Quantum Gravity and the Titius-Bode Rule

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Today, in the areas of theoretical foundation and experimental verification, there is great interest in the properties of a future Quantum Gravity. The ultimate goal of the present research is to contribute to the creation of a definitive, universally acceptable, theory of Quantum Gravity. In the present paper, the two hundred and fifty years of history of the empirical Titius-Bode rule is investigated, under the assumption that this rule is key evidence for a quantum feature in the already long-known classical gravity.

Keywords: Titius-Bode rule, Bohr-Sommerfeld quantization, de Broglie matter wave, Quantum Gravity.

1. Introduction

Currently there is increased interest in the properties of Quantum Gravity, in order to establish its foundation in both theoretical and experimental areas. The currently favored Standard Model, which unifies the basic physical interactions into a single theoretical framework, does not contain the gravitational interaction. The root of this problem is the fact that the most advanced theory of gravity, Einstein's general relativity theory (GRT), cannot, either in vision, or in mathematical formulation, be reconciled with the philosophy of modern quantum mechanics (QM).

In recent decades, most attempts to unify gravity and the other fundamental physical interactions into a common theoretical description have been associated with the various attempts to increase the dimension of the well-known four-dimensional (relativistic) space-time. These so-called string theories and membrane theories use very complicated mathematical tools; in addition, the experimental support can seem hopeless, since its assumed effects appear in unobservably small space-time domains.

By simpler theoretical considerations, the quantum of the gravitational field, if it exists at all, is a spin-2, massless particle that would be called the 'graviton', by analogy with the photon. The direct detection of the graviton has so far been unavailable due to its estimated extremely low energy. It seems that in the other fields successfully used methods of QM and quantum field theories did not lead to breakthrough results. Over time, the recurring failures of these attempts incite us to approach the problem with completely different physical considerations.

About two hundred and fifty years ago, the so-called Titius-Bode rule (T-B rule) for the known planets was found. It describes the approximate distances of the planets from the Sun with an exponentially quantized function [1- 3]. The semi-major axes of the planets in astronomical unit are approximately:

$$a_n \cong 0.4 + 0.3 \cdot 2^n ; \; (n = -\infty, 0, 1, 2, \ldots) \quad . \tag{1}$$

In the case of the innermost planet (Mercury) the exponent n is minus infinity (in this case the second term is zero), but for the other planets the second term in the formula includes non-negative exponents of integer values. Especially in the case of Earth, $n = 1$, and the formula gives a unit value for the Sun-Earth distance, according to the definition of the astronomical distance unit. For details, we find a lot of information on the Internet.

Table 1 gives the results of T-B rule calculations including the real and calculated planetary distances; the relative errors of the calculated values are shown as percentages. The standard deviation of the calculated distances is very high; it is about 33%.

Table 1. Demonstration of the Titius-Bode rule

Planet	Real distance	Calculated distance	Relative error
Mercury	0.39	0.4	2.56%
Venus	0.72	0.7	2.78%
Earth	1	1	0%
Mars	1.52	1.6	5.26%
Ceres	2.77	2.8	1.08%
Jupiter	5.2	5.2	0%
Saturn	9.54	10	4.82%
Uranus	19.2	19.6	2.08%
Neptune	30.06	38.8	29.08%
Pluto	39.44	77.2	95.74%

The aim of the present work is a new, alternative physical interpretation of T-B rule, which would in the future be the starting point for a Quantum Gravity Theory.

Over the past centuries, and over in recent decades, a number of attempts have been made to decipher of the physical background of T-B rule. Unfortunately disturbing evidence exists that, for the moons of the planets in the Solar System, the T-B rule only partially or not at all satisfied in some cases, but the exponential distribution for distance seems to be their common property [4].

Physicists, astronomers cannot see any new physical law in the T-B rule, since the Solar System developed over billions of years trough chaotic, dissipative processes, a series of random mass collisions, which played a crucial role in the generation of the Solar System. However, some physicists accept the existence of some kind of regularity trends, which are interpreted as 'path resonances' [5]. In their view, this regularity is the direct consequence of the long gravitational couplings between the planets, which produced for the orbital radii simple rational fractions as: 1:2, 2:3, 2:5, *etc*.

