Question 331: A short note on pi

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Abstract. This note presents some results on pi.

1. Introduction: Two Polynomials

$$p(z) = 15625z^3 - 60000z^2 + 403392z - 32768 \tag{1}$$

$$q(z) = 15625z^3 - 20625z^2 + 10587z - 1331 \tag{2}$$

- Real Roots:

$$p(z) = 0 \wedge z \in \mathbb{R} \Rightarrow z = z_p \tag{3}$$

$$z_p = \frac{512}{2101 + 1107\sqrt[3]{7} + 549\sqrt[3]{49}} = \frac{32}{25} - \frac{216}{125}\sqrt[3]{7} + \frac{72}{125}\sqrt[3]{49} \tag{4}$$

$$q(z) = 0 \wedge z \in \mathbb{R} \Rightarrow z = z_q \tag{5}$$

$$z_q = \frac{1331}{3529 + 678\sqrt[3]{7} + 696\sqrt[3]{49}} = \frac{11}{25} + \frac{6}{125}\sqrt[3]{7} - \frac{12}{125}\sqrt[3]{49} \tag{6}$$

- Formula:

$$\pi = 2\sin^{-1}\left(\sqrt[6]{z_p}\right) + 2\sin^{-1}\left(\sqrt[6]{z_q}\right) \tag{7}$$

2. Taxicab Numbers

- Bernard Frenicle de Bessy (1657):

$$1729 = 1^3 + 12^3 = 9^3 + 10^3 \tag{8}$$

$$4101 = 9^3 + 15^3 = 2^3 + 16^3 \tag{9}$$

- Related formulas:

$$\pi = 2\sin^{-1}\left(\sqrt{\frac{1}{1729}}\right) + 2\sin^{-1}\left(24\sqrt{\frac{3}{1729}}\right) \tag{10}$$

$$\pi = 2\sin^{-1}\left(27\sqrt{\frac{1}{1729}}\right) + 2\sin^{-1}\left(10\sqrt{\frac{10}{1729}}\right) \tag{11}$$

$$\pi = 2\sin^{-1}\left(\frac{9}{2}\sqrt{\frac{1}{114}}\right) + 2\sin^{-1}\left(\frac{5}{2}\sqrt{\frac{5}{38}}\right) \tag{12}$$

$$\pi = 2\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{1}{57}}\right) + 2\sin^{-1}\left(\frac{16}{3}\sqrt{\frac{2}{57}}\right) \tag{13}$$

$$\pi = 2\sin^{-1}\left(\frac{1}{540\sqrt{10}}\right) + 2\sin^{-1}\left(\frac{1457}{1296\sqrt{3}}\right) + 2\sin^{-1}\left(\frac{1999}{480\sqrt{30}}\right) \tag{14}$$

$$\pi = 2\sin^{-1}\left(\frac{4}{405\sqrt{15}}\right) + 2\sin^{-1}\left(\frac{3371}{960\sqrt{15}}\right) + 2\sin^{-1}\left(\frac{725}{1728}\right) \tag{15}$$

3. Tribonacci Constant

❖ Tribonacci constant T :

$$T^3 - T^2 - T - 1 = 0 \tag{16}$$

$$T = \frac{1}{3}\left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\right) \tag{17}$$

$$\frac{1}{T} = \frac{1}{3}\left(-1 + \sqrt[3]{17 + 3\sqrt{33}} + \sqrt[3]{17 - 3\sqrt{33}}\right) \tag{18}$$

$$T = \frac{1}{3} + \frac{1}{3}\sqrt[3]{38 + 12\sqrt[3]{38 + 12\sqrt[3]{38 + \ldots}}} \tag{19}$$

$$T = \frac{1}{3} + \frac{1}{3}\sqrt{12 + \frac{38}{\sqrt{12 + \frac{38}{\sqrt{12 + \frac{38}{\sqrt{12 + \ldots}}}}}}} \tag{20}$$

$$T=\frac{1}{2}+\frac{1}{2}\sqrt{5+\frac{8}{1+\sqrt{5+\frac{8}{1+\sqrt{5+...}}}}} \tag{21}$$

$$\frac{1}{T}=-\frac{1}{3}+\frac{1}{3}\sqrt[3]{34-6\sqrt[3]{34-6\sqrt[3]{34-...}}} \tag{22}$$

