Units and Class Numbers extracted from regulators of small number fields

L. Karami
lulu.karami82@gmail.com

Abstract

This submission demonstrates how to use the analytic class number formula to express certain quotients of Dedekind's Eta function as a unit raised to the power of a quoteint of class numbers, for particular number fields. It includes a loose derivation for some special cases of reciprocity laws and the Fourier series of particular Eisenstein series.

1 Gauss Sums and Reciprocity

1.1 Preliminaries

Consider the field

$$\mathbb{Z}/p\mathbb{Z}$$

It is known when p is a prime, the multiplicative group

$$(\mathbb{Z}/p\mathbb{Z})^{\times} \simeq Z_{p-1}$$

is isomorphic to the cyclic group of order $p-1$, which implies a single non-zero α in the multiplicative group generates the whole group. A Dirichlet Character, χ_p, is a multiplicative group homomorphism

$$\chi_p : \mathbb{Z}/p\mathbb{Z} \mapsto \mathbb{C}^{\times}$$

where p indicates the **modulus**. By fermats little thorem,

$$\begin{aligned} \alpha^p &\equiv \alpha \pmod p \\ \alpha^{p-1} &\equiv 1 \pmod p \end{aligned}$$

which tells us the group homomorphism χ

$$\chi_p(\alpha)^{p-1} = 1 \implies \chi_p(\alpha) = \zeta_{p-1}^k$$

for some integer k and ζ_{p-1} a $p-1$'th primitive root of unity, giving $p-1$ distinct flavors for the characters χ_p. The trivial homomorphism

$$\chi_p(n) = 1$$

for a non-zero n in $\mathbb{Z}/p\mathbb{Z}$ is known as the **trivial** character. Regardless of the character, $\chi_p(p)$ is always zero.

1.2 Orthogonality of characters

Let $c(\chi_p)$ be the constant defined by

$$c(\chi_p) = \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \chi_p(n)$$

Then the following holds on c;

$$c(\chi_p) = \begin{cases} 0 & \text{if } \chi \text{ is not the trivial character} \\ p-1 & \text{if } \chi \text{is the trivial character} \end{cases}$$

Proof

Assume χ is trivial. Then $c(\chi_p)$ is just

$$\sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} 1 = p-1$$

In other words, it counts the number of non-zero elements in $\mathbb{Z}/p\mathbb{Z}$.

Assume χ is non-trivial. Consider the map

$$n \mapsto \alpha \cdot n$$

It is one to one and onto for n and α elements of $(\mathbb{Z}/p\mathbb{Z})^\times$ and α non-zero. Since χ is non-trivial, that means there exists a non-zero α such that

$$\chi_p(\alpha) \neq 1$$

Now consider the quantity

$$\chi_p(\alpha) \cdot c(\chi_p) = c'(\chi_p)$$

It can be rewritten as the following sums

$$\begin{aligned} c'(\chi_p) = &\chi_p(\alpha) \cdot \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n) \\ \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(\alpha) \cdot \chi_p(n) = &\sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n \cdot \alpha) \\ \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n \cdot \alpha) = &\sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n) \end{aligned}$$

where the second summation equality holds because χ is a homormorphism under multiplication and the last summation equality holds because the multiplication map by α affects the sum only by rearrangement of the order of the terms. This gives

$$\chi_p(\alpha) \cdot c(\chi_p) = c(\chi_p) \implies c(\chi_p) = 0$$

since α can be taken as a generator of the multiplicative group, which forces $\chi_p(\alpha)$ to never be equal to 1 if χ_p is non-trivial.

1.3 Gauss Sum properties

A **Gauss Sum** of a character χ_p and modulus p is defined to be

$$\tau(\chi_p) = \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n) \cdot \zeta_p^n$$

which is usually seen alongside with a similar sum

$$\tau(\chi_p, v) = \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n) \cdot \zeta_p^{v \cdot n}$$

where v is taken to be an integer. $\tau(\chi_p)$ is **the** Gauss Sum of χ_p. The character χ_p here will never be considered to be the trivial character.

For the latter sum under the restrictions on v and χ, the sum obeys

$$\tau(\chi_p, v) = \bar{\chi}_p(v) \cdot \tau(\chi_p)$$

where $\bar{\chi}_p$ is the **conjugate** of χ_p.

Proof

Assume

$$v \equiv 0 \pmod p$$

This gives

$$\begin{aligned}
\tau(\chi_p, 0) &= \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n) \cdot \zeta_p^{0 \cdot n} \\
&= \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n) \\
&= 0 \\
&= \bar{\chi}_p(0) \cdot \tau(\chi_p)
\end{aligned}$$

which holds due to the orthogonality of χ_p non-trivial and

$$\begin{aligned}
\chi_p(p) &= \chi_p(0) \\
\bar{0} &= 0
\end{aligned}$$

Now, assume

$$v \neq 0 \pmod p$$

which implies

$$\chi_p(v) \neq 0$$

as well as the multiplication map by v being one to one and onto within $\mathbb{Z}/p\mathbb{Z}$. As of right now, we have

$$\begin{aligned}
\tau(\chi_p, v) &= \chi_p(v) \cdot \bar{\chi}_p(v) \cdot \tau(\chi_p, v) \\
&= \bar{\chi}_p(v) \cdot \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n \cdot v) \cdot \zeta_p^{v \cdot n} \\
&= \bar{\chi}_p(v) \cdot \sum_{m \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(m) \cdot \zeta_p^{m} \\
&= \bar{\chi}_p(v) \cdot \tau(\chi_p)
\end{aligned}$$

Although simple, the property of $\tau(\chi_p, v)$ just proven is important when working within finite rings
Along with the properties investigated on τ, it also obeys

$$\mid \tau(\chi_p) \mid^2 = p$$

where $\mid \cdot \mid$ is the absolute value of a complex number $\cdot$.

