Solution for a Special case of the Toeplitz conjecture

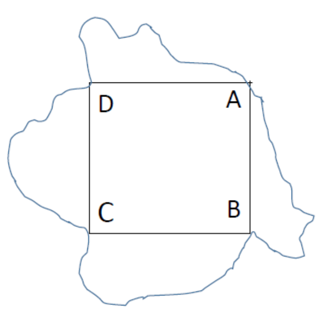
Prashanth R. Rao

Abstract: According to Toeplitz conjecture or the inscribed square conjecture, every simple closed curve in a plane must have atleast one set of four points on it that belong to a square. This conjecture remains unsolved for a general case although it has been proven for some special cases of simple closed curves. In this paper, we prove the conjecture for a special case of a simple closed curve derived from two simple closed curves, each of which have exactly only one set of points defining a square. The Toeplitz solution squares of two parent simple closed curves have the same dimensions and share exactly one common vertex and the adjacent sides of the two squares form a right angle. The derived simple closed curve is formed by eliminating this common vertex (that belonged to the two solutions squares to begin with) and connecting other available points on the parent curves. We show that this derived simple closed curve has atleast one solution square satisfying the Toeplitz conjecture.

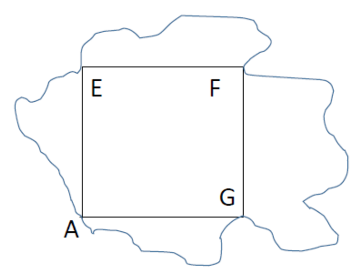
Results:

We start with two simple closed curves, each with exactly one solution of four points that form a square and the two squares have sides with identical length. (Figure 1).

Figure 1



Simple closed curve 1 with exactly one Toeplitz solution Square A-B-C-D

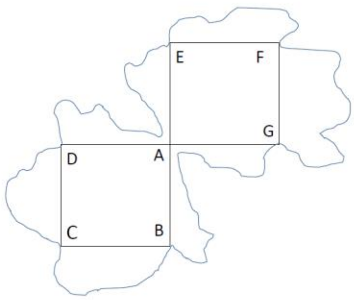


Simple closed curve 2 with exactly one Toeplitz solution Square A-E-F-G

The two starting simple closed curves

Next we merge these two squares at one common vertex as shown in Figure 2 and derive a new simple closed curve by excluding the common vertex and connecting available points on the two starting curves. Therefore the six points D,C,B,E,F,G lie on this curve and the common vertex of the initial squares, point A is excluded from it. Since the two squares are juxtaposed so that points DAG and EAB are collinear and the two lines DAG and EAB are at right angles to one another. Therefore angles DAE, DAB, EAG and GAB are right angles.

Figure 2



Simple closed loop obtained by merging the two simple closed curved curves but excluding A as a point on the derived simple closed curve

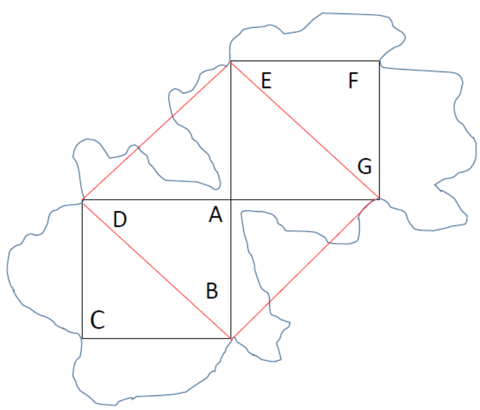
The derived simple closed curve must have atleast one Toeplitz solution square D-E-G-B.

Let the length of the sides of the two starting squares be "s". Then since DAE, EAG, GAB and BAD are right angles therefore the corresponding segments DE, EG, GB and DB must represent the hypotenuses of the right angles triangles DAE, EAG, GAB and BAD. Therefore each of these segments DE, EG, GB and DB have length "$s\sqrt{2}$ ".

Consider triangle EBD, EB is equal to "2s". So EB must represent the hypotenuse of the triangle EBD where the other two sides are each of length " $s\sqrt{2}$ ". So angle EDB is a right angle. Similarly angle DEG, EGB, GBD, BDE are all right angles.

Therefore DEGB must be a square and therefore represents atleast one Toeplitz solution of the derived simple closed curve.

Figure 3



The derived simple closed curve has atleast one Toeplitz solution D-E-B-G