- Pi formulas:

$$\pi=4\tan^{-1}\left(\frac{1}{T}\right)+4\tan^{-1}\left(\frac{1}{T^2}\right) \tag{23}$$

$$\pi=4\tan^{-1}(T)-4\tan^{-1}\left(\frac{1}{T^2}\right) \tag{24}$$

$$\pi=4\tan^{-1}(T-1)+4\tan^{-1}\left(\frac{2}{T}-1\right) \tag{25}$$

$$\pi=\frac{4}{3}\tan^{-1}(T)+\frac{4}{3}\tan^{-1}(T^2) \tag{26}$$

$$\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}T^{-2n-1}\left(1+T^{-2n-1}\right) \tag{27}$$

$$\pi=4\sum_{n=0}^{\infty}\frac{c_n}{n+1}T^{-n-1} \tag{28}$$

$$c_n=-c_{n-2}-2c_{n-3}-3c_{n-4}-2c_{n-5}-c_{n-6} \tag{29}$$

$$c_n=\{1,2,2,-4,-9,-8,6,...\} \tag{30}$$

$$\pi=4\sum_{n=0}^{\infty}\frac{c_n}{n+1}T^{-n-1} \tag{31}$$

$$c_n=-c_{n-2}-c_{n-4}-c_{n-6}\quad ,n\geq 6 \tag{32}$$

$$c_n=\{1,2,-1,0,1,-2,...\} \tag{33}$$

$$\pi=3\sum_{n=0}^{\infty}T^{-n}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{m}{k}\binom{k}{n-k-m}\binom{2m}{m}\frac{2^{-2m}}{2m+1} \tag{34}$$

$$\pi=4\int_0^T\frac{3x^2-2x-1}{1+x^2+2x^3-x^4-2x^5+x^6}\,dx \tag{35}$$

4. **Triangular Root of** 2

❖ Constant R_2 :

$$R_2 = \frac{\sqrt{17}-1}{2} \tag{36}$$

$$R_2 = \sqrt{4+\sqrt{4+\sqrt{4+\dots}}} - 1 \tag{37}$$

$$R_2 = \sqrt{4-\sqrt{4-\sqrt{4-\dots}}} \tag{38}$$

$$R_2 = \cfrac{4}{1+\cfrac{4}{1+\cfrac{4}{1+\dots}}} \tag{39}$$

$$\pi = 16\int_0^{R_2} \frac{1+2x}{16+x^2+2x^3+x^4}\,dx \tag{40}$$

$$\pi = 16R_2\int_0^{1} \frac{1+2R_2x}{16+\left(4-R_2\right)x^2+\left(10R_2-8\right)x^3+\left(20-9R_2\right)x^4}\,dx \tag{41}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{R_2}\right)+4\tan^{-1}\left(\frac{R_2\left(R_2-1\right)}{4}\right) \tag{42}$$

$$\pi = 12\int_{1/\phi}^{R_2} \frac{1+2x}{10-2x-x^2+2x^3+x^4}\,dx \quad ,\phi = \frac{1+\sqrt{5}}{2} \tag{43}$$

$$\pi = 12\int_{-\phi}^{R_2} \frac{1+2x}{10-2x-x^2+2x^3+x^4}\,dx \quad ,\phi = \frac{1+\sqrt{5}}{2} \tag{44}$$

5. **Nested Radical** S_5

❖ Constant S_5 :

$$S_5 = \frac{\sqrt{21}+1}{2} \tag{45}$$

$$S_5 = \sqrt{5+\sqrt{5+\sqrt{5+\dots}}} \tag{46}$$

$$S_5 = 1+\sqrt{5-\sqrt{5-\sqrt{5-...}}} \tag{47}$$

$$\pi = 20\int_0^{S_5} \frac{2x-1}{25+x^2-2x^3+x^4}\,dx \tag{48}$$

$$\pi = 20S_5\int_0^1 \frac{2S_5x-1}{25+(5+S_5)x^2-(10+12S_5)x^3+(30+11S_5)x^4}\,dx \tag{49}$$

$$\pi = 16\int_\phi^{S_5} \frac{2x-1}{17+2x-x^2-2x^3+x^4}\,dx \quad ,\phi=\frac{1+\sqrt{5}}{2} \tag{50}$$

$$\pi = 16\int_{-1/\phi}^{S_5} \frac{2x-1}{17+2x-x^2-2x^3+x^4}\,dx \quad ,\phi=\frac{1+\sqrt{5}}{2} \tag{51}$$