Some physicists, including the author of present work, think a deeper physical law lies behind the T-B rule. On the Internet, and in recognized astronomical journals as well, there are many scientific articles related to the theoretical modeling of this mysterious behavior of the planet's orbits. In the present work we are trying to understand the T-B rule by a supposed quantum property of the gravity. It should be noted, that we are not the first to combine the gravity with the macroscopic manifestations of QM, see [6-10].

2. The Exponential Approximation

Many authors conclude that the distribution of the distances of the planets around the Sun has the mathematical essence of a typical exponential distribution [11-13]. The physicists concerned have tried to explain the situation by different physical theories, but no really reassuring and generally accepted theory exists today. The planetary system was formed over billions of years, and we all can agree that in this long time period random processes played a decisive role. However, this long period may also have allowed a presently unknown property of gravity to form the exponential distribution of the distances for the majority of planets and planet's moons, with limited accuracy of course. In this light, it is an obvious thing to fit to the known planetary distances an exponential function, which in the recent past has also been realized in many other places. However, we cannot speak about a final canonized result in this respect.

In present work we have carried out the fitting procedure for the planetary distances of the Solar System, assuming the exponential distribution. Using the real distance data from **Table 1**., the obtained best result is:

$$a_n = a_0\alpha^n ;\ (n = 1, 2, 3, \ldots 10) \quad , \tag{2.1}$$

where $n = 1$ belongs to the distance of the most inner planet Mercury (*i.e.* semi-major axis of the ellipse). The further planetary distances belong to the powers $n = 2, 3, 4, \ldots$ *etc*. The result of the math fitting is:

$$a_0 = 0.2108\ldots;\ \alpha = 1.7078\ldots;\ \sigma = 0.130\ldots = 13\%. \tag{2.2}$$

It cannot be said that the obtained standard deviation σ is too large or too small, but if we insist that the exponential distribution cannot be a coincidence; the 13% standard deviation supports our belief.

Of course, additional examination has been carried out wherein we omitted from the calculation some 'irregular' planets. Regrettably, this way led to no significant improvement of the (2.1) exponential rule. Now we exemplify it with two calculations. In first example we omitted from the calculation Uranus and Neptune:

$$a_0 = 0.2211\ldots;\ \alpha = 1.6799\ldots;\ \sigma = 0.099\ldots = 9.9\%. \tag{2.3}$$

In the second example, we omitted from the calculation five planets: Venus, Mars, Saturn, Uranus, and Neptune. The matching result is:

$$a_0 = 0.2207\ldots;\ \alpha = 1.6778\ldots;\ \sigma = 0.052\ldots = 5.2\%. \tag{2.4}$$

The obtained results indicate that the constants a_0 and α did not change significantly by omitting of 'irregular' planets; however, the accuracy of the fitted exponential model improved noticeably. These facts ultimately contribute to our belief that behind the T-B rule, an unknown, but surely important, real physical law hides.

3. Refining the Titius-Bode Rule

The fitted exponential functions of the planetary distances described in the previous Section contain only one 'quantum number'. In this relation, however, an important question arises: Can there be such exponential functions with two or more quantum numbers which are capable of calculating planetary distances more accurately than the simple exponential formula (2.1)? The mathematical relevance of the question is whether we are able to discover such functions. The other side of this question is far more important; namely, whether there can be found such a multi-quantum-variable function, which, in one way or another, can be connected to a real or perceivably real physical explanation. On the Internet we have found this kind of functions for descriptions of planetary distances completed more or less with the analysis of the physical background; for example [14-17].

Regarding to the mathematical point of view, we recently found surprisingly good mathematical functions for the high precision description of planetary distances. Each of these has two quantum numbers, which we named as principal quantum number (n) and orbital quantum number (j) by analogy to the well-known quantum numbers of the hydrogen atom. The functions studied to date and considered successful for the quantized distance follow:

1. The first example of the distance function contains two fitting parameters:

$$a_n \cong \mathrm{a}_0(\alpha^n + \alpha^j) ;\ n = 1, 2, \ldots N;\ j = 0, 1, \ldots N - 1 . \tag{3.1}$$

In the fitting procedure we have taken into account all the distance data from the **Table 1**. The obtained fitting parameters and the standard deviation of the model are:

$$a_0 = 0.143913\ldots;\ \alpha = 1.746846\ldots;\ \sigma = 0.0271\ldots = 2.71\%. \tag{3.2}$$

To understand the above statement correctly, each planet's distance is assigned two quantum numbers that are specific to the given planet. This surprisingly simple formula, depending on only two fitting parameters, gives very good values for the real planetary distances. The only problem with this formula is that in the case $j = 0$, it does not return to the quantized exponential function (2.1) that we studied in the previous Section.