Proof

The norm squared of τ can be manipulated from the sum to yield

$$\begin{aligned}
\mid \tau(\chi_p) \mid^2 &= \bar{\tau}(\chi_p) \cdot \tau(\chi_p) \\
&= \overline{\Big(\sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n) \cdot \zeta_p^{\cdot n} \Big)} \cdot \Big(\sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n) \cdot \zeta_p^{n} \Big) \\
&= \sum_{(m,n) \in (\mathbb{Z}/p\mathbb{Z})^{2\times}} \bar{\chi}_p(n) \chi_p(m) \cdot \zeta_p^{m-n}
\end{aligned}$$

where

$$(\mathbb{Z}/p\mathbb{Z})^{2\times}$$

is the cartesian product of $(\mathbb{Z}/p\mathbb{Z})^\times$ with itself. Since n is always non-zero in $\mathbb{Z}/p\mathbb{Z}$ within the sum, we can sum over the multiplication map

$$(m, n) \mapsto (n \cdot m, n)$$

to give

$$\begin{aligned}
\sum_{(m,n) \in (\mathbb{Z}/p\mathbb{Z})^{2\times}} \bar{\chi}_p(n) \chi_p(m) \cdot \zeta_p^{m-n} &= \sum_{(m,n) \in (\mathbb{Z}/p\mathbb{Z})^{2\times}} \bar{\chi}_p(n) \cdot \chi_p(n) \chi_p(m) \cdot \zeta_p^{m \cdot n - n} \\
&= \sum_{(m,n) \in (\mathbb{Z}/p\mathbb{Z})^{2\times}} \chi_p(m) \cdot \zeta_p^{n \cdot (m-1)} \\
&= \sum_{m=1}^{p-1} \chi_p(m) \sum_{n=1}^{p-1} \zeta_p^{n \cdot (m-1)}
\end{aligned}$$

The last double sum can be manipulated even further to give

$$\begin{aligned}\sum_{m=1}^{p-1}\chi_p(m)\sum_{n=1}^{p-1}\zeta_p^{n\cdot(m-1)} &= \sum_{m=1}^{p-1}\chi_p(m)\cdot\Big((\sum_{n=1}^{p}\zeta_p^{n\cdot(m-1)})-1\Big)\\ &=\Big(\sum_{m=1}^{p-1}\chi_p(m)\cdot\sum_{n=1}^{p}\zeta_p^{n\cdot(m-1)}\Big)-\Big(\sum_{m\in(\mathbb{Z}/p\mathbb{Z})^\times}\chi_p(m)\Big)\\ &=\sum_{m=1}^{p-1}\chi_p(m)\sum_{n=1}^{p}\zeta_p^{n\cdot(m-1)}\end{aligned}$$

where the last summation equality holds because of the orthogonality of a non-trivial character. Now, the interior sum of the last double sum obeys

$$\sum_{n=1}^{p}\zeta_p^{n\cdot(m-1)} = \begin{cases} p & \text{if } n\cdot(m-1)\equiv 0 \pmod p\\ 0 & \text{else}\end{cases}$$

for n, m integers. This follows from manipulations on

$$\sum_{n=1}^{p}\zeta_p^{v\cdot n}$$

and by letting v be equal to $n\cdot(m-1)$. Now, since n is non-zero in $\mathbb{Z}/p\mathbb{Z}$, this means that $(m-1)$ has to be 0 in $\mathbb{Z}/p\mathbb{Z}$ for the interior sum to be non-zero. This happens only when $m=1$. The sum now reads

$$\begin{aligned}\sum_{m=1}^{p-1}\chi_p(m)\sum_{n=1}^{p}\zeta_p^{n\cdot(m-1)} =& \chi_p(1)p+\sum_{m=2}^{p-1}\chi_p(m)\sum_{n=1}^{p}0\\ =& p\end{aligned}$$

since $\chi_p(1)$ is always 1, which folllows easily by just squaring 1. This completes the proof of

$$\mid\tau(\chi_p)\mid^2 = p$$

for χ a non-trivial character of prime modulus p.

1.4 Gauss Sums and some Reciprocity

Gauss Sums have some significance in reciprocity. For example, consider the residue character on odd primes p, q

$$\left(\frac{q}{p}\right)_2 = q^{\frac{p-1}{2}} \pmod p$$

This character is also known as the **Legendre Symbol**. When p is fixed and q is variable, then the residue symbol becomes the unique quadratic character defined by

$$\begin{aligned} \chi_p : \mathbb{N} &\rightarrow \mathbb{C}^\times \\ \chi_p(\alpha) &= -1 \end{aligned}$$

for a generator α of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$. It is of more interest when q is fixed and p is variable, especially when trying to determine how a rational prime ideal (p) within the ideals of a quadratic field $\mathbb{Q}(\sqrt{q})$ factors. This can be reformulated as

$$\begin{aligned} \left(\frac{q}{p}\right)_2 = 1 &\implies (p) = \mathfrak{p} \cdot \bar{\mathfrak{p}} \\ \left(\frac{q}{p}\right)_2 = -1 &\implies (p) \text{ is a prime ideal} \end{aligned}$$

It turns out if q is fixed and p is variable, then the symbol is wholly defined by the residue class p falls in within either the ring $\mathbb{Z}/q\mathbb{Z}$ or $\mathbb{Z}/4q\mathbb{Z}$.

1.5 Quadratic Reciprocity

The law of quadratic reciprocity precisely states

$$\left(\frac{p}{q}\right)_2 \cdot \left(\frac{q}{p}\right)_2 = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}$$

for odd primes p, q and

$$\left(\frac{2}{p}\right)_2 = \chi_8(p)$$

where χ_8 is a non-trivial character $\pmod 8$ with

$$\begin{aligned}\chi_8(\pm 1) =& 1\\ \chi_8(\pm 3) =& -1\end{aligned}$$

Proof

Assume $q = 2$. Consider the character on $(\mathbb{Z}/8\mathbb{Z})^\times$ defined by

$$\begin{aligned}\chi_8(\pm 1) =& 1\\ \chi_8(\pm 3) =& -1\\ \chi_8(2) =& 0\end{aligned}$$

Then the Gauss Sum of this character has the following expression

$$\begin{aligned}\tau(\chi_8) =& \sum_{n=1}^{7} \chi_8(n)\zeta_8^n\\ =& 2\sqrt{2}\end{aligned}$$

While working within the cyclotomic field $\mathbb{Z}(\zeta_8)$ modulo a prime p, the gauss sum of this character behaves in an interesting manner under the map

$$\alpha :\mapsto \alpha^p$$

Precisely, while working in $\mathbb{Z}(\zeta_8)/p\mathbb{Z}(\zeta_8)$

$$\begin{aligned}\tau^p(\chi_8) =& (2\sqrt{2})^p\\ =& 2 \cdot (\sqrt{2})^p\\ =& 2^{\frac{p-1}{2}}(2\sqrt{2})\\ =& \left(\frac{2}{p}\right)_2 \cdot \tau(\chi_8)\end{aligned}$$

But if we consider the property within $\pmod p$

$$(a+b)^p = a^p + b^p \pmod p$$

then the Gauss Sum under this map also yields

$$\begin{aligned}\tau^p(\chi_8) =& \Big(\sum_{n=1}^{7}\chi_8(n)\zeta_8^n\Big)^p \\ =& \sum_{n=1}^{7}\chi_8^p(n)\zeta_8^{p\cdot n} \\ =& \chi_8(p)\cdot\sum_{n=1}^{7}\chi_8(n)\zeta_8^n\end{aligned}$$

