(m_0^2c^4 - E^2) - c^2(L^2 - (k/c)^2)}}{\sqrt{m_0^2c^4 - E^2}} = \frac{Kc}{\sqrt{m_0^2c^4 - E^2}} \;;\; J^2 = L'^2 + K^2 \tag{25}$$

In classical physics $r(t)$ and $\phi(t)$ are cyclic functions of time with period T. In the relativistic domain, there is a precession of the perihelion, such that $r(t)$ and $\phi(t)$ have different periods, T_r and T_ϕ. Sommerfeld's (postulated) quantum rules [Sommerfeld, 1916], with n_ϕ and n_r integers, are

$$\int_t^{t+T_r} (\gamma m_0 \dot{r}^2)\mathrm{d}t \int_{r(t)}^{r(t+T_r)} p_r \mathrm{d}r = \oint p_r \mathrm{d}r = n_r h \tag{26}$$

$$\int_t^{t+T_\phi} L\dot{\phi}\mathrm{d}t = \int_0^{2\pi} L\mathrm{d}\phi = \oint L\mathrm{d}\phi = \; 2\pi L = n_\phi h \;\; (\Rightarrow L' = \hbar\sqrt{n_\phi^2 - \alpha^2}) \tag{27}$$

Sommerfeld gave two methods to compute equation (26). We reproduce them is Annex. The final result is (with $E < m_0c^2$ for $V < 0$):

$$\oint p_r \mathrm{d}r = 2\pi(J - L') = n_r h \Leftrightarrow J = \frac{Ek/c}{\sqrt{m_0^2c^4 - E^2}} = \frac{E\alpha\hbar}{\sqrt{m_0^2c^4 - E^2}} = (n_r + \sqrt{n_\phi^2 - \alpha^2})\hbar \tag{28}$$

This can be rewritten

$$E^2\alpha^2 = (n_r + \sqrt{n_\phi^2 - \alpha^2})^2(m_0^2c^4 - E^2) \Leftrightarrow E^2(\alpha^2 + (n_r + \sqrt{n_\phi^2 - \alpha^2})^2) = (n_r + \sqrt{n_\phi^2 - \alpha^2})^2 m_0^2c^4 \tag{29}$$

$$E^2 = \frac{m_0^2c^4}{\frac{\alpha^2 + (n_r + \sqrt{n_\phi^2 - \alpha^2})^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}} = \frac{m_0^2c^4}{1 + \frac{\alpha^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}} \Leftrightarrow E = \frac{m_0c^2}{\sqrt{1 + \frac{\alpha^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}}} \tag{30}$$

This last equation is called "fine structure of Hydrogen". In Dirac's theory, n_ϕ is replaced by $j + 1/2$ with $j = n_\phi \pm 1/2$ (see later). The energy levels are then the same : it was the great triumph of Dirac's theory that it reproduced Sommerfeld's energy levels.

With equations (28) and (25), it is interesting to rewrite the energy in a new form

$$E = \frac{m_0c^2}{\sqrt{1+\frac{(\alpha\hbar)^2}{J^2}}} = m_0c^2\frac{\sqrt{J^2}}{\sqrt{J^2+(\alpha\hbar)^2}} = m_0c^2\sqrt{\frac{J^2+(\alpha\hbar)^2-(\alpha\hbar)^2}{J^2+(\alpha\hbar)^2}} = m_0c^2\sqrt{1-\frac{(\alpha\hbar)^2}{K^2+L^2}} \tag{31}$$

III – A 7$^{\text{th}}$ quantum model of matter

Inspired by Bohr, Sommerfeld, Schrodinger, Pauli, Dirac and Fock, we now suggest a new quantum model of matter. Since our definition of the quantum wave function is based on the action, we first establish some new results. Our first postulate is assumed to be written in spherical coordinates r, θ, φ (with $r = \sqrt{x^2+y^2+z^2}$). We will then express the action as $S = S(t, r, \theta, \varphi)$. In this coordinates system, the periods $T_r \neq T_\phi$ (used in polar coordinates) are now $T_r \neq T_\theta \neq T_\varphi$. H refers to the Hamiltonian ($= E$). We start from

$$H = \gamma m_0c^2 - \frac{k}{r} = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - \frac{k}{r} = \frac{m_0c^2(1-v^2/c^2)+m_0v^2}{\sqrt{1-v^2/c^2}} - \frac{k}{r} \tag{32}$$

$$H = \frac{m_0v^2}{\sqrt{1-v^2/c^2}} + m_0c^2\sqrt{1-v^2/c^2} - \frac{k}{r} = \vec{p}.\vec{v} + m_0c^2\sqrt{1-v^2/c^2} - \frac{k}{r} = \vec{p}.\vec{v} - \mathcal{L} \tag{33}$$