2. The second example of the distance function contains three fitting parameters (**Table 2.**):

$$a_n \cong a_0 \alpha^n \beta^{-j};\ (n = 1,2,...N;\ j = 0,1,...N-1)\ . \qquad (3.3)$$

It can be seen that in the case $j = 0$, this formula returns the tested exponential function (2.1) in the previous Section. The fitted result is also accurate for the planetary distances; the relative standard deviation is around 2.3%:

$$\begin{gathered} a_0 = 0.220153...;\ \alpha = 1.813371...; \\ \beta = 1.157176...;\ \sigma = 0.0228... = 2.28\%\ . \end{gathered} \qquad (3.4)$$

Table 2. Generalization of T-B rule
using the double quantum-numbered calculation model of (3.3).

Planet	n	j	Real distance	Calculated distance	Relative error
Mercury	1	0	0.39	0.3992	2.36%
Venus	2	0	0.72	0.7239	0.55%
Earth	3	2	1	0.9804	-1.96%
Mars	4	3	1.52	1.5363	1.07%
Ceres	5	3	2.77	2.7859	0.57%
Jupiter	6	3	5.2	5.0518	-2.85%
Saturn	7	3	9.54	9.1608	-3.97%
Uranus	8	2	19.2	19.2230	0.12%
Neptune	9	3	30.06	30.1236	0.21%
Pluto	10	5	39.44	40.7939	3.43%

It is important to mention that this latter introduced distance function exactly corresponds to our newly established quantized model of gravity based on the old quantum theory of Bohr-Sommerfeld.

Remark: The fitting procedure of the above shown distance functions has been realized by the Monte-Carlo method. For calculating the standard deviation of the fitted planet distances, we used the usual method:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\left[(a_n - a'_n)/a_n\right]^2}\quad . \qquad (3.5)$$

In this formula the real planet distances are represented by a_n, the calculated planet distances are represented by a'_n, and finally N is the number of the planets have been involved in the calculation.

4. The 'Wave Nature' of the Matter

In the previous Section, the planet's distances were described by a double quantum-numbered formula that reminds us of the quantum mechanical model of the hydrogen atom. The simple fact that the planets occupy approximately exponentially quantized orbits around the Sun does not itself imply any genetic link to the Quantum Mechanics (QM). However, we are going to show that the exponential distribution of the planetary orbits intrinsically connected to the previous version of the QM; namely to the old quantum theory. This new recognition is indeed a possible scientific direction to a really solid foundation of the long-sought quantum theory of gravity. In the following we show that our unusual way leads to the surely real physical interpretation of the Titius-Bode law.

The starting point of the old quantum theory is known as Bohr-Sommerfeld (B-S) quantization theory [19-21]. In the B-S theory, the quantization of a closed physical system can be realized by the following rule:

$$S = \oint p_i dq_i = n_i h;\ (i = 1,2,3,...;\ n_i = 1,2,3,...), \qquad (4.1)$$

where p_i are the momentum components of the particles, q_i are the coordinates of the particles, i counts the number of degrees of freedom of the system, and h is Planck's constant. The quantum numbers n_i are positive integers, and the integral is taken over one period of the motion at constant energy (as described by the Hamiltonian). The integral S is an area in the so-called 'phase space', and has dimension of 'action', and is quantized in units of Planck's constant. For this reason, Planck's constant was often called '*elementary quantum of action*'.

The initial successes of the B-S theory fed high hopes for understanding quantum phenomena by using this theory. It successfully produced the known quantized energy levels of harmonic oscillator, and perhaps the most important result was the clarification of Bohr's atomic model. A little later, A. Sommerfeld developed a relativistic formulation for Bohr's model, also on the principle of the B-S quantization. The 'relativistic atom-model' led to the interpretation of the fine structure of the hydrogen spectrum, which remains appropriate up to today. At the same time, despite all efforts, the B-S quantization was not suitable for the description of spectra of two- or many-electron atoms. For the general solution we had to wait until the middle of the 1920's, the birth of QM.