Which is due to the fact that $(\pm1)^p = \pm1$, since χ_8 only takes those values... as well as the properties established earlier on $\tau(\chi_q, v)$! This gives the following equality;

$$\Big(\frac{2}{p}\Big)_2 \cdot \tau(\chi_8) = \chi_8(p)\cdot\tau(\chi_8) \implies \Big(\frac{2}{p}\Big)_2 = \chi_8(p)$$

In general for odd primes p, q, the law of quadratic reciprocity can be proven by considering the Quadratic Gauss Sum

$$\tau(\chi_p) = \sum_{n\in\mathbb{Z}/p\mathbb{Z}} \chi_p(n)\zeta_p^n$$

in the setting of

$$\mathbb{Z}(\zeta_p)/q\mathbb{Z}(\zeta_p)$$

of the character χ_p, which happens to coincide with the symbol

$$\chi_p(q) = \Big(\frac{q}{p}\Big)_2.$$

The absolute value squared of this quantity

$$\begin{aligned}\mid\tau(\chi_p)\mid^2 =& \chi_p(-1)\cdot\tau(\chi_p)\cdot\tau(\bar{\chi}_p) \\ =& \chi_p(-1)\cdot\tau^2(\chi_p) \\ =& p\end{aligned}$$

where the second to last equality holds due to the property of χ_p is a purely real character, hence being its own conjugate, as well as the property of $\tau(\chi_p, v)$. Now, within $\mathbb{Z}(\zeta_p)/q\mathbb{Z}(\zeta_p)$, this particular quantity under the map

$$\alpha :\mapsto \alpha^q$$

satisfies

$$\begin{aligned}
\tau^q(\chi_p) \equiv& \Big(\tau^2(\chi_p)\Big)^{\frac{q-1}{2}} \cdot \tau(\chi_p) \\
\equiv& \Big(\chi_p(-1) \cdot p\Big)^{\frac{q-1}{2}} \cdot \tau(\chi_p) \\
\equiv& (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \cdot p^{\frac{q-1}{2}} \cdot \tau(\chi_p) \\
\equiv& (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \cdot \Big(\frac{p}{q}\Big)_2 \cdot \tau \pmod q
\end{aligned}$$

where the last equality holds since this was all done in a finite ring where the prime $q = 0$, as well as

$$\begin{aligned}
\chi_p(-1) =& \Big(\frac{-1}{p}\Big)_2 \\
=& (-1)^{\frac{p-1}{2}}
\end{aligned}$$

It seems that the power map 'dislodges' the residue symbol from the Gauss Sum. Like done with the $q = 2$ case, the particular quantity can be seen from the point of view of

$$\begin{aligned}
\tau^q(\chi_p) =& \Big(\sum_{n \in \mathbb{Z}/p\mathbb{Z}} \chi_p(n) \cdot \zeta_p^n\Big)^q \\
=& \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \chi_p^q(n) \cdot \zeta_p^{q \cdot n} \\
=& \tau(\chi_p, q) \\
=& \bar{\chi}_p(q) \cdot \tau(\chi_p) \\
=& \chi_p(q) \cdot \tau(\chi_p) \\
=& \Big(\frac{q}{p}\Big)_2 \cdot \tau
\end{aligned}$$

which holds again due to χ_p a real character and the property on $\tau(\chi_p, v)$ established earlier. Both reformulations of the same particular quantity give the relationship

$$(-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}\cdot\left(\frac{p}{q}\right)_2=\left(\frac{q}{p}\right)_2$$

which can be polished to give the standard formulation of Quadratic Reciprocity;

$$\left(\frac{p}{q}\right)_2\cdot\left(\frac{q}{p}\right)_2=(-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$$

1.5.1 Gauss Sums and some Cubic Symbols

Here, some Cubic Reciprocity will be dealt with.
Let

$$p\equiv q\equiv 1 \pmod 3$$

be primes. Let

$$\pi,\theta\in\mathbb{Z}(\zeta_3)$$

such that

$$\begin{aligned}|\ \pi\ |^2&=p\\ |\ \theta\ |^2&=q\end{aligned}$$

which are guaranteed to exist since $\mathbb{Z}(\zeta_3)$ is a PID and primes of the form $3k+1$ split into two quadratic integers within this ring. Let

$$\begin{aligned}\left(\frac{\theta}{\pi}\right)_3&=\theta^{\frac{\mathrm{N}\pi-1}{3}} \pmod \pi\\ &=\theta^{\frac{p-1}{3}} \pmod \pi\\ &\in\{1,\zeta_3,\zeta_3^2\}\end{aligned}$$

be the **Cubic Residue Symbol** and its image in $\mathbb{Z}(\zeta_3)$. Then there is a reciprocity law along the lines of

$$\left(\frac{\mathfrak{h}(\pi)}{\theta}\right)_3=\left(\frac{q}{\bar{\pi}}\right)_3$$

where $\mathfrak{h}$ satisfies

$$\mathfrak{h}(\pi) = \frac{\bar{\pi}}{\pi}$$

where π^{-1} is taken to be the multiplicative inverse of π modulo θ when present within the residue symbol, and $\bar{\pi}$ is the conjugate of π.

There is some ambiguity due to the presence of 6 units within $\mathbb{Z}(\zeta_3)$, precisely

$$\{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$$

which can change the value of composition of $\mathfrak{h}$; the character $\left(\frac{\bar{\cdot}}{\cdot}\right)$, and the residue symbol (character), $\left(\frac{\cdot}{\theta}\right)_3$. For now the unit ambiguity will be ignored.

Proof

Since p is a $3k+1$ prime, there exists a pair of **cubic** Dirichlet Characters of modulus p, conjugate to each other, such that

$$\chi_p(\alpha) = \zeta_3$$

for α a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$. Similar to the quadratic character, the cubic residue symbol with a bottom index of either π or $\bar{\pi}$, and top index of α coincides, with some ambiguity, with χ_p.

This proof will also be like that of the proof of the Law of Quadratic Reciprocity, by using a Gauss Sum to try to dislodge two different residue symbol expressions.