6. **Ramanujan Nested Radical** R

❖ Constant R :

$$R=\sqrt{5+\sqrt{5+\sqrt{5-\sqrt{5+\sqrt{5+\sqrt{5+\sqrt{5-...}}}}}}} \tag{52}$$

$$R=\frac{2+\sqrt{5}+\sqrt{15-6\sqrt{5}}}{2} \tag{53}$$

$$R^4-4R^3-4R^2+31R-29=0 \tag{54}$$

$$\pi = 116\int_0^R \frac{4x^3-12x^2-8x+31}{841+961x^2-248x^3-232x^4+94x^5+8x^6-8x^7+x^8}\,dx \tag{55}$$

$$\pi = 4(5\phi-4)\int_0^R \frac{1+2\phi-2x}{41-15\phi+(5+8\phi)x^2-(2+4\phi)x^3+x^4}\,dx \quad ,\phi=\frac{1+\sqrt{5}}{2} \tag{56}$$

7. **Calabi Triangle Constant** C_{CR}

❖ Calabi Constant C_{CR} :

$$C_{CR}=\frac{1}{3}+\frac{\left(-23+3i\sqrt{237}\right)^{1/3}}{3\sqrt[3]{4}}+\frac{11}{3\left(2\left(-23+3i\sqrt{237}\right)\right)^{1/3}} \tag{57}$$

$$2x^3-2x^2-3x+2=0 \quad ,x=C_{CR} \tag{58}$$

$$C_{CR}=\frac{1}{3}+\frac{1}{6}\sqrt{66-\frac{92}{\sqrt{66-\frac{92}{\sqrt{66-...}}}}} \tag{59}$$

$$\pi=8\int_{0}^{C_{CR}}\frac{3+4x-6x^2}{4+9x^2+12x^3-8x^4-8x^5+4x^6}\,dx \tag{60}$$

$$\pi=3\sin^{-1}\left(\frac{3C_{CR}-1}{\sqrt{22}}\right)+\sin^{-1}\left(\frac{23}{11\sqrt{22}}\right) \tag{61}$$

8. **The Numbers** A,B,C

❖ The number A :

$$A=-\frac{2-\sqrt{3}}{3}-\frac{1}{3}\sqrt[3]{\alpha+\beta\sqrt[3]{\alpha+\beta\sqrt[3]{\alpha+...}}} \tag{62}$$

$$\alpha=124-66\sqrt{3} \quad , \quad \beta=30-12\sqrt{3} \tag{63}$$

$$\pi=-12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{A^2-1}{2}\right)^n\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}(-1)^k A^{2k+1} \tag{64}$$

❖ The number B :

$$B=-\frac{2-\sqrt{3}}{3}+\left(q+p\left(q+p\left(q+...\right)^{3/2}\right)^{3/2}\right)^{-1/2} \tag{65}$$

$$p=\frac{112-41\sqrt{3}}{117} \quad , \quad q=\frac{15+6\sqrt{3}}{26} \tag{66}$$

$$\pi=12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1-B^2}{2}\right)^n\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}(-1)^k B^{2k+1} \tag{67}$$

$$\pi=12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}(-1)^k B^{2n-2k-1}\left(1-B^2\right)^{2k+1} \tag{68}$$

❖ The number C :

$$C=-\frac{2-\sqrt{3}}{3}+p+q\left(p+q\left(p+...\right)^3\right)^3 \tag{69}$$

$$p=\frac{112-41\sqrt{3}}{117} \quad , \quad q=\frac{15+6\sqrt{3}}{26} \tag{70}$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1-C^2}{2}\right)^n\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}(-1)^k C^{2k+1} \qquad (71)$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}(-1)^k C^{2n-2k-1}\left(1-C^2\right)^{2k+1} \qquad (72)$$