The first real breakthrough in this field came in 1925, when appeared the epoch-making article of W. Heisenberg in which he gave a really operable mathematical background for the description of quantum phenomena. Heisenberg assigned infinite-dimensional matrices to the physical quantities (coordinates, momentums); hence, the name of his theory is matrix mechanics. In 1926, E. Schrödinger found another version of QM, which has been named wave mechanics. It was equivalent to the Heisenberg's matrix mechanics, as Schrödinger himself showed from the beginning. The basic idea of wave mechanics came from French physicist Louis de Broglie, who already in 1924 created the theory of electron waves, at that time without any remarkable scientific echo [22]. Today, de Broglie's concept of matter waves is fully accepted by the physicist's community.

De Broglie's idea of matter waves was based on the theory of relativity. In 1900 M. Planck showed that the most experimentally known laws of the thermal (black body) radiation can be interpreted only by quantized energy radiation:

$$E = h\nu \equiv \hbar\omega\ . \qquad (4.2)$$

Here h is Planck's constant, ν is the frequency of the thermal (electromagnetic) radiation, $\hbar = h/2\pi$ and $\omega = 2\pi\nu$. In relativity, the energy and the three components of momentum form a four-vector (c is the speed of light):

$$p_\mu = \left\{ E/c, p_x, p_y, p_z \right\} \quad . \tag{4.3}$$

Planck's law of thermal radiation ties frequency ω to the energy E according to (4.2), in this relation whether what physical quantities can be associated with the momentum components? In the relativity, the electromagnetic wave assigned to the *wave number four-vector* (shortly wave four-vector), which its first component is just equal to the frequency of the electromagnetic wave divided by the light speed:

$$k_\mu = \left\{ k_0 = \omega/c,\ k_x = 2\pi/\lambda_x,\ k_y = 2\pi/\lambda_y,\ k_z = 2\pi/\lambda_z \right\}. \tag{4.4}$$

The λ 's are the spatial components of the wavelength. The relativistic generalization of Planck's law by the above statements can be only the next (it was recognized by de Broglie):

$$p_\mu = \hbar k_\mu. \tag{4.5}$$

The rest mass of the electromagnetic waves (the mass of photons) is zero; the four-momentum squared satisfies the following equation:

$$p_\mu p^\mu = \hbar^2 k_\mu k^\mu = 0. \tag{4.6}$$

De Broglie supposed that this equation must also be met for the rest massive particles, especially electrons. According to relativity, the above equation will change into the following form:

$$p_\mu p^\mu = \hbar^2 k_\mu k^\mu = m^2 c^4. \tag{4.7}$$

In this case this equation associates the mass with some kind of wave, which is called matter wave today. De Broglie's important outcome can be found in the majority of textbooks in a simplified form (this is the de Broglie wavelength):

$$\lambda = h/p = h/mv, \tag{4.8}$$

where p is the momentum, m is the mass and v is the speed of the particle. A simple calculation can easily show that the wavelengths of the macroscopic bodies are unobservably short. However, in the case of the electron having very small mass, its matter wave can be detect with interference experiment [23].

5. The 'Wave-Gravity' Hypothesis

After the overwhelming success of the initial results of the obscured preliminary quantum theory, the Bohr-Sommerfeld (B-S) quantization rule remained only a curiosity of physics history. Understandably, de Broglie's theory of matter waves was not taken into account later in the obsolete B-S quantization method. However, this old quantization method is able to give a new, very interesting physical outcome. Continuing the use of relativistic notation, the B-S quantization of the matter waves can be written in the following simple form:

$$S = \oint p_\mu dx^\mu = \hbar \oint k_\mu dx^\mu = \sum_{\mu=0}^{3} n_{(\mu)} = nh;\ (n = 1,2,3,...). \tag{5.1}$$

By this condition, the 'action integral' S associated to the matter wave can be only a whole-number multiple of the Planck's constant h. In the usual procedure of the B-S quantization the momentum components must express in function of the coordinates, the only question that remains is what the matter-wave vector dependence on the space-time coordinates is. It is important to note that both sides of Eq. (5.1) have *action dimension* (energy $\times$ time), so the *loop integral* can only be a dimensionless quantity. Clearly, the simplest choice of the matter-wave vector satisfying the (5.1) condition is the following:

$$k_\mu \Rightarrow 2\pi \left\{ 1/x_0, 1/x_1, 1/x_2, 1/x_3 \right\} \quad , \tag{5.2}$$

where the space-time four-vector usually has the form:

$$x_\mu = \left\{ x_0,\ x_1,\ x_2,\ x_3 \right\} = \left\{ ct,\ x,\ y,\ z \right\} \quad . \tag{5.3}$$