Let

$$\tau(\chi_p) = \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \chi_p(n)\zeta_p^n$$

where χ_p is a cubic character. Since τ is a $\mathbb{Z}$-linear combination of roots of unity of the form

$$\begin{aligned} \chi_p(n) \cdot \zeta_p^n =& \zeta_3^k \zeta_p^n \\ =& \zeta_{3p}^{k'} \end{aligned}$$

this implies

$$\tau(\chi_p) \in \mathbb{Z}(\zeta_{3p})$$

Now, any element σ^k of the group

$$\mathrm{Aut}(\mathbb{Q}(\zeta_{3p})/\mathbb{Q}(\zeta_3))$$

which sends

$$\begin{aligned}\sigma^k : \zeta_3 \mapsto& \zeta_3 \\ \sigma^k : \zeta_p \mapsto& \zeta_p^k\end{aligned}$$

acts on τ as

$$\begin{aligned}\sigma^k(\tau) =& \sigma^k\Big(\sum_{n\in\mathbb{Z}/p\mathbb{Z}} \chi_p(n)\zeta_p^n\Big) \\ =& \sum_{n\in\mathbb{Z}/p\mathbb{Z}} \chi_p(n)\cdot\sigma^k(\zeta_p^n) \\ =& \sum_{n\in\mathbb{Z}/p\mathbb{Z}} \chi_p(n)\zeta_p^{k\cdot n} \\ =& \bar{\chi}_p(k)\cdot\tau\end{aligned}$$

Any σ in $\mathrm{Aut}(\mathbb{Q}(\zeta_{3p})/\mathbb{Q}(\zeta_3))$ fixes $\chi_p(n)$ for any n since χ is a cubic character. The last equality holds due to the properties established on $\tau(\chi, v)$ earlier. Since τ is a Gauss Sum of a cubic character, then by cubing τ and acting on it by a σ, it yields

$$\begin{aligned}\sigma^k(\tau^3) =& \Big(\sigma^k(\tau)\Big)^3 \\ =& \Big(\chi_p(k)\tau\Big)^3 \\ =& \tau^3\end{aligned}$$

Which implies

$$\tau^3 \in \mathbb{Z}(\zeta_3)$$

Precisely, τ^3 is an **Eisenstein Integer**.
Let

$$\tau^3 = \epsilon$$

Then we have

$$\begin{aligned} |\epsilon|^2 =& |\tau^3|^2 \\ =& p^3 \end{aligned}$$

where the last equality holds due to the absolute value property on τ. The field norm of $\mathbb{Z}(\zeta_3)$, N, coinciding with the norm squared of a complex number as well as $\mathbb{Z}(\zeta_3)$ being a PID both imply

$$\begin{aligned} \tau^3 =& \bar{\pi}^l \cdot \pi^r \\ r + l =& 3 \\ r, l \in& \mathbb{Z}^+ \end{aligned}$$

If one of r, l is zero and the other 3, then τ would have to be of the form

$$\tau(\chi_p) = \zeta_3^k \pi$$

which would be completely fixed under any

$$\sigma \in \mathrm{Aut}(\mathbb{Q}(\zeta_{3p}/\mathbb{Q}(\zeta_3))$$

which is a contradiction since

$$\begin{aligned} \sigma^\alpha(\tau) = & \bar{\chi}_p(\alpha)\tau(\chi_p) \\ =& \zeta_3^2 \tau(\chi_p) \end{aligned}$$

where α is a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ and $\chi_p(\alpha)$ was chosen to be ζ_3. Hence, since there exists a σ that does not fix $\tau(\chi_p)$, this implies

$$\begin{aligned} \tau(\chi_p) =& \bar{\pi}\pi^2 \text{ or } \bar{\pi}^2\pi \\ =& p\pi \text{ or } p\bar{\pi} \end{aligned}$$

One point of ambiguity is deciding this.

With a fuzzy picture of what $\tau^3(\chi_p)$ looks like in $\mathbb{Z}(\zeta_3)$, it is now time to work with the power map

$$f_q : \alpha \mapsto \alpha^q$$

in some suitable quotient ring, where q is a $3k+1$ prime satisfying

$$\begin{aligned} \mathrm{N}\theta =& q \\ \theta \in& \mathbb{Z}(\zeta_3) \end{aligned}$$

The ring of choice of course is

$$R = \mathbb{Z}(\zeta_{3p})/\theta\mathbb{Z}(\zeta_{3p})$$

All within R, where $q = 0$, the map f_q acts on $\tau(\chi_p)$ as follows;

$$\begin{aligned} f_q(\tau(\chi_p)) =& \tau^q(\chi_p) \\ =& \Big(\tau^3(\chi_p)\Big)^{\frac{q-1}{3}} \cdot \tau(\chi_p) \\ =& \Big(\frac{p \cdot \pi}{\theta}\Big)_3 \cdot \tau(\chi_p) \end{aligned}$$

where whatever prime that pops out from the cubing of the Gauss Sum shall be denoted as π and its conjugate as $\bar{\pi}$. Just like with the Quadratic Character, the power map dislodges an expression involving the residue symbol. Simultaneously, f_q acting on τ also yields

$$\begin{aligned} f_q(\tau(\chi_p)) =& \Big(\sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n)\zeta_p^n\Big)^q \\ =& \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p^q(n) \cdot \zeta_p^{n \cdot q} \\ =& \sum_{n \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p(n)\zeta_p^{n \cdot q} \\ =& \bar{\chi}_p(q) \cdot \tau(\chi_p) \\ =& \Big(\frac{\mathrm{N}\theta}{\bar{\pi}}\Big)_3 \cdot \tau_p \text{ or } \Big(\frac{\mathrm{N}\theta}{\pi}\Big)_3 \cdot \tau_p \end{aligned}$$

where the ambiguity in the last line is due to uncertainty of which Eisenstein prime, π or $\bar{\pi}$, should be used as the modulus. The goal of determining this is somewhat irrelevant for now. Now, by refining the top index premier dislodged residue symbol

$$\begin{aligned} p \cdot \pi =& \bar{\pi} \cdot \pi^2 \\ =& \bar{\pi} \cdot \frac{\pi^3}{\pi} \end{aligned}$$

it yields a nicer expression within the residue symbol

$$\begin{aligned}\left(\frac{p\cdot\pi}{\theta}\right)_3 =&\left(\frac{\pi^3}{\theta}\right)_3\cdot\left(\frac{\frac{\bar{\pi}}{\pi}}{\theta}\right)_3\\ =&\left(\frac{\pi}{\theta}\right)_3^3\cdot\left(\frac{\frac{\bar{\pi}}{\pi}}{\theta}\right)_3\\ =&\cdot\left(\frac{\frac{\bar{\pi}}{\pi}}{\theta}\right)_3\\ =&\left(\frac{\mathfrak{h}(\pi)}{\theta}\right)_3\end{aligned}$$

and by equating the two results of the image of $\tau(\chi_p)$ under f_q within the quotient ring $\mathbb{Z}(\zeta_{3p})/q\mathbb{Z}(\zeta_{3p})$ and double checking using the Law of Cubic Reciprocity, it gives

$$\left(\frac{\mathfrak{h}(\pi)}{\theta}\right)_3 =\left(\frac{q}{\bar{\pi}}\right)_3$$

for primes p, q of the form $3k+1$. Regardless though of which prime appears in the bottom index of the RHS just above, what is of interest are polynomials of the form

$$L(X) =(1-\left(\frac{q}{\bar{\pi}}\right)_3\cdot X)\cdot(1-\left(\frac{q}{\pi}\right)_3\cdot X)$$

where q is a prime of the form $9k+1$ in order to rid the residue symbol of the unit problem and the ambiguity of choosing the 'proper' character. Also of interest is replacing q by a composite number n of the form $9k+1$ with no prime factors of the form $3k+2$. This too eliminates any ambiguity of units throwing off the value of the residue symbol, although there is still some ambiguity when it comes to properly choosing the proper character symbol to place within the product representation of the polynomial $EL_{p,n}(X)$.