Here $\mathcal{L} = -m_0c^2\sqrt{1-v^2/c^2}+\frac{k}{r}$ is the usual Lagrangian. It can be found in the famous "Landau and Lifschitz" [Landau, Ed 1975]. Its properties are well known, and the action S is then given by :

$$S = \int \mathcal{L}\mathrm{d}t = \int \vec{p}.\vec{v} - H\mathrm{d}t = -Ht + \int \vec{p}.\vec{v}\mathrm{d}t \tag{34}$$

In spherical coordinates, with

$$\vec{r} = (r,0,0)\ ;\ \dot{\vec{r}} = \vec{v} = (\dot{r}, r\dot{\theta}, rsin\theta\dot{\varphi})\ ;\ \vec{L} = (r,0,0)\wedge\gamma m_0(\dot{r}, r\dot{\theta}, rsin\theta\dot{\varphi}) = \gamma m_0(0, r^2\dot{\theta}, r^2sin\theta\dot{\varphi}) \tag{35}$$

The Lagrangian is explicitly

$$\mathcal{L} = -m_0c\sqrt{c^2-v^2} + \frac{k}{r} = -m_0c\sqrt{c^2-\dot{r}^2-r^2\dot{\theta}^2-r^2sin^2\theta\dot{\varphi}^2} + \frac{k}{r} \tag{36}$$

The action is, according to equation (34) (and writing $L_\theta = \gamma m_0r^2\dot{\theta}, L_\varphi = \gamma m_0r^2sin^2\theta\dot{\varphi}$)

$$S = -Ht + \int \gamma m_0(\dot{r}^2 + r^2\dot{\theta}^2 + r^2sin^2\theta\dot{\varphi}^2)\mathrm{d}t = -Ht + \int \gamma m_0\dot{r}^2 + L_\theta\dot{\theta} + L_\varphi\dot{\varphi}\ \mathrm{d}t \tag{37}$$

with a change of variables we now obtain the desired expression for the action $S = S(t, r, \theta, \varphi)$

$$S = -Ht + \int \gamma m_0\dot{r}\mathrm{d}r + \int L_\theta\mathrm{d}\theta + \int L_\varphi\mathrm{d}\varphi = -Ht + \int p_r(r)\mathrm{d}r + \int L_\theta(\theta)\mathrm{d}\theta + \int L_\varphi\mathrm{d}\varphi \tag{38}$$

L_φ is constant since $\mathcal{L}$ does not depend on φ. From the right hand side of above we deduce $H + \frac{\partial S}{\partial t} = 0$, which is the usual Hamilton-Jacobi equation, here extended to the relativistic domain. From equations (36) and (38) we deduce

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial \mathcal{L}}{\partial \dot{r}}\ ;\ L_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial S}{\partial \theta}\ ;\ L_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial S}{\partial \varphi} \tag{39}$$

$p_r(r)$ is given by (20). From (35) and the definition of L_θ, L_φ, the squared norm of the angular momentum is given by

$$L^2 = L_\theta^2 + \frac{{L_\varphi}^2}{sin^2\theta} \Leftrightarrow L_\theta\dot{\theta} + L_\varphi\dot{\varphi} = \frac{L_\theta^2}{\gamma m_0 r^2} + \frac{L_\varphi^2}{\gamma m_0 r^2 sin^2\theta} = \frac{L^2}{\gamma m_0 r^2} = L\dot{\phi} \tag{40}$$

The right hand side of above gives the relation between our spherical coordinates system and the polar coordinates system (or cylindrical coordinates system with $z = 0$) used by Sommerfeld. We have seen with (38) that the action can be seperated : $S(t, r, \theta, \varphi) = -Ht + S_r(r) + S_\theta(\theta) + S_\varphi(\varphi)$. Considering now our third postulate,

$\mathbf{P_{QM}}$: $\Psi(\vec{x}, t) = \int \hat{\Psi}(\vec{k}, w) e^{iS(\vec{x},t)/\hbar} \mathrm{d}\vec{k}\mathrm{d}w$

we write in spherical coordinates

$\mathbf{P_{QM}}$: $\Psi(t, r, \theta, \varphi) = \int \hat{\Psi}(\vec{k}, w) e^{iS(t,r,\theta,\varphi))/\hbar} \mathrm{d}\vec{k}\mathrm{d}w = \int \hat{\Psi}(\vec{k}, w) e^{i(-Ht+S_r(r)+S_\theta(\theta)+S_\varphi(\varphi))/\hbar} \mathrm{d}\vec{k}\mathrm{d}w$