❖ Relations:

$$x^6+4x^5-x^4+3x^2-4x+1=0 \quad ,x=A,B,C \qquad (73)$$

$$x^3+\left(2-\sqrt{3}\right)x^2-x+2-\sqrt{3}=0 \quad ,x=A,B,C \qquad (74)$$

9. The numbers α,β,γ

❖ The numbers α,β,γ :

$$\sqrt{3}\tanh\alpha-\tan\alpha=0\wedge 0<\alpha<1\Rightarrow\alpha=0.888539... \qquad (75)$$

$$\left(\sqrt{2}+1\right)\tanh\beta-\tan\beta=0\wedge 1<\beta<2\Rightarrow\beta=1.091769... \qquad (76)$$

$$\left(2+\sqrt{3}\right)\tanh\gamma-\tan\gamma=0\wedge 1<\gamma<2\Rightarrow\gamma=1.266367... \qquad (77)$$

❖ Recurrences:

$$\alpha_{n+1}=\tan^{-1}\left(\sqrt{3}\tanh\alpha_n\right) \quad ,\alpha_1=1\Rightarrow\alpha_n\to\alpha \qquad (78)$$

$$\beta_{n+1}=\tan^{-1}\left(\left(\sqrt{2}+1\right)\tanh\beta_n\right) \quad ,\beta_1=1\Rightarrow\beta_n\to\beta \qquad (79)$$

$$\gamma_{n+1}=\tan^{-1}\left(\left(2+\sqrt{3}\right)\tanh\gamma_n\right) \quad ,\gamma_1=1\Rightarrow\gamma_n\to\gamma \qquad (80)$$

❖ Pi formulas:

$$\pi=6\alpha\int_0^1\frac{\sinh(\alpha x)-\sin(\alpha x)}{\left(\sinh(\alpha x)\right)^2+\left(\sin(\alpha x)\right)^2}dx \qquad (81)$$

$$\pi=4\beta\int_0^1\frac{\sinh(\beta x)-\sin(\beta x)}{\left(\sinh(\beta x)\right)^2+\left(\sin(\beta x)\right)^2}dx \qquad (82)$$

$$\pi=3\gamma\int_0^1\frac{\sinh(\gamma x)-\sin(\gamma x)}{\left(\sinh(\gamma x)\right)^2+\left(\sin(\gamma x)\right)^2}dx \qquad (83)$$

$$\pi = 3\alpha + 3\sum_{n=1}^{\infty} \frac{1}{n} e^{-2n\alpha} \sin(2n\alpha) \tag{84}$$

$$\pi = 3\alpha + \frac{3\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{e^{-(6n+2)\alpha}}{3n+1} + \frac{e^{-(6n+4)\alpha}}{3n+2} \right) \tag{85}$$

$$\pi = \frac{8}{3}\beta + \frac{8}{3}\sum_{n=1}^{\infty} \frac{1}{n} e^{-2n\beta} \sin(2n\beta) \tag{86}$$

$$\pi = \frac{8}{3}\beta + \frac{4\sqrt{2}}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{e^{-(8n+2)\beta}}{4n+1} + \frac{\sqrt{2}\, e^{-(8n+4)\beta}}{4n+2} + \frac{e^{-(8n+6)\beta}}{4n+3} \right) \tag{87}$$

$$\pi = \frac{12}{5}\gamma + \frac{12}{5}\sum_{n=1}^{\infty} \frac{1}{n} e^{-2n\gamma} \sin(2n\gamma) \tag{88}$$

$$\begin{aligned} \pi &= \frac{12}{5}\gamma \\ &+ \frac{6}{5}\sum_{n=0}^{\infty} (-1)^n \left(\frac{e^{-(12n+2)\gamma}}{6n+1} + \frac{\sqrt{3}\, e^{-(12n+4)\gamma}}{6n+2} + \frac{2e^{-(12n+6)\gamma}}{6n+3} + \frac{\sqrt{3}\, e^{-(12n+8)\gamma}}{6n+4} + \frac{e^{-(12n+10)\gamma}}{6n+5} \right) \end{aligned} \tag{89}$$

10. Complex series for pi

❖

$$\pi = 4 - 4\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Im}\left(\left(1 - (1+i)e^{-1-i}\right)^n \right) \tag{90}$$

$$c_n = \operatorname{Im}\left(\left(1 - (1+i)e^{-1-i}\right)^n \right) \quad , n \in \mathbb{N} \tag{91}$$

$$c_{n+2} = A c_{n+1} - B c_n \quad , n \in \mathbb{N} \tag{92}$$

$$A = 2 - 2e^{-1}(\sin 1 + \cos 1) \tag{93}$$

$$B = 1 + 2e^{-2} - 2e^{-1}(\sin 1 + \cos 1) \tag{94}$$

$$c_1 = e^{-1}(\sin 1 - \cos 1) \tag{95}$$

$$c_2 = 2e^{-1}(\sin 1 - \cos 1)\left(1 - e^{-1}(\sin 1 + \cos 1)\right) \tag{96}$$

❖

$$\pi = 4u - 4\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Im}\left(\left(1 - u(1+i)e^{-u(1+i)}\right)^n \right) \tag{97}$$

$$u = 0.862404795\ldots \quad (98)$$

$$f(x) = \left|1 - (x + xi)e^{-x-xi}\right| \quad ,0 < x < 1 \quad (99)$$

$$f(u) < f(x) \ ,\forall x \in (0,1) \quad (100)$$

$$f(u) = \min_{0<x<1} f(x) = 0.4882\ldots \quad (101)$$

❖

$$\pi = 3\sqrt{3} - 3\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1 - \left(1 + i\sqrt{3}\right)e^{-1-i\sqrt{3}}\right)^n\right) \quad (102)$$

$$\pi = \frac{3\sqrt{3}}{2} - 3\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1 - \left(\frac{1 + i\sqrt{3}}{2}\right)e^{-\left(1+i\sqrt{3}\right)/2}\right)^n\right) \quad (103)$$

$$\pi = \frac{3\sqrt{3}}{2} - \frac{3i}{2} + 3i\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \left(\frac{1 + i\sqrt{3}}{2}\right)e^{-\left(1+i\sqrt{3}\right)/2}\right)^n \quad (104)$$

$$\pi = \frac{12\sqrt{3}}{7} - \frac{3}{7}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1 - 4\left(1 + i\sqrt{3}\right)e^{-4\left(1+i\sqrt{3}\right)}\right)^n\right) \quad (105)$$

❖

$$\pi = 3u\sqrt{3} - 3\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1 - u\left(1 + i\sqrt{3}\right)e^{-u\left(1+i\sqrt{3}\right)}\right)^n\right) \quad (106)$$

$$u = 0.658836741\ldots \quad (107)$$

$$f(x) = \left|1 - x\left(1 + i\sqrt{3}\right)e^{-x\left(1+i\sqrt{3}\right)}\right| \quad ,0 < x < 1 \quad (108)$$

$$f(u) < f(x) \ ,\forall x \in (0,1) \quad (109)$$

$$f(u) = \min_{0<x<1} f(x) = 0.3274\ldots \quad (110)$$

❖

$$\pi = 6 - 6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1 - \left(\sqrt{3} + i\right)e^{-\sqrt{3}-i}\right)^n\right) \quad (111)$$

$$\pi = 3 - 6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1 - \left(\frac{\sqrt{3} + i}{2}\right)e^{-\left(\sqrt{3}+i\right)/2}\right)^n\right) \quad (112)$$

$$\pi = 3 - 3\sqrt{3}i + 6i\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \left(\frac{\sqrt{3} + i}{2}\right)e^{-\left(\sqrt{3}+i\right)/2}\right)^n \quad (113)$$

❖

$$\pi = 6u - 6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-u\left(\sqrt{3}+i\right)e^{-u\left(\sqrt{3}+i\right)}\right)^{n}\right) \tag{114}$$

$$u = 0.558047018... \tag{115}$$

$$f\left(x\right) = \left|1-x\left(\sqrt{3}+i\right)e^{-x\left(\sqrt{3}+i\right)}\right| \quad ,0<x<1 \tag{116}$$

$$f\left(u\right) < f\left(x\right) \ , \forall x \in \left(0,1\right) \tag{117}$$

$$f\left(u\right) = \min_{0<x<1} f\left(x\right) = 0.5758... \tag{118}$$

❖

$$\pi = 2\sqrt{3} - 6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\left(1+\frac{i}{\sqrt{3}}\right)e^{-\left(1+\frac{i}{\sqrt{3}}\right)}\right)^{n}\right) \tag{119}$$