1.5.2 Gauss Sum's and some Quartic Symbols

Just like with Cubic Gauss Sums and some Cubic Residue there is ambiguity. It can be shown using very similar methods that

$$\begin{aligned}\left(\frac{\frac{\bar{\pi}}{\pi}}{\theta}\right)_4 =&\left(\frac{\mathrm{N}\theta}{\bar{\pi}}\right)_4 \text{ or } \left(\frac{\mathrm{N}\theta}{\pi}\right)_4\\ =&\left(\frac{\mathfrak{h}(\pi)}{\theta}\right)_4\end{aligned}$$

for π and θ **Gaussian Primes** with non-square prime norms p, q, respectively, with residue symbol defined as

$$\left(\frac{\cdot}{\theta}\right)_4 = \left(\,\cdot\,\right)^{\frac{\mathrm{N}\theta - 1}{4}} \pmod{\theta}$$

2 Eisenstein Series

2.1 Introduction to Eisenstein Series

The classical **Eisenstein Series** are defined as

$$\mathrm{G}_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+n\tau)^{2k}}$$

where k is taken to be a fixed positive integer greater than or equal to 1 and $\mathrm{G}_{2k}(\tau)$ is taken to be a function of τ. This sum definition is a sum over the complex lattice Λ;

$$\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\}$$

When k is strictly greater than 1, the $\mathrm{G}_{2k}(\tau)$ satisfies

$$\mathrm{G}_{2k}(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{2k} \cdot \mathrm{G}_{2k}(\tau)$$

when a, b, c, d are integers satisfying $ad - bc = 1$. When k is 1, it does not satisfy the above transformation formula due to conditional convergence of the sum. It is interesting to see that $\mathrm{G}_{2k}(\tau)$ admits a fourier series. Particularly,

$$\mathrm{G}_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi\imath)^{2k}}{(2k-1)!} \cdot \sum_{n\geq 1} \sigma_{2k-1}(n) e^{2\pi\imath n\tau}$$

where

$$\sigma_{2k-1}(n) = \sum_{d\mid n} d^{2k-1}$$

for $d \mid n$ meaning 'd divides n' and $\zeta(s)$ Riemann's Zeta Function.

Proof

Using the formula

$$\sum_{n\in\mathbb{Z}} \frac{1}{\tau+n} = \pi\cot(\pi\tau)$$

where the sum is taken as

$$\sum_{n\in\mathbb{Z}} \frac{1}{\tau+n} = \lim_{T\to\infty} \sum_{n=-T}^{T} \frac{1}{\tau+n}$$

to assure convergence. Using the exponential form of $\pi\cot(\pi\tau)$, it yields the fourier series

$$\begin{aligned}\pi\cot(\pi\tau) &= -\pi\imath - \frac{2\pi\imath}{e^{-2\pi\imath\tau}-1}\\ &= -\pi\imath - 2\pi\imath\sum_{n\geq 1} e^{2\pi\imath n\tau}\end{aligned}$$

which gives the equality

$$\sum_{n\in\mathbb{Z}} \frac{1}{\tau+n} = -\pi\imath - 2\pi\imath\sum_{n\geq 1} e^{2\pi\imath n\tau}$$

By differentiating both sides $2k-1$ times, for k a positive integer, this yields

$$\begin{aligned}\frac{d^{2k-1}}{d\tau^{2k-1}}\Big(\sum_{n\in\mathbb{Z}} \frac{1}{\tau+n}\Big) &= \sum_{n\in\mathbb{Z}} \frac{d^{2k-1}}{d\tau^{2k-1}}(\tau+n)^{-1}\\ &= (-1)^{2k-1}(2k-1)!\cdot\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^{2k}}\\ &= \frac{d^{2k-1}}{d\tau^{2k-1}}\Big(-\pi\imath - 2\pi\imath\sum_{n\geq 1} e^{2\pi\imath n\tau}\Big)\\ &= -(2\pi\imath)^{2k}\cdot\sum_{n\geq 1} n^{2k-1}e^{2\pi\imath n\tau}\end{aligned}$$

The right hand side of the first equality, particularly the sum

$$(-1)^{2k-1}(2k-1)!\cdot\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^{2k}}$$

is beginning to look like the definition for $\mathrm{G}_{2k}(\tau)$. Refining some scalars on the sum yields a fourier series

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^{2k}} &= \frac{(2\pi\imath)^{2k}}{(2k-1)!}\cdot\sum_{n\geq 1} n^{2k-1}e^{2\pi\imath n\tau}\\ &= f_{2k}(\tau)\end{aligned}$$

which is beginning to look like the fourier series mentioned for G_{2k}. Since

$$\cot(-z) = -\cot(z)$$

it follows the $(2k-1)^{\text{th}}$ derivative of cotangent is an **even** function if k is an integer, and so is any scalar multiple of it; aka $f_{2k}(\tau) = f_{2k}(-\tau)$. Remember this. By breaking up the sum

$$\begin{aligned}
G_{2k}(\tau) = & \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+n\tau)^{2k}} \\
=& \Big(\sum_{m\in\mathbb{Z}\setminus 0} f_{2k}(m\tau) \Big) + \Big(\sum_{n\in\mathbb{Z}\setminus 0} \frac{1}{n^{2k}} \Big) \\
=& 2\zeta(2k) + 2\sum_{m\geq 1} f_{2k}(m\tau)
\end{aligned}$$