Using the above definitions, the B-S quantization condition can be written:

$$S = \hbar \oint k_\mu dx^\mu = h \oint \frac{dx_\mu}{x_\mu} = h \sum_{\mu=0}^{3} n_{(\mu)} = nh;\ (n = 1,2,3,...). \tag{5.4}$$

Surprisingly, Planck's constant with its microscopic property 'vanishes' from the relativistic B-S quantization, so its dominant role in the atomic, molecular, nuclear, particle, *etc*., physics, here becomes irrelevant. In this situation, we shall use the B-S quantization rule for macroscopic physical system in the following format:

$$S = D_{(\mu)} \oint \frac{dx_\mu}{x_\mu} = C \sum_{\mu=0}^{3} n_{(\mu)} = Cn > 0;\ (n = 1,2,3,...), \tag{5.5}$$

where C is an 'action' dimensioned, but as yet unknown, constant having only positive value. Applying the usual space-time metric, this condition can be written:

$$\begin{aligned} S = S_T + S_R &= D_{(0)} \oint \frac{dx_0}{x_0} - D_{(a)} \sum_{a=1}^{3} \oint \frac{dx_a}{x_a} \\ &= Cn > 0\ ; \quad (n = 1,2,3,...). \end{aligned} \tag{5.6}$$

This quantum condition is equivalent to the following two conditions:

$$S_T = D_{(0)} \oint \frac{dx_0}{x_0} = Cn\ ; \quad (n = n_{(0)} = 1,2,3,..)\ ; \tag{5.6a}$$

$$\begin{aligned} S_R &= -D_{(a)} \sum_{a=1}^{3} \oint \frac{dx_a}{x_a} = -C \sum_{a=1}^{3} n_{(a)}; \\ &\equiv -Cj \quad (j = 0,1,2,..,n-1)\ . \end{aligned} \tag{5.6b}$$

The requirement for the j quantum variable in (5.6b) assures that the action S in (5.6) will be positive in all circumstances.

Firstly we investigate the time-component (5.6a). The loop integral in this case means that the movement is periodical with finite periods:

$$S_T = D_T \oint \frac{dx_0}{x_0} \equiv D_T \int_{T_0}^{T} \frac{dt}{t} = Cn; \qquad (5.7)$$
$$(n = 1,2,3,...;\ D_T \equiv D_{(0)}) .$$

It is useful to replace the time variable for the distance variable with the help of a simple integral transformation. Supposing that the velocity of the matter wave is equal to a constant v, we can introduce new variables:

$$R_0 = vT_0\ ;\ R_A = vT;\ dt = dr / v\ ;\ 1/t = v/r\ , \qquad (5.8)$$

which leads to the following quantum condition being equivalent to (5.7):

$$S_T = D_T \int_{T_0}^{T} \frac{dt}{t} = D_T \int_{R_0}^{R_A} \frac{dr}{r} = Cn;\ \ (n = 1,2,3,...) . \qquad (5.9)$$

It is important to mention the transformed integral does not depend on the velocity of the matter wave. On the other hand, knowing that the T-B rule is related to the gravitational central-force Solar System, we can suppose the spatial quantum condition (5.6b) depends on only the radial distance from the gravitational center:

$$S_R = D_{(a)} \sum_{a=1}^{3} \oint \frac{dx_a}{x_a} = D_R \int_{R_0}^{R_B} \frac{dr}{r} \qquad (5.10)$$
$$= Cj\ ;\ \ (j = 0,1,2,...,n-1) .$$

Based on the above, the total macroscopic action integral will be:

$$S = D_T \int_{R_0}^{R_A} \frac{dr}{r} - D_R \int_{R_0}^{R_B} \frac{dr}{r} = C(n-j) > 0\ ; \qquad (5.11)$$
$$(n = 1,2,3,..;\ j = 0,1,2,...,n-1) .$$