Since $r(t)$, $\theta(t)$ and $\varphi(t)$ are cyclic variables $S_r(r)$, $S_\theta(\theta)$ and $S_\varphi(\varphi)$ are cyclic functions :

$$S_\varphi(\varphi(t)) = S_\varphi(\varphi(t + T_\varphi)) \Leftrightarrow e^{i(\int_{\varphi(t_0)}^{\varphi(t)} L_\varphi \mathrm{d}\varphi)/\hbar} = e^{i(\int_{\varphi(t_0)}^{\varphi(t+T_\varphi)} L_\varphi \mathrm{d}\varphi)/\hbar} \tag{41}$$

$$1 = e^{i2\pi n_\varphi} = e^{i(\int_{\varphi(t)}^{\varphi(t+T_\varphi)} L_\varphi \mathrm{d}\varphi)/\hbar} \Leftrightarrow \oint L_\varphi \mathrm{d}\varphi = n_\varphi h \tag{42}$$

and similarily

$$S_\theta(\theta(t)) = S_\theta(\theta(t + T_\theta)) \Leftrightarrow e^{i(\int_{\theta(t_0)}^{\theta(t)} L_\theta \mathrm{d}\theta)/\hbar} = e^{i(\int_{\theta(t_0)}^{\theta(t+T_\theta)} L_\theta \mathrm{d}\theta)/\hbar} \tag{43}$$

$$1 = e^{i2\pi n_\theta} = e^{i(\int_{\theta(t)}^{\theta(t+T_\theta)} L_\theta \mathrm{d}\theta)/\hbar} \Leftrightarrow \oint L_\theta \mathrm{d}\theta = n_\theta h \tag{44}$$

and finally,

$$S_r(r(t)) = S_r(r(t + T_r)) \Leftrightarrow e^{i(\int_{r(t_0)}^{r(t)} p_r(r)\mathrm{d}r)/\hbar} = e^{i(\int_{r(t_0)}^{r(t+T_r)} p_r(r)\mathrm{d}r)/\hbar} \tag{45}$$

$$1 = e^{i2\pi n_r} = e^{i(\int_{r(t)}^{r(t+T_r)} p_r(r)\mathrm{d}r)/\hbar} \Leftrightarrow \oint p_r \mathrm{d}r = n_r h \tag{46}$$

We recognize here "Sommerfeld's quantum rules", (46) producing the required energy levels. In equation (46), the integral over dr is made from aphelion to perihelion and from perihelion to aphelion, symmetrically, while the integral over the angles coordinates is made in one direction. Indeed, $-\varphi$ and $-\theta$ are possible values, while $-r$ is forbidden, especially in the action $S(t, r, \theta, \varphi)$ given in equation (38). We then restrict our quantum number : $n_r > 0$ but make no such constraint on $n_\varphi, n_\theta, n_\phi$. Note that $n_\varphi, n_\theta, n_\phi$ ar not independant but linked by (40), n_φ refering to L_z and n_ϕ refering to L. We observe that (for $x, y, z \neq 0$):

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} = \begin{bmatrix} yz(p_z/z - p_y/y) \\ zx(p_x/x - p_z/z) \\ xy(p_y/y - p_x/x) \end{bmatrix} \tag{47}$$

$L_z = L_\varphi \neq 0 \Rightarrow p_y/y \neq p_x/x \Rightarrow L^2 > L_\varphi^2$ and $|n_\phi| > |n_\varphi|$. This last relation can be deduced from equation (40)(left hand side) and the definition of L_θ too. Clearly, equation (23) and its solution are only defined for $L \neq 0 \Leftrightarrow |n_\phi| \geq 1$. In the classical limit (with $L' \to L,\ n_\phi^2 - \alpha^2 \to n_\phi^2$), the energy levels (30), will produces Schrodinger's energy levels :

$$E = \frac{m_0c^2}{\sqrt{1 + \frac{\alpha^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}}} \approx m_0c^2 - \frac{\alpha^2 m_0 c^2}{2(n_r + |n_\phi|)} \to \epsilon = -\frac{\alpha^2 m_0 c^2}{2(n_r + |n_\phi|)} = -\frac{\alpha^2 m_0 c^2}{2n_s}\ ;\ n_s = n_r + |n_\phi| \tag{48}$$

Here n_s is Schrodinger's main quantum number. For a given value of $|n_\phi|$ there are two possible values of n_ϕ and $2|n_\phi|-1$ possible values of n_φ. This point justifies that our degeneracy is twice Schrodinger's degeneracy (see Table 1) since in his model $n_\phi<0$ is forbidden, making the wavefunction divergent. On the contrary, n_φ can be either positive or negative in both models (Schrodinger's model and the model introduced here). The fact that half of the observed degeneracy was missing in Schrodinger's theory was called "duplexity phenomena" by Dirac [Dirac, 1928]. There is no missing degeneracy in the present model since the degeneracy is, in the classical limit, $2n_s^2$:

Table 1 : Degeneracy for Hydrogen, in the classical limit

n_s	n_r	n_ϕ	n_φ	Degeneracy
1	0	±1	0	$2=2n_s^2$
2	1	±1	0	$2+6=8=2n_s^2$
	0	±2	$0,\pm1$	
	2	±1	0	
3	1	±2	$0,\pm1$	$2+6+10=18=2n_s^2$
	0	±3	$0,\pm1,\pm2$	

In general, the degeneracy is $2\sum_{n_\phi=1}^{n_\phi=n_s}2|n_\phi|-1=4\frac{n_s(n_s+1)}{2}-2n_s=2n_s^2$

Considering a non interacting system, in cartesian coordinates, the action S, according to equation (33) will take the form

$$S(x,y,z,t)=-Ht+\int p_xv_x+p_yv_y+p_zv_z\ \mathrm{d}t=-Ht+p_x(x-x_0)+p_y(y-y_0)+p_z(z-z_0) \tag{49}$$

Then Ψ will take the form, corresponding to the definition of a 4D Fourier transform,

$$\Psi(t,x,y,z)=\int\hat{\Psi}(\vec{k},w)e^{iS(t,x,y,z)/\hbar}\mathrm{d}\vec{k}\mathrm{d}w=\int\hat{\Psi}(\vec{k},w)e^{i(-Ht+p_x(x-x_0)+p_y(y-y_0)+p_z(z-z_0))/\hbar}\mathrm{d}\vec{k}\mathrm{d}w \tag{50}$$

$$\Psi(t,x,y,z)=\int\hat{\Psi}(k_x,k_y,k_z,w)e^{i(-wt+k_x(x-x_0)+k_y(y-y_0)+k_z(z-z_0))}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z\mathrm{d}w \tag{51}$$

which is the fundamental solution of both Klein-Gordon equation (with $H^2=p^2c^2+m_0^2c^4$) and Schrodinger equation (with $H=\frac{p^2}{2m_0}$) :

$$-\hbar^2\frac{\partial^2\Psi}{\partial t^2}=-\hbar^2c^2(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2})+m_0^2c^4\Psi\quad(Klein-Gordon) \tag{52}$$

$$i\hbar\frac{\partial\Psi}{\partial t}=\frac{-\hbar^2}{2m_0}(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2})=H_s\Psi\quad(Schrodinger) \tag{53}$$

H_s is Schrodinger's hamiltonian for a free particle. By definition, we have

$$\int\int\int\int|\hat{\Psi}(k_x,k_y,k_z,w)|^2\mathrm{d}w\ \mathrm{d}k_x\ \mathrm{d}k_y\ \mathrm{d}k_z\ <\infty\quad(can\ be\ normalized\ to\ unity) \tag{54}$$

Then, in this theory, a free particle obeys a wave-equation : free particles exhibit a wave-like behavior. With equations (38),(39) and (40), equation (20) can be written :

$$E^2-m_0^2c^4=p_r^2c^2+\frac{L^2c^2-k^2}{r^2}-\frac{2Ek}{r}=(\frac{\partial S}{\partial r})^2c^2+\frac{(\frac{\partial S}{\partial\phi})^2c^2-k^2}{r^2}-\frac{2Ek}{r} \tag{55}$$

With $\Psi(\vec{x},t)=\int\hat{\Psi}(\vec{k},w)e^{iS(\vec{x},t)/\hbar}\mathrm{d}\vec{k}\mathrm{d}w$ we deduce, $\frac{\partial S}{\partial r}=-i\hbar\frac{\partial\Psi}{\Psi\partial r}$, $\frac{\partial S}{\partial\phi}=-i\hbar\frac{\partial\Psi}{\Psi\partial\phi}$

$$(E^2-m_0^2c^4)\Psi^2=-\hbar^2(\frac{\partial\Psi}{\partial r})^2c^2+\frac{-\hbar^2(\frac{\partial\Psi}{\partial\phi})^2c^2-k^2\Psi^2}{r^2}-\frac{2Ek}{r}\Psi^2 \tag{56}$$

or, in cartesian coordinates,

$$(E+\frac{k}{r})^2\Psi^2 = -\hbar^2(\frac{\partial\Psi}{\partial x})^2c^2 - \hbar^2(\frac{\partial\Psi}{\partial y})^2c^2 - \hbar^2(\frac{\partial\Psi}{\partial z})^2c^2 + m_0^2c^4\Psi^2 \tag{57}$$

$$[(i\hbar\frac{\partial}{\partial t}+\frac{k}{r})\Psi]^2 = -\hbar^2(\frac{\partial\Psi}{\partial x})^2c^2 - \hbar^2(\frac{\partial\Psi}{\partial y})^2c^2 - \hbar^2(\frac{\partial\Psi}{\partial z})^2c^2 + m_0^2c^4\Psi^2 \tag{58}$$

Introducing the notation of quantum field theory : $A^\mu = (V/c, A_x, A_y, A_z)$, $A_\mu = (V/c, -A_x, -A_y, -A_z)$ with $V = -k/r$ (and $\vec{A} = 0$ for central potential) we rewrite,

$$0 = -[(i\hbar\frac{\partial}{\partial ct}-\frac{V}{c})\Psi]^2+[(-i\hbar\frac{\partial}{\partial x}-A_x)\Psi]^2+[(-i\hbar\frac{\partial}{\partial y}-A_y)\Psi]^2+[(-i\hbar\frac{\partial}{\partial z}-A_z)\Psi]^2+m_0^2c^2\Psi^2 \tag{59}$$

Introducing the operateur $D^\mu = \hbar\frac{\partial}{\partial x^\mu} - iA^\mu = \partial^\mu - iA^\mu$ with $x^\mu = (ct, x, y, z)$ and the usual convention of summation : $x^\mu x_\mu = c^2t^2 - x^2 - y^2 - z^2$ the previous equation can be written

$$0 = -(D^\mu\Psi)(D_\mu\Psi) + m_0^2c^2\Psi^2 = (D^\mu\Psi)(D_\mu\Psi) - m_0^2c^2\Psi^2 \tag{60}$$

or

$$((\partial^\mu - iA^\mu)\Psi)((\partial_\mu - iA_\mu)\Psi) - m_0^2c^2\Psi^2 = 0 \tag{61}$$

In student books on Quantum field theories, Equation (60) and (61) are known as the Lagrangian densities of a real scalar field or Klein-Gordon field. An example can be found in the course "Gauge field theory" of Cambridge university [Ben Gripaios, 2016]. Applying a variationnal principal to this Lagrangian density will produce Klein-Gordon equation in an electromagnetic field. In the present work, the left hand side of (61) is not a Lagrangian density, but (61) our equation for Ψ. Solving it directly gives $\mathbf{P_{QM}}$ since we produced it from our third postulate. As we have seen above, for a central potential, (61) implies the relativistic energy levels of atoms, their degeneracy, and two angular momenta.

"The present model should be sufficient to describe quantitatively the quantum tunneling. We quote [Merzbacher, 2002] dicussing theearly history of quantum tunneling :
In the nuclear case, the strong attractive forces inside the nucleus, still of mysterious origin in 1928, and the external Coulomb repulsion combine to form the potential barrier. Sketched in figure 6, this barrier was, of course, quite unlike a rectangular barrier or even the triangular barrier of figure 4 used for field emission, and the calculation had to be appropriately modified. The critically important exponent in the formula for the transmission coefficient was expressed as the phase (or action) integral in units of Plancks constant, where the limits r1 and r2 are the inner and outer classical turning points for an alpha particle with energy E. As shown in figure 7, the tunneling theories of1928 reproduced remarkably well the empirical relationship, established by Geiger and John Nuttall in 1912, between the decay rate and the energy of the emitted alpha particle, and at last provided firm evidence for the validity of quantum mechanics in the nuclear domain. This model, produced in 1928, and neglecting relativistic effect, was based on the planetary model of the atom, which is, for us, included in the action and then in the wavefunction (we used relativistic equations). The correspondance with experimental data is reproduced below (figure 7 for Merzbacher):

Figure 2 : Comparaison of planetary model with experiment to justify quantum tunneling

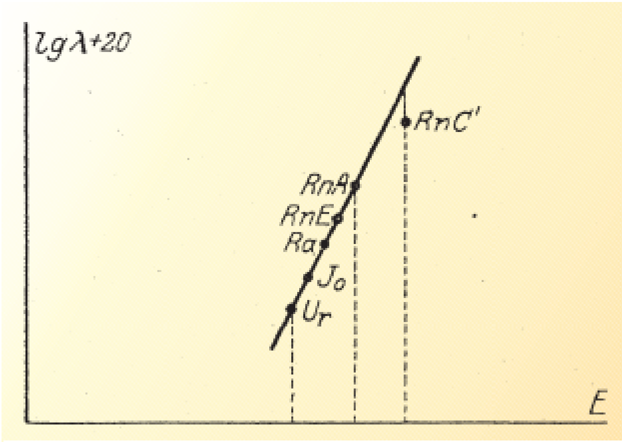


FIGURE 7. GEORGE GAMOW'S SEMILOG PLOT compares his approximate formula for the decay constant λ versus the alpha-particle energy E, and the empirical data of Hans Geiger and John Nuttall for the radioactive uranium series. The excellent agreement was the greatest triumph of the early days of the tunnel effect. (From ref. 12).

IV – Gravitation

We start from $\mathbf{P_{Grav}}$: the conservd energy is given by

$$E = \gamma m_0 c^2 + \frac{GM\gamma m_0}{r} = \gamma m_0 c^2 (1 - \frac{GM}{c^2 r}) \tag{62}$$

This equation is only valid in the low field approximation. In cartesian coordinates it can be written

$$E = \frac{m_0 c^2}{1 - (\frac{\dot{x}}{c})^2 - (\frac{\dot{y}}{c})^2} (1 - \frac{GM}{c^2 \sqrt{x^2 + y^2}}) \tag{63}$$

from which we deduce

$$\frac{dE}{dt} = (\ddot{x}\dot{x} + \ddot{y}\dot{y})\gamma^3 m_0 (1 - \frac{GM}{c^2 r}) + (x\dot{x} + y\dot{y})\gamma m_0 \frac{GM}{c^2 \sqrt{x^2 + y^2}^3} = 0 \tag{64}$$

$$\frac{dE}{dt} = \dot{x}[\gamma^3 m_0 \ddot{x}(1 - \frac{GM}{c^2 r}) + \frac{GM\gamma x}{r^3}] + \dot{y}[\gamma^3 m_0 \ddot{x}(1 - \frac{GM}{c^2 r}) + \frac{GM\gamma y}{r^3}] = 0 \tag{65}$$

$$dE = dx[\gamma^3 m_0 \ddot{x}(1 - \frac{GM}{c^2 r}) + \frac{GM\gamma x}{r^3}] + dy[\gamma^3 m_0 \ddot{x}(1 - \frac{GM}{c^2 r}) + \frac{GM\gamma y}{r^3}] = 0 \tag{66}$$

We recall

$$\frac{dp_x}{dt} = \frac{d(\gamma m_0 \dot{x})}{dt} = \gamma m_0 \ddot{x} + \gamma^3 m_0 \ddot{x}(\dot{x}/c)^2 = \gamma^3 m_0 \ddot{x}((1 - \dot{x}^2) + \dot{x}^2) = \gamma^3 m_0 \ddot{x} \tag{67}$$

and similarly

$$\frac{dp_y}{dt} = \gamma^3 m_0 \ddot{y} \tag{68}$$

to rewrite (66)

$$dE = dx[\frac{dp_x}{dt}(1 - \frac{GM}{c^2 r}) + \frac{GM\gamma x}{r^3}] + dy[\frac{dp_y}{dt}(1 - \frac{GM}{c^2 r}) + \frac{GM\gamma y}{r^3}] = 0 \tag{69}$$

and then deduce two independant equations of motion

$$\frac{dp_x}{dt} = \frac{GM\gamma}{(1-\frac{GM}{c^2r})r^3}x = F_x \quad ; \quad \frac{dp_y}{dt} = \frac{GM\gamma}{(1-\frac{GM}{c^2r})r^3}y = F_y \tag{70}$$

This gives back the classical equations of motion when $\gamma \approx 1$ and $1 - GM/(c^2r) \approx 1$. Then, the angular momentum L is conserved since

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{p}\wedge\vec{r})}{dt} = \frac{d\vec{p}}{dt}\wedge\vec{r} + \frac{d\vec{r}}{dt}\wedge\vec{p} = \vec{F}\wedge\vec{r} + \vec{v}\wedge\gamma m_0\vec{v} = 0 + 0 \tag{71}$$

Going back to (62) we write

$$E^2 = (1-\frac{GM}{c^2r})^2(\gamma m_0c^2)^2 = (1-\frac{GM}{c^2r})^2(p_r^2c^2 + \frac{L^2c^2}{r^2} + m_0^2c^4) \tag{72}$$

$$E^2 = (1-\frac{GM}{c^2r})^2((\gamma m_0\dot{r})^2c^2 + \frac{L^2c^2}{r^2} + m_0^2c^4) \tag{73}$$

with $L = \vec{p}\wedge\vec{r} = \gamma m_0 r^2\frac{d\phi}{dt} \Leftrightarrow \dot{r} = \frac{dr}{dt} = \frac{Ldr}{\gamma m_0 r^2 d\phi}$ we now write, dividing (73) by L^2c^2

$$(\frac{E}{Lc})^2 = (1-\frac{GM}{c^2r})^2((\frac{dr}{r^2d\phi})^2 + \frac{1}{r^2} + (\frac{m_0c^2}{Lc})^2) \tag{74}$$

or

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{r^2d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{GM}{c^2r})^2(\frac{1}{r^2} + (\frac{m_0c^2}{Lc})^2) \tag{75}$$