Using the fourier series of $f_{2k}(\tau)$, this gives us

$$\begin{aligned}
\sum_{m\geq 1} f_{2k}(m\tau) \equiv & \sum_{m\geq 1}\sum_{n\geq 1} n^{2k-1} e^{2\pi \imath n\cdot m\tau} \\
\equiv & \sum_{n\geq 1} c_{2k-1}(n) e^{2\pi\imath n\tau}
\end{aligned}$$

where

$$\begin{aligned}
c_{2k-1}(n) = & \sum_{d\cdot k = n} d^{2k-1} \\
=& \sigma_{2k-1}(n)
\end{aligned}$$

is the **Sum of Divisors** function of power $2k-1$, and the equivalence, $\equiv$, was used to indicate an ommision of the scalar $\dfrac{(2\pi\imath)^{2k}}{(2k-1)!}$. Putting all the pieces together, this is enough to show

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi\imath)^{2k}}{(2k-1)!} \cdot \sum_{n\geq 1} \sigma_{2k-1}(n) e^{2\pi\imath n\tau}$$

2.2 Twisted Eisenstein Series

It is most natural to follow up on the classical Eisenstein Series with their twisted cousins.
Let the Eisenstein Series of a primitive Dirichlet Character χ_p of modulus p be defined as the Fourier Series

$$\mathrm{G}_{2k}(\chi_p,\tau) = \frac{2\cdot\tau(\bar{\chi}_p)}{(2k-1)!}\cdot\left(\frac{2\pi\imath}{p}\right)^{2k}\cdot\sum_{n\geq 1}\sigma_{2k-1}(n)\chi_p(n)e^{\frac{2\pi\imath n\tau}{p}}$$

where $\tau(\bar{\chi}_p)$ is a scalar defined by

$$\tau(\chi_p) = \sum_{n\in\mathbb{Z}/p\mathbb{Z}}\chi_p(n)\zeta_p^n$$
$$\zeta_p = e^{\frac{2\pi\imath}{p}}$$

which is also known as the **Gauss Sum** of the character χ_p.
Then this twisted Eisenstein Series admits the definition

$$\mathrm{G}_{2k}(\chi_p,\tau) = p^{-2k}\sum_{n\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(n)\mathrm{G}_{2k}(\frac{\tau+n}{p})$$

where $\mathrm{G}_{2k}(\tau)$ is the classical Eisenstein series of weight $2k$.

Proof

The proof is straightforward. All it takes is to expand

$$p^{-2k}\sum_{n\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(n)\mathrm{G}_{2k}(\frac{\tau+n}{p})$$

as a double sum by rewriting G_{2k} as a Fourier Series. It looks like

$$\begin{aligned}
p^{-2k}\sum_{n\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(n)\mathrm{G}_{2k}(\frac{\tau+n}{p}) =& p^{-2k}\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\cdot\Big(2\zeta(2k)+2\frac{(2\pi\imath)^{2k}}{(2k-1)!}\cdot\sum_{n\geq 1}\sigma_{2k-1}(n)e^{2\pi\imath n\cdot(\frac{\tau+l}{p})}\Big)\\
=& p^{-2k}\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\cdot\Big(2\frac{(2\pi\imath)^{2k}}{(2k-1)!}\cdot\sum_{n\geq 1}\sigma_{2k-1}(n)e^{2\pi\imath n\cdot(\frac{\tau+l}{p})}\Big)\\
=& p^{-2k}\cdot 2\frac{(2\pi\imath)^{2k}}{(2k-1)!}\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\cdot\Big(\sum_{n\geq 1}\sigma_{2k-1}(n)e^{2\pi\imath n\cdot(\frac{\tau+l}{p})}\Big)\\
=& \frac{2}{(2k-1)!}\cdot\Big(\frac{2\pi\imath}{p}\Big)^{2k}\cdot\sum_{n\geq 1}\sigma_{2k-1}(n)e^{\frac{2\pi\imath n\tau}{p}}\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\Big(\bar{\chi}_p(l)\zeta_p^{n\cdot l}\Big)\\
=& \frac{2}{(2k-1)!}\cdot\Big(\frac{2\pi\imath}{p}\Big)^{2k}\cdot\sum_{n\geq 1}\tau(\bar{\chi}_p,n)\sigma_{2k-1}(n)e^{\frac{2\pi\imath n\tau}{p}}\\
=& \frac{2\cdot\tau(\bar{\chi}_p)}{(2k-1)!}\cdot\Big(\frac{2\pi\imath}{p}\Big)^{2k}\cdot\sum_{n\geq 1}\sigma_{2k-1}(n)\chi_p(n)e^{\frac{2\pi\imath n\tau}{p}}
\end{aligned}$$

Where in the second line, The constant coefficient $\zeta(2k)$ drops due to the orthogonality property of a non-trivial Dirichlet Character χ, and in the last line the property of

$$\tau(\bar{\chi}_p,v)=\chi_p(v)\cdot\tau(\bar{\chi}_p),v\in\mathbb{Z}$$

was used.

2.2.1 Twisted Eisenstein Series as a Lattice Sum

The twisted Eisenstein Series admits the lattice sum

$$\mathrm{G}_{2k}(\chi_p, \tau) = \sum_{(m,n)\in\mathbb{Z}^2\backslash 0} \frac{\bar{\chi}_p(m)\cdot\chi_p(n)}{(m+n\tau)^{2k}}$$

Proof

The approach taken here is to find a Fourier Series

$$\sum_{(m,n)\in\mathbb{Z}^2\backslash 0} \frac{\bar{\chi}_p(m)\cdot\chi_p(n)}{(m+n\tau)^{2k}} = \sum_{n\geq 1} c_{2k-1}(n)e^{\frac{2\pi\imath n\tau}{p}}$$

and show that

$$c_{2k-1}(n) = \frac{2\cdot\tau(\bar{\chi_p})}{(2k-1)!}\cdot\Big(\frac{2\pi\imath}{p}\Big)^{2k}\cdot\sigma_{2k-1}(n)\chi_p(n)$$

which would imply the Twisted Eisenstein Series Lattice sum representation. This is best accomplished by an approach done for the lattice sum of the Classical Eisenstein Series. For starters, consider

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \frac{\bar{\chi}_p(m)}{m+\tau} &= \sum_{l\in\mathbb{Z}/p\mathbb{Z}}\sum_{m\in\mathbb{Z}} \bar{\chi}_p(l)\cdot\Big(\frac{1}{p\times m+l+\tau}\Big)\\
&= \sum_{l\in\mathbb{Z}/p\mathbb{Z}} \frac{\bar{\chi}_p(l)}{p} \sum_{m\in\mathbb{Z}/p\mathbb{Z}} \frac{1}{m+(\frac{\tau+l}{p})}\\
&= \sum_{l\in\mathbb{Z}/p\mathbb{Z}} \frac{\bar{\chi}_p(l)}{p}\cdot\pi\cot\big(\pi\cdot(\frac{\tau+l}{p})\big)\\
&= \sum_{l\in\mathbb{Z}/p\mathbb{Z}} \frac{\bar{\chi}_p(l)}{p}\Big(-\pi\imath - 2\pi\imath\sum_{m\geq 1} e^{2\pi\imath m(\frac{\tau+l}{p})}\Big)
\end{aligned}$$