Introducing the constants:

$$C_T = C / D_T\ ;\ \ C_R = C / D_R\ , \qquad (5.12)$$

the (5.11) quantum condition takes a simple form:

$$\int_{R_0}^{R_A} \frac{dr}{r} - \int_{R_0}^{R_B} \frac{dr}{r} = C_T n - C_R j > 0\ ; \qquad (5.13)$$
$$(n = 1,2,3,..;\ j = 0,1,2,...,n-1) .$$

The evaluation of the integrals leads to the following result:

$$\ln(R_A / R_B) = C_T n - C_R j > 0\ ; \qquad (5.14)$$
$$(n = 1,2,3,..;\ j = 0,1,2,...,n-1),$$

which is equivalent to the following exponential form:

$$R_A = R_B \exp(C_T n - C_R j) = R\alpha^n \beta^{-j}\ ;$$
$$\alpha^n = \exp(C_T n)\ \ ;\ \ \beta^{-j} = \exp(-C_R j)\ ; \qquad (5.15)$$
$$(n = 1,2,3,..;\ j = 0,1,2,...,n-1).$$

This final result in case $j = 0$ is the same that we have gotten empirically for the planet's distances in (2.1):

$$a_n = a_0 \alpha^n\ ;\ \ (a_n \equiv R_A;\ a_0 = R_B;\ n = 1,2,3,...) . \qquad (5.16)$$

The entire (5.15) formula is the same as the double quantum-numbered planet distance function that we presented in Section 3 with the formula (3.3):

$$a_n = a_0 \alpha^n \beta^{-j}; \qquad (5.17)$$
$$(a_n \equiv R_A;\ a_0 = R_B;\ n = 1,2,3,..;\ j = 0,1,2,...,n-1) .$$

With this simple approach we have successfully given the real physical background of the exponential distance distribution of the planets of our Sun System. This solution is based on the old Bohr-Sommerfeld quantum theory. The calculation of the planet's orbits remained on the level of classic mechanics applying gravitational theory of Newton. Nevertheless, our model seems to be more than the classical physics, and may be a first step for the future foundation of the long hoped quantum gravity theory. Having regard to the fact that the essence of our introduced model is closely linked to the de Broglie matter wave theory, we have named our new model 'Wave-Gravitational Theory' (WGT). This term emphasizes that our model is far from the original Quantum Mechanics as known today. At this level, there is no sense to introduce the basic concepts of the QM; however, some initiative exists to do this, for example in [15].

6. Summary

In this study we have given the Titius-Bode rule a possible physical interpretation that can in the future be a starting point for a final quantum gravity theory. The basis for the presented theory is the relativistic extension of the old Bohr-Sommerfeld quantization. We have involved the well-known matter wave theory of de Broglie for the imposition of the B-S quantum condition; namely, joining the momentum four-vector with the matter wave four-vector. From this simple quantum condition, not surprisingly, the Planck's constant has been eliminated. This fact opens up the possibility for the extension of this quantization method to macroscopic physical systems, especially the Solar System. By this method we have obtained the *allowed orbits* of planets, which have been experienced very much earlier and remained to date an unsolvable mystery named as Titius-Bode rule. We have called this new theoretical construction by short name 'Wave-Gravitational Theory' (WGT), which basically remained on the foundation of classical mechanics. The undeniable simplicity and accuracy of WGT is surprising.

In recent years, in the newly explored 'exosolar systems 'contain 'exoplanets', which also have exponential orbit distributions by the observations [24-29]. All the results of this study and the recently observed orbital-distributions of the

exoplanets further strengthen our belief in the true physical origin of the T-B rule.

References

[1] http://en.wikipedia.org/wiki/Titius%E2%80%93Bode_law.

[2] M.M. Nieto, 1972: The Titius-Bode Law of Planetary Distances: Its History and Theory (Pergamon Press, Oxford. (1), pp 59-62. (1982).

[3] http://space.unibe.ch/staff/thomas/planetenphysik_1_10.pdf.

[4] P. Lynch, "On the significance of the Titius-Bode law for the distribution of the planets", Monthly Notice of the Royal Astronomical Society, **341**, Issue 4, pp. 1174-1178. (2003). http://adsabs.harvard.edu/abs/2003MNRAS.341.1174L.

[5] S.F. Dermott, "Bode's Law and the resonant structure of the solar system", Nature Physical Science **244**, (132), pp. 18-21 (1973).

