The difference of any real transcendental number and complex number e^i is always a complex transcendental number.

Charanjeet Singh Bansrao

bansrao357@gmail.com

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From Euler's formula,

$$e^{ix} = \cos x + i\sin x$$

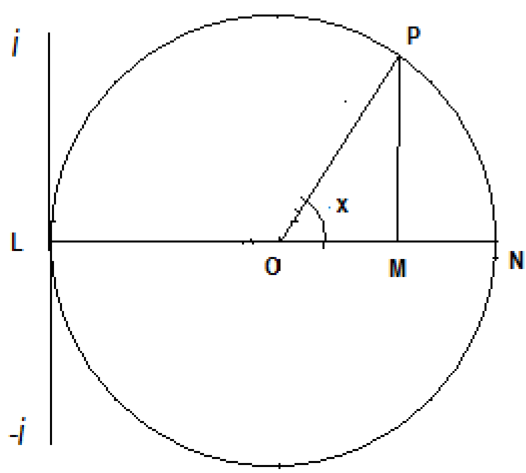
we can derive the following equation

$$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2}$$

Where , a, c and x are real numbers such that

$$x = \cos^{-1}\frac{2c-a}{a} \text{ and } c = \frac{a(\cos x+1)}{2}$$

The above equation can be obtained as follows. Consider the following circle on a complex plane with center O touching the imaginary axis at zero.



Let length LM is c and MN is b then diameter LN should be c+b and radius OP or ON is $\frac{c+b}{2}$. In this way, the length OM is

$$OM = ON - MN$$

$$= \frac{c+b}{2} - b = \frac{c-b}{2}$$

Using Pythagoras theorem, the length PM can be obtained as follows:

$$PM = \sqrt{OP^2 - OM^2}$$

$$PM = \sqrt{\left[\frac{c+b}{2}\right]^2 - \left[\frac{c-b}{2}\right]^2} = \sqrt{cb}$$

Hence $$\sin x = \frac{PM}{OP} = \frac{2\sqrt{cb}}{c+b} \quad (1)$$

and $$\cos x = \frac{OM}{OP} = \frac{c-b}{c+b} \quad (2)$$

Insert (1) and (2) in Euler's formula to get

$$e^{ix} = \frac{c-b}{c+b} + i\,\frac{2\sqrt{cb}}{c+b}$$

$$(c+b)e^{ix} = c - b + 2i\sqrt{cb}$$

Let $c + b = a$ then $b = a - c$; so we have,

$$ae^{ix} = c - (a - c) + 2i\sqrt{c(a-c)}$$

Or $$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2} \quad (3)$$

Lemma 1. If x is algebraic and $x \in \left\{\cos^{-1}\frac{2c-a}{a}\right\}$ then a is algebraic and c is transcendental.

Proof: Consider equation 2

$$\cos x = \frac{c-b}{c+b}$$

Or $\cos x = \frac{c-b}{a} = \frac{2c-a}{a}$ because $b = a - c$.

Therefore $x = \cos^{-1}\frac{2c-a}{a}$

According to Lindemann's theorem, for all algebraic values of x the trigonometric function $\cos x$ is transcendental. But $\frac{2c-a}{a}$ can always be transcendental only if a is algebraic and c is transcendental. This implies that set of algebraic values of x is subset of $\left\{\cos^{-1}\frac{2c-a}{a}\right\}$ when a is algebraic and c is transcendental.

It is not impossible to make number a algebraic of any desired value by adding some unknown transcendental number b in c. Similarly we can obtain any desired algebraic value of x by adjusting the value of the number a.

Lemma 2. $2i\sqrt{ca-c^2}$ is always transcendental if a is algebraic and c is transcendental.

Proof. Let $ca-c^2$ = y

Where we assume y is algebraic. We get the following quadratic equation:

$$c^2 \quad -ac + y = 0$$

Therefore $$c = \frac{-(-a) \pm\sqrt{(-a)^2 -4y}}{2} \qquad (4)$$

Since c is transcendental and a is algebraic then equation (4) can only be transcendental if y is transcendental. Therefore our assumption that y or $ca -c^2$ is algebraic is wrong .Hence term $2i\sqrt{ca-c^2}$ is transcendental.

Proposition: $c-e^i$ is a complex transcendental number where c is any real transcendental number.

Proof: We can re-write equation 3 as follows:

$$a = 2i\sqrt{ca-c^2} \ +2c - ae^{ix} \qquad (5)$$

Let a is algebraic and c is transcendental.

We have $x = \cos^{-1}\frac{2c-a}{a}$

According to lemma 1 we can make x alegbraic of value of one radian by adjusting the value of a. Hence equation 5 becomes—

$$a = 2i\sqrt{ca-c^2} \ +2c - ae^{i}$$

Since a is algebraic therefore both the terms $2i\sqrt{ca - c^2}$ and $2c - ae^i$ should be algebraic or transcendental.

But from lemma 2 we know $2i\sqrt{ca - c^2}$ is transcendental therefore $2c - ae^i$ is also transcendental.

We can adjust the values of coefficients 2 and a to make them equal to some algebraic number n without affecting the value of the term $2c - ae^i$. Hence we can write --

$$n(c - e^i) = 2c - ae^i$$

In this way we conclude that the difference $c - e^i$ is a complex transcendental number where c is any real transcendental number.