❖

$$\pi = -4i\sum_{n=1}^{\infty}\frac{2^{n}}{n}\left(\sin z\right)^{2n} \tag{120}$$

$$z = \frac{1}{2}\sin^{-1}\left(\frac{1}{\sqrt[4]{2}}\right) + \frac{i}{2}\cosh^{-1}\left(\sqrt{\frac{2+\sqrt{2}}{2}}\right) \tag{121}$$

❖

$$\pi = -4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\cos\left(\frac{1+i}{3}\right)\right)^{n}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{\left(1+i\right)/3}{\cos\left(\left(1+i\right)/3\right)}\right)^{n}\right) \tag{122}$$

$$\pi = -4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\sin\left(\frac{1+i}{2}\right)\right)^{n}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{\left(1+i\right)/2}{\sin\left(\left(1+i\right)/2\right)}\right)^{n}\right) \tag{123}$$

$$\pi = -4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\sin\left(1+i\right)\right)^{n}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1+i}{\sin\left(1+i\right)}\right)^{n}\right) \tag{124}$$

$$\pi = -4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\tan\left(\frac{1+i}{2}\right)\right)^{n}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{\left(1+i\right)/2}{\tan\left(\left(1+i\right)/2\right)}\right)^{n}\right) \tag{125}$$

$$\pi = -4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\tan\left(\frac{1+i}{3}\right)\right)^{n}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{\left(1+i\right)/3}{\tan\left(\left(1+i\right)/3\right)}\right)^{n}\right) \tag{126}$$

$$\pi=-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\cosh\left(\frac{1+i}{2}\right)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{(1+i)/2}{\cosh((1+i)/2)}\right)^{n}\right) \qquad (127)$$

$$\pi=-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\sinh\left(\frac{1+i}{3}\right)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{(1+i)/3}{\sinh((1+i)/3)}\right)^{n}\right) \qquad (128)$$

$$\pi=-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\tanh\left(\frac{1+i}{2}\right)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{(1+i)/2}{\tanh((1+i)/2)}\right)^{n}\right) \qquad (129)$$

$$\pi=-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\tanh(1+i)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1+i}{\tanh(1+i)}\right)^{n}\right) \qquad (130)$$

❖

$$\pi=-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\left(\frac{1+i}{2}\right)\Gamma\left(\frac{1+i}{2}\right)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1}{\Gamma((1+i)/2)}\right)^{n}\right) \qquad (131)$$

$$\pi=-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\Gamma\left(\frac{3+i}{2}\right)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1}{\Gamma((1+i)/2)}\right)^{n}\right) \qquad (132)$$

$$\pi=-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\left(\frac{1+i}{3}\right)\Gamma\left(\frac{1+i}{3}\right)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1}{\Gamma((1+i)/3)}\right)^{n}\right) \qquad (133)$$

$$\pi=-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\Gamma\left(\frac{4+i}{3}\right)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1}{\Gamma((1+i)/3)}\right)^{n}\right) \qquad (134)$$

$$\pi=4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\Gamma(1+i)\right)^{n}\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\Gamma(2+i)\right)^{n}\right) \qquad (135)$$

Remark: $\Gamma(x)$,Gamma function.

❖

$$\pi=3\sqrt{3}+3\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(g(1)\right)^{n}\right) \qquad (136)$$

$$\pi=\frac{9\sqrt{3}}{4}+3\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(g\left(\frac{3}{4}\right)\right)^{n}\right) \qquad (137)$$

$$\pi = \frac{3\sqrt{3}}{2} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(g\left(-1\right)\right)^{n}\right) \tag{138}$$

$$\pi = 3\alpha\sqrt{3} + 3\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(g\left(\alpha\right)\right)^{n}\right) \tag{139}$$

$$f\left(z\right) = 1 - \frac{e^{z}}{z} \quad , z \in \mathbb{C} - \{0\} \tag{140}$$

$$g\left(x\right) = f\left(x + x\sqrt{3}\,i\right), x \in \mathbb{R} \tag{141}$$

$$\alpha = 0.67782142248... \tag{142}$$

$$\alpha := \begin{cases} \alpha > 0 \\ \left|g\left(\alpha\right)\right| < \left|g\left(x\right)\right|, \forall x, x > 0 \end{cases} \tag{143}$$

❖

$$\pi = \frac{4\left(2+\sqrt{3}\right)}{5} + \frac{12}{5}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(h\left(\frac{1}{3}\right)\right)^{n}\right) \tag{144}$$

$$\pi = \frac{24\left(2+\sqrt{3}\right)}{29} + \frac{12}{29}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(h\left(2\right)\right)^{n}\right) \tag{145}$$

$$\pi = \frac{6\left(2+\sqrt{3}\right)}{7} - \frac{12}{7}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(h\left(-\frac{1}{2}\right)\right)^{n}\right) \tag{146}$$

$$f\left(z\right) = 1 - \frac{e^{z}}{z} \quad , z \in \mathbb{C} - \{0\} \tag{147}$$

$$h\left(x\right) = f\left(x + x\left(2+\sqrt{3}\right)i\right), x \in \mathbb{R} \tag{148}$$

❖

$$\pi = 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\cos\left(\frac{1+i}{2}\right)\right)^{n}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\left(\frac{1+i}{2}\right)\cos\left(\frac{1+i}{2}\right)\right)^{n}\right) \tag{149}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\cosh\left(\frac{1+i}{3}\right)\right)^{n}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\left(\frac{1+i}{3}\right)\cosh\left(\frac{1+i}{3}\right)\right)^{n}\right) \tag{150}$$

❖

$$\pi = 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\cos\left(a+ai\right)\right)^{n}\right) \tag{151}$$

$$\pi = -4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1}{\cos(a+ai)}\right)^n\right) \quad (152)$$

$$\tan(a)\tanh(a)=1 \ , a = 0.93755203... \quad (153)$$

$$x_{n+1} = \tan^{-1}\left((\tanh x_n)^{-1}\right) \quad , x_1 = 1 \Rightarrow x_n \to a = 0.93755203... \quad (154)$$

❖

$$\pi = 6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left((1-\cos(a+ai))^n\right) \quad (155)$$

$$\pi = -6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1}{\cos(a+ai)}\right)^n\right) \quad (156)$$

$$\tan(a)\tanh(a)=1/\sqrt{3} \ , a = 0.74168834... \quad (157)$$

$$x_{n+1} = \tan^{-1}\left(\left(\sqrt{3}\tanh x_n\right)^{-1}\right) \quad , x_1 = 1 \Rightarrow x_n \to a = 0.74168834... \quad (158)$$

❖

$$\pi = 8\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left((1-\cos(a+ai))^n\right) \quad (159)$$

$$\pi = -8\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1}{\cos(a+ai)}\right)^n\right) \quad (160)$$

$$\tan(a)\tanh(a)=\sqrt{2}-1 \ , a = 0.63537231... \quad (161)$$

$$x_{n+1} = \tan^{-1}\left(\left(\left(\sqrt{2}+1\right)\tanh x_n\right)^{-1}\right) \quad , x_1 = 1/2 \Rightarrow x_n \to a = 0.63537231... \quad (162)$$

❖

$$\pi = 12\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left((1-\cos(a+ai))^n\right) \quad (163)$$

$$\pi = -12\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\frac{1}{\cos(a+ai)}\right)^n\right) \quad (164)$$

$$\tan(a)\tanh(a)=2-\sqrt{3} \ , a = 0.51480086... \quad (165)$$

$$x_{n+1} = \tan^{-1}\left(\left(\left(2+\sqrt{3}\right)\tanh x_n\right)^{-1}\right) \quad , x_1 = 1/2 \Rightarrow x_n \to a = 0.51480086... \quad (166)$$

❖

$$\pi = -6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\sin\left(a+ai\right)\right)^{n}\right) \tag{167}$$

$$\tan\left(a\right) = \sqrt{3}\tanh\left(a\right) \ \ , a = 0.88853943... \tag{168}$$

$$x_{n+1} = \tan^{-1}\left(\sqrt{3}\tanh x_n\right) \quad , x_1 = 1 \Rightarrow x_n \to a = 0.88853943... \tag{169}$$

❖

$$\pi = -8\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-\sin\left(a+ai\right)\right)^{n}\right) \tag{170}$$

$$\tan\left(a\right) = \left(\sqrt{2}+1\right)\tanh\left(a\right) \ \ , a = 1.09176902... \tag{171}$$

$$x_{n+1} = \tan^{-1}\left(\left(\sqrt{2}+1\right)\tanh x_n\right) \quad , x_1 = 1 \Rightarrow x_n \to a = 1.09176902... \tag{172}$$

References

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