[6] J. Giné, "On the origin of the gravitational quantization: The Titius--Bode Law", Chaos, Solitons & Fractals **32**, (2), pp. 363–369, (2007). http://web.udl.es/usuaris/t4088454/ssd/Prepublicaciones/PS/bode2.pdf.

[7] V. Perinova, *et al.*, "Distribution of distances in the solar system", http://vixra.org/pdf/0805.0002v1.pdf.

[8] A.G. Agnese, R. Festa, "Discretization on the cosmic scale inspired from the Old Quantum Mechanics", Hadronic J. **21,** pp. 237–253. (1998).

[9] R. Louise, "A postulate leading to the Titius-Bode law", Moon and the Planets **26**, (1), pp. 93-96. (1982). http://adsabs.harvard.edu/full/1982M%26P....26...93L.

[10] R. Louise, "Quantum formalism in gravitation quantitative application to the Titius-Bode law", Moon and the Planets **27**, (1982) http://adsabs.harvard.edu/full/1982M%26P....27...59L.

[11] J.L. Ortiz, *et al.*, "Possible patterns in the distribution of planetary formation regions", Monthly Notices of the Royal Astronomical Society, **379**, (3), pp. 1222-1226. (2007).

[12] W. Hayes & S. Tremaine, "Fitting Selected Random Planetary Systems to Titius-Bode Laws", Icarus **135**, (2) 549-557. (1998). http://arxiv.org/pdf/astro-ph/9710116v1.pdf.

[13] http://en.wikipedia.org/wiki/Dermott's_law.

[14] V. Pankovic, *et al.*, "De Broglie-Like Wave in the Planetary Systems", (2009) http://arxiv.org/pdf/0903.1729.pdf.

[15] F.A. Scardigli, "Quantum-Like Description of the Planetary System", Foundations of Physics **37** (8), pp. 1278-1295. (2007), http://iopscience.iop.org/1742-6596/67/1/012038.

[16] Z. Zawislawski, *et al.*, "On the regularities of the mean distances of secondary bodies in the solar system", Astronomical and Astrophysical Transactions **19**, pp. 177-190. (2000), http://www.cbk.waw.pl/~kosek/z/zartykul.pdf.

[17] S.I. Ragnarsson, "Planetary distances: a new simplified model", Astronomy and Astrophysics, **301**, p. 609. (1995), http://adsabs.harvard.edu/full/1995A%26A...301..609R.

[18] D. Sarkadi, "The structure of the physical mass", http://www.scribd.com/doc/64984275/????

[19] http://en.wikipedia.org/wiki/Old_quantum_theory.

[20] A. Messiah, , **Quantum Mechanics,** Vol. I, Chapt. I, Section IV), English translation from French by G. M. Temmer (North Holland, John Wiley & Sons. Cf., 1966).

[21] H. Krag, "Relativity and quantum theory from Sommerfeld to Dirac", *Ann. Phys. (Leipzig)* **9**, 11–12, 961 – 974. (2000).

[22] http://en.wikipedia.org/wiki/Louis_de_Broglie.

[23] http://en.wikipedia.org/wiki/Davisson%E2%80%93Germer_experiment.

[24] Zs. Tóth, I. Nagy, "Dynamical stability of the Gliese 581 exoplanetary system", http://arxiv.org/pdf/1302.1322v1.pdf.

[25] J.D.Flores-Gutierrez, C.Garcia-Guerra, "A variant of the Titius-Bode Law", Revista Mexicana de Astronomia y Astrofisica, **47**, 173–184. (2011). http://www.scielo.org.mx/pdf/rmaa/v47n1/v47n1a12.pdf.

[26] C. Lovis, *et al.*, The HARPS search for southern extra-solar planets. http://arxiv.org/pdf/1011.4994v1.pdf.

[27] Chang, H-Y., Titius-Bode's Relation and Distribution of Exoplanets. Journal of Astronomy and Space Sciences **27**, (1), pp. 1-10 (2010), http://www.janss.kr/Upload/files/JASS/27_1_01.pdf.

[28] European Southern Observatory: "Richest Planetary System Discovered", http://www.eso.org/public/news/eso1035/???.

[29] A. Poveda & P. Lara, "The Exo-planetary System of 55 Cancri and the Titius-Bode Law", http://arxiv.org/abs/0803.2240.