A quick breath of fresh air is good. Okay, back to summation; I left off at

$$\begin{aligned}
\sum_{l\in\mathbb{Z}/p\mathbb{Z}} \frac{\bar{\chi}_p(l)}{p}\Big(- \pi\imath - 2\pi\imath \sum_{m\geq 1} e^{2\pi\imath m(\frac{\tau+l}{p})}\Big) &= \sum_{l\in\mathbb{Z}/p\mathbb{Z}} \frac{\bar{\chi}_p(l)}{p}\Big(- 2\pi\imath \sum_{m\geq 1} e^{2\pi\imath m(\frac{\tau+l}{p})}\Big) \\
&= -\Big(\frac{2\pi\imath}{p}\Big)\cdot \sum_{l\in\mathbb{Z}/p\mathbb{Z}}\sum_{m\geq 1} \bar{\chi}_p(l)\zeta_p^{m\cdot l} e^{\frac{2\pi\imath m\tau}{p}} \\
&= -\tau(\bar{\chi}_p)\cdot\Big(\frac{2\pi\imath}{p}\Big)\cdot\sum_{m\geq 1}\chi_p(m)e^{\frac{2\pi\imath m\tau}{p}}
\end{aligned}$$

Whew! This was a lot for me to type out. Okay. This gives a Fourier Series for

$$\sum_{m\in\mathbb{Z}} \frac{\bar{\chi}_p(m)}{m+\tau} = -\tau(\bar{\chi}_p)\cdot\Big(\frac{2\pi\imath}{p}\Big)\cdot\sum_{m\geq 1}\chi_p(m)e^{\frac{2\pi\imath m\tau}{p}}$$

and by differentiating $2k-1$ times and adjusting the RHS by a scalar, these steps yield

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \frac{\bar{\chi}_p(m)}{(m+\tau)^{2k}} =& \frac{\tau(\bar{\chi}_p)}{(2k-1)!}\cdot\Big(\frac{2\pi\imath}{p}\Big)^{2k}\cdot\sum_{m\geq 1} m^{2k-1}\chi_p(m)e^{\frac{2\pi\imath m\tau}{p}} \\
=& f_{2k}(\chi_p,\tau)
\end{aligned}$$

while noting

$$\begin{aligned}
f_{2k}(\chi_p,-\tau) =& \sum_{m\in\mathbb{Z}} \frac{\bar{\chi}_p(m)}{(m-\tau)^{2k}} \\
=& \sum_{m\in\mathbb{Z}} \frac{\bar{\chi}_p(m)}{(-m+\tau)^{2k}} \\
=& \bar{\chi}_p(-1)\cdot\sum_{m\in\mathbb{Z}} \frac{\bar{\chi}_p(-m)}{(-m+\tau)^{2k}} \\
=& \bar{\chi}_p(-1)\cdot\sum_{-m\in\mathbb{Z}} \frac{\bar{\chi}_p(m)}{(m+\tau)^{2k}} \\
=& \bar{\chi}_p(-1)\cdot f_{2k}(\bar{\chi}_p,\tau)
\end{aligned}$$

which is the same as saying 'f_{2k}' is odd if $\bar{\chi}_p(-1) = -1$ or even if $\bar{\chi}_p(-1) = 1$. Either way, it won't matter. To cap off the proof, we need to compare the original 'weighted' lattice sum to that of a sum over $f_{2k}(\tau)$, which looks like

$$\begin{aligned}
\sum_{(m,n)\in\mathbb{Z}^2\setminus 0} \frac{\bar{\chi}_p(m)\cdot\chi_p(n)}{(m+n\tau)^{2k}} =& \sum_{n\in\mathbb{Z}\setminus 0} \chi_p(n)\cdot f_{2k}(\chi_p, n\tau) \\
=& \sum_{n\geq 1} \Big(\chi_p(n)\cdot f_{2k}(\chi_p, n\tau) + \chi_p(-n)\cdot f_{2k}(\chi_p, -n\tau)\Big) \\
=& \sum_{n\geq 1} (1+\bar{\chi}_p^2(-1))\cdot\chi_p(n)\cdot f_{2k}(\chi_p, n\tau) \\
=& 2\sum_{n\geq 1} \chi_p(n)\cdot f_{2k}(\chi_p, n\tau) \\
=& \frac{2\cdot\tau(\bar{\chi}_p)}{(2k-1)!}\cdot\left(\frac{2\pi\imath}{p}\right)^{2k}\cdot\sum_{n\geq 1}\sum_{m\geq 1} m^{2k-1}\chi_p(m\cdot n)e^{\frac{2\pi\imath nm\tau}{p}} \\
=& \frac{2\cdot\tau(\bar{\chi}_p)}{(2k-1)!}\cdot\left(\frac{2\pi\imath}{p}\right)^{2k}\cdot\sum_{n\geq 1}\chi_p(n)\sigma_{2k-1}(n)e^{\frac{2\pi\imath n\tau}{p}} \\
=& \mathrm{G}_{2k}(\chi_p, \tau)
\end{aligned}$$

where the last equality holds since the fourier series are identical. This completes the proof.

2.3 Slightly more twisted Eisenstein Series

Consider a similair Eisenstein series defined by the lattice sum

$$\mathrm{G}_{2k}(\psi_q, \mathbb{Z}(\alpha), \tau) = \sum_{m,n\in\mathbb{Z}^2\setminus(0,0)} \frac{\bar{\psi}_q(m+n\alpha)\psi_\pi(m+n\bar{\alpha})}{(m+n\tau)^{2k}}$$

where α is an imaginary quadratic of the form

$$\begin{aligned}
\alpha &= \frac{-1+\sqrt{-D}}{2} && \text{if } D\equiv 3 \pmod 4 \\
\alpha &= \sqrt{-D} && \text{if } D\equiv 1 \pmod 4
\end{aligned}$$

and ψ_q is a multiplicative character of modulus q, whether it be a quadratic integer or a rational one, on the multiplicative group

$$\psi_q : (\mathbb{Z}(\alpha)/(q))^\times \to \mathbb{C}^\times$$

satisfying

$$\psi_q(\gamma)\psi_q(\beta) = \psi_q(\gamma\beta)$$

If the **Minimial polynomial** of α

$$
\begin{aligned}
f(X) =& X^2 + D && \text{if } D \equiv 1 \pmod 4 \\
f(X) =& X^2 + X + \frac{D+1}{4} && \text{if } D \equiv 3 \pmod 4
\end{aligned}
$$

is reducible in $\mathbb{F}_p$ and the modulus of the character ψ is an irrational prime in $\mathbb{Z}(\alpha)$ with field norm equal to p, then this twisted Eisenstein series can be rewritten as

$$\omega \cdot (c\tau + d)^{-2k} \mathrm{G}_{2k}(\psi_\pi, \mathbb{Z}(\alpha), \frac{a\tau + b}{c\tau + d}) = \mathrm{G}_{2k}(\chi_p, \tau)$$

for some non-zero scalar ω, the quantity π being precisely the modulus of the character ψ, and the transformation above satisfies