In the solar system $\frac{GM}{c^2r} \approx 10^{-6}$ then we can write

$$(1-\frac{2GM}{c^2r})(\frac{dr}{r^2d\phi})^2 \approx (\frac{E}{Lc})^2 - (1-\frac{2GM}{c^2r})(\frac{1}{r^2} + (\frac{m_0c^2}{Lc})^2) \tag{76}$$

and compare with the equation of motion in general relativity (See C Magnan reproducing J.A. Wheeler and S. Weinberg for an example [Magnan, 2007]):

$$(\frac{dr}{r^2d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{2GM}{c^2r})(\frac{1}{r^2} + (\frac{m_0c^2}{Lc})^2) \tag{77}$$

Our postulate $\mathbf{P_{Grav}}$ is approximatly analogous to general relativity, and allows us to describe gravitation and electromagnetism with strictly the same principles. Especially, the two previous equations have the same roots for $dr/d\phi = 0$. Accuracies of experimental data for general relativity are given in table 2

Table 2 : Experimental accuracy for measurements in general relativity

Reference	Measurement	Accuracy (exp)	$\frac{2GM}{c^2r}$
Hafele -Keating (1972)	Gravitationnal time dilation on Earth	$\approx 10^{-1}$	$\approx 10^{-9}$
Pound-Rebka (1959)	Gravitationnal redshift on Earth	$\approx 10^{-1}$	$\approx 10^{-9}$
Vessot et al. (1980)	Gravitationnal redshift on Earth	$\approx 10^{-4}$	$\approx 10^{-9}$
Shapiro (1968)	Gravitationnal time delay induced by Sun	$\approx 10^{-1}$	$\approx 10^{-6}$
Clemence (1947)	Mercury perihelion	$\approx 10^{-1}$	$\approx 10^{-8}$
TMET (1973)	Light Deflection by Sun (during eclipse)	$\approx 10^{-1}$	$\approx 10^{-6}$

TMET refers to "Texas Mauritanian Eclipse Team".

In each case, the extra term $\frac{2GM}{c^2r}$ is negligible when compared to unity or experimental uncertainties

Conclusion :

In part I, we examined classical Kepler's problem, gave some new links involving Planck's law, Kepler's third law, and Heiseinberg uncertainties relation. We established some geometrical properties of the three involved angular momenta.

In part II Vand III, we recalled Sommerfeld's model of matter, gave a new quantum model of atoms justifying their relativistic energy levels, their degeneracy, and the wave like behavior of the electron. All historical models of atoms (Bohr, Sommerfeld, Pauli, Schrodinger, Dirac, Fock) were discussed.

In part VI , we gave a new relativistic theory of gravitation, which we compared to general relativity, showing a similar behavior in the low field limit. Everything came from 3 very simple postulates..

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Annex : Sommerfeld′s integral

We reproduce here [Sommerfeld, 1916] to compute equation (2). Comments are suppressed.

$$p_r = \gamma m_0 \dot{r} = \gamma m_0 \frac{dr}{d\phi}\dot{\phi} = \frac{L}{r^2}\frac{dr}{d\phi}\ ;\ dr = \frac{dr}{d\phi}d\phi\ ;\ p_r dr = L(\frac{dr}{rd\phi})^2 d\phi \tag{78}$$

We have seen under equation (23) that (we fix $\phi_0 = 0$) that $u = 1/r$ takes the form

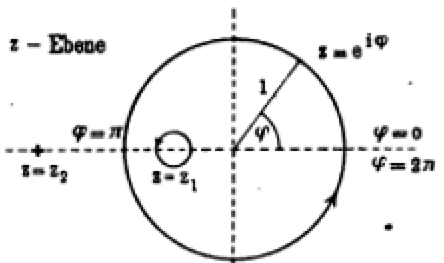
$$u = A(1 + ecos(\Gamma\phi) \Rightarrow \frac{dr}{rd\phi} = \frac{e\Gamma sin(\Gamma\phi)}{1 + ecos(\Gamma\phi)} \tag{79}$$

From this result we deduce, with $\varphi = \Gamma\phi$

$$\oint p_r dr = L\int_0^{2\pi}(\frac{e\Gamma sin(\Gamma\phi)}{1 + ecos(\Gamma\phi)})^2 d\phi = L\Gamma e^2\int_0^{2\pi}(\frac{sin(\varphi)}{1 + ecos(\varphi)})^2 d\varphi \tag{80}$$

We recall $L\Gamma = L'$. Sommerfeld uses ϵ for e, γ for Γ :

Fig. 100.



$$(1)\quad J_1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{1+\varepsilon\cos\varphi}$$

$$z = e^{i\varphi}$$

$$J_1 = \frac{1}{2\pi i}\oint\frac{dz}{z\left[1+\frac{\varepsilon}{2}\left(z+\frac{1}{z}\right)\right]} = \frac{1}{\pi i\varepsilon}\oint\frac{dz}{z^2+2\eta z+1}$$

$$\eta = \frac{1}{\varepsilon};\qquad \begin{cases} z_1 = -\eta+\sqrt{\eta^2-1}\ \ldots\ldots -z_1 < 1 \\ z_2 = -\eta-\sqrt{\eta^2-1}\ \ldots\ldots -z_2 > 1 \end{cases}$$

$$J_1 = \frac{1}{\pi i\varepsilon}\oint\frac{dz}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{\pi i\varepsilon(z_1-z_2)}\left(\oint\frac{dz}{z-z_1}-\oint\frac{dz}{z-z_2}\right)$$

$$J_1 = \frac{2\pi i}{\pi i\varepsilon(z_1-z_2)}\qquad z_1-z_2 = 2\sqrt{\eta^2-1} = \frac{2}{\varepsilon}\sqrt{1-\varepsilon^2},$$

$$J_1 = \frac{1}{\sqrt{1-\varepsilon^2}}$$

$$\boxed{J_2 = \frac{\varepsilon^2}{2\pi}\int_0^{2\pi}\frac{\sin^2\varphi}{(1+\varepsilon\cos\varphi)^2}\,d\varphi} = \frac{\varepsilon}{2\pi}\,\frac{\sin\varphi}{1+\varepsilon\cos\varphi}\Bigg|_0^{2\pi} - \frac{\varepsilon}{2\pi}\int_0^{2\pi}\frac{\cos\varphi}{1+\varepsilon\cos\varphi}$$

$$J_2 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{1+\varepsilon\cos\varphi}-1\right)d\varphi = J_1 - 1 = \frac{1}{\sqrt{1-\varepsilon^2}}-1.$$

with $e = K/J(= c/a)$ we deduce from equation (123) and the above result :

$$\oint p_r dr = 2\pi L'(\frac{1}{\sqrt{1-e^2}} - 1) = 2\pi L'(\frac{J}{\sqrt{J^2-K^2}} - 1) = 2\pi L'(\frac{J}{L'} - 1) = 2\pi(J - L') \tag{81}$$

In the classical limit $L' \to L$. The second method starts from the equation (20)

$$E^2 - m_0^2 c^4 = p_r^2 c^2 + \frac{L^2c^2 - k^2}{r^2} - \frac{2Ek}{r} \Leftrightarrow p_r = \sqrt{[(\frac{E}{c})^2 - m_0^2 c^2] + 2\frac{Ek/c^2}{r} - \frac{L'^2}{r^2}} = \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}} \tag{82}$$

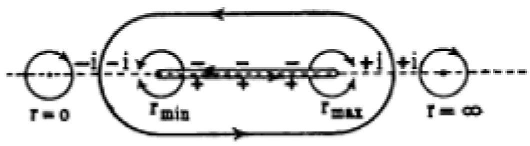
From which we deduce,

$$\oint p_r dr = \oint \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}} dr \tag{83}$$

This integral must be performed from the aphelion to the perihelion (r_{min} and r_{max}), Sommerfeld computed this integral over the complex plane by means of the residue theorem.

$$J_s = \oint \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}}\, dr.$$

Fig. 101.



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$$r = 0. \quad \Rightarrow \quad \sqrt{C}\int \frac{dr}{r}\left(1 + \frac{B}{C} r + \cdots\right). \quad \Rightarrow \quad -2\pi i \sqrt{C}.$$

$$s = \frac{1}{r}, \quad dr = -\frac{ds}{s^2} \qquad J_s = -\int \sqrt{A + 2Bs + Cs^2}\,\frac{ds}{s^2}$$

$$= -\sqrt{A}\int\left(1 + \frac{B}{A}s + \cdots\right)\frac{ds}{s^2}.$$

$$\Rightarrow \quad +2\pi i \frac{B}{\sqrt{A}}.$$

$$\Rightarrow \quad J_s = -2\pi i\left(\sqrt{C} - \frac{B}{\sqrt{A}}\right)$$

We deduce,

$$\oint p_r dr = -2\pi i(iL' - iJ) = 2\pi(J - L') \;\; (or\ 2\pi(J - L)\ in\ classical\ limit) \tag{84}$$