

The difference of any real transcendental number and complex number e^i is always a complex transcendental number.

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From Euler's formula,

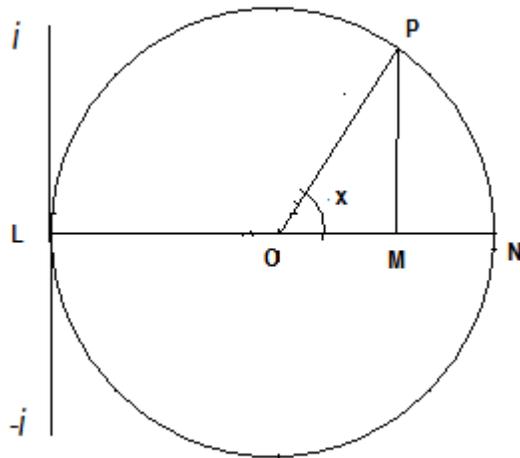
$$e^{ix} = \cos x + i \sin x$$

we can derive the following equation

$$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2}$$

where , a , c and x are real numbers.

The above equation can be obtained as follows. Consider the following circle on a complex plane with center O touching the imaginary axis at zero.



Let length LM is c and MN is b then diameter LN should be $c+b$ and radius OP or ON is $\frac{c+b}{2}$. In this way, the length OM is

$$\begin{aligned} OM &= ON - MN \\ &= \frac{c+b}{2} - b = \frac{c-b}{2} \end{aligned}$$

Using Pythagoras theorem, the length PM can be obtained as follows:

$$PM = \sqrt{OP^2 - OM^2}$$

$$PM = \sqrt{\left[\frac{c+b}{2}\right]^2 - \left[\frac{c-b}{2}\right]^2} = \sqrt{cb}$$

Hence $\sin x = \frac{2\sqrt{cb}}{c+b}$ (1)

and $\cos x = \frac{c-b}{c+b}$ (2)

Insert (1) and (2) in Euler's formula to get

$$e^{ix} = \frac{c-b}{c+b} + i \frac{2\sqrt{cb}}{c+b}$$

$$(c+b)e^{ix} = c-b + 2i\sqrt{cb}$$

Let $c+b = a$ then $b = a-c$; so we have,

$$ae^{ix} = c - (a-c) + 2i\sqrt{c(a-c)}$$

Or $ae^{ix} = 2c - a + 2i\sqrt{ca - c^2}$ (3)

Lemma 1. If x is algebraic and $x \in \left\{ \cos^{-1} \frac{2c-a}{a} \right\}$ then a is algebraic and c is transcendental.

Proof: Consider equation 2

$$\cos x = \frac{c-b}{c+b}$$

Or $\cos x = \frac{c-b}{a} = \frac{2c-a}{a}$ because $b = a-c$.

Therefore $x = \cos^{-1} \frac{2c-a}{a}$

According to Lindemann's theorem for all algebraic values of x the trigonometric function $\cos x$ is transcendental. But $\frac{2c-a}{a}$ can always be transcendental only if a is algebraic and c is transcendental. This implies that set of algebraic values of x is subset of $\left\{ \cos^{-1} \frac{2c-a}{a} \right\}$ when a is algebraic and c is transcendental.

It is not impossible to make number a algebraic of any desired value by adding some unknown transcendental number b in c . Similarly we can obtain any desired value of x by adjusting the value of number a .

Lemma 2. $2i\sqrt{ca - c^2}$ is always transcendental if a is algebraic and c is transcendental.

Proof. Let $ca - c^2 = y$

Where we assume y is algebraic. We get the following quadratic equation:

$$c^2 - ac + y = 0$$

Therefore
$$c = \frac{-(-a) \pm \sqrt{(-a)^2 - 4y}}{2} \quad (4)$$

Since c is transcendental and a is algebraic then equation (4) can only be transcendental if y is transcendental. Therefore our assumption that y or $ca - c^2$ is algebraic is wrong. Hence term $2i\sqrt{ca - c^2}$ is transcendental.

Theorem: $c - e^i$ is a complex transcendental number where c is any real transcendental number.

Proof : We can re-write equation 3 as follows:

$$a = 2i\sqrt{ca - c^2} + 2c - ae^{ix} \quad (5)$$

Let a is algebraic and c is transcendental.

We have $x = \cos^{-1} \frac{2c-a}{a}$

According to lemma 1 we can make x equals to one radian by adjusting the value of a . Hence equation 5 becomes—

$$a = 2i\sqrt{ca - c^2} + 2c - ae^i$$

Since a is algebraic therefore both the terms $2i\sqrt{ca - c^2}$ and $2c - ae^i$ should be algebraic or transcendental.

But from lemma 2 we know $2i\sqrt{ca - c^2}$ is transcendental therefore $2c - ae^i$ is also transcendental.

We can adjust the values of coefficients 2 and a to make them equal to some algebraic number n without affecting the value of the term $2c - ae^i$. Hence we can write --

$$n(c - e^i) = 2c - ae^i$$

In this way we conclude that the difference $c - e^i$ is a complex transcendental number where c is any real transcendental number.