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

Proof

Consider the matrix and its' associated linear transformation

$$\gamma_0 = \begin{pmatrix} \bar{\alpha} & \alpha \\ 1 & 1 \end{pmatrix}$$

where α is a solution to $f(X)$ **Modulo p** . γ_0 has a non-zero determinant in characteristic p, as long as p does not divide the discriminant of the field

$$K = \mathbb{Q}(\alpha)$$

γ_0 acts on τ as follows;

$$\gamma_0 \circ \tau = \frac{\bar{\alpha}\tau + \alpha}{\tau + 1}$$

and within the Eisenstein series, the behavior of the lattice sum looks like

$$\mathrm{G}_{2k}(\psi_\pi, \mathbb{Z}(\alpha), \gamma_0 \circ \tau) = (\tau + 1)^{2k} \cdot \sum_{m,n \in \mathbb{Z}^2 \setminus (0,0)} \frac{\bar{\psi}_q(m + n\alpha)\psi_\pi(m + n\bar{\alpha})}{\big((m + n\alpha) + (m + n\bar{\alpha})\tau\big)^{2k}}$$

One would like to stop here, rearrange the order of summation, and match it with that of $\mathrm{G}_{2k}(\chi_p, \tau)$. Bad news is, the determinant of γ_0

$$\begin{aligned} \mid \gamma_0 \mid =& \bar{\alpha} - \alpha \\ =& \sqrt{\Delta_K} \end{aligned}$$

is not equal to 1. But it is non-zero in characteristic p, which means it can be modified to give something that can be worked with to rearrange the summation and match it with $\mathrm{G}_{2k}(\chi_p, \tau)$. The modified version of γ_0, call it γ, is precisely

$$\gamma = \begin{pmatrix} \lambda\alpha + 1 & \alpha \\ \lambda & 1 \end{pmatrix}$$

where the quantity λ is an integer satisfying

$$\lambda^{-2} \equiv \Delta_K \pmod{p}$$

The transformation γ was found by taking the transformation γ_0, multiplying the left column by a non-zero element of $\mathbb{Z}/p\mathbb{Z}$ such that the determinant of this modified transformation was at least 1 modulo p. From there, an element of $\mathrm{SL}(2, \mathbb{Z})$, the γ presented, was found such that the entries were equivalent to the modified transformation modulo p. What this all looks like is the following;

$$\bar{\chi}(\lambda)(\lambda\tau + 1)^{-2k} \cdot \mathrm{G}_{2k}(\psi_\pi, \mathbb{Z}(\alpha), \gamma \circ \tau) = \mathrm{G}_{2k}(\chi_p, \tau)$$

Where the replacement of the characters is justified since the fields $\mathbb{Z}/p\mathbb{Z}$ and $\mathcal{O}/(\pi)$ are isomorphic; both are fields of characteristic p. The only α that will serve purpose for the remainder of this piece is specifically those for which the ring of integers

$$\mathcal{O}_K = \mathbb{Z}(\alpha)$$

is a Principal Ideal Domain.

2.4 Eisenstein Series of Weight 2

Consider the Eisenstein Series

$$\mathrm{G}_2(\chi_p,\tau) = 2\tau(\bar{\chi}_p)\cdot\Big(\frac{2\pi\imath}{p}\Big)^2\cdot\sum_{n\geq 1}\chi_p(n)\sigma_1(n)e^{\frac{2\pi\imath n\tau}{p}}$$

and the accompanied indefinite integral

$$\int \mathrm{G}_2(\chi_p,\tau)d\tau = \mathrm{G}_0(\chi_p,\tau)+\mathrm{c}$$

where c is a constant of integration and $\mathrm{G}_0(\chi_p,\tau)$ is defined as

$$\mathrm{G}_0(\chi_p,\tau) = 2\tau(\bar{\chi}_p)\cdot\Big(\frac{2\pi\imath}{p}\Big)\cdot\sum_{n\geq 1}\chi_p(n)\frac{\sigma_1(n)}{n}e^{\frac{2\pi\imath n\tau}{p}}$$

Let $\eta(\tau)$ be defined as

$$\eta(\tau) = e^{\frac{\pi\imath\tau}{12}}\cdot\prod_{n\geq 1}\big(1-e^{2\pi\imath n}\big)$$

Then the accompanied indefintie integral is satisfies

$$\int \mathrm{G}_2(\chi,\tau)d\tau = -\frac{4\pi\imath}{p}\cdot\sum_{p\geq n\geq 1}\bar{\chi}_p(n)\mathrm{ln}\Big(\eta(\frac{\tau+n}{p})\Big)+c$$

where the sum on the RHS is a sum over n and c is a constant of integration that may or may not differ from the previous c constant.

Proof

The proof will follow by finding the Fourier Series of

$$-\frac{4\pi\imath}{p}\cdot\sum_{p\geq n\geq 1}\bar{\chi}_p(n)\mathrm{ln}\Big(\eta(\frac{\tau+n}{p})\Big)$$

To start, recall

$$\frac{1}{1-X}=\sum_{n\geq 0}X^n \implies -\ln(1-x)=\sum_{n\geq 1}\frac{X^n}{n}$$

for suitable X. The logarithm of $\eta(\tau)$ has the following Fourier series;

$$\begin{aligned}
-\ln\big(\eta(\tau)\big) &= -\ln\big(e^{\frac{\pi\imath\tau}{12}} \cdot \prod_{n\geq 1}(1-e^{2\pi\imath n\tau})\big) \\
&= -\frac{\pi\imath\tau}{12} + \sum_{n\geq 1} -\ln\big(1-e^{2\pi\imath n\tau}\big) \\
&= -\frac{\pi\imath\tau}{12} + \sum_{n\geq 1}\sum_{l\geq 1}\frac{e^{2\pi\imath nl\tau}}{l} \\
&= -\frac{\pi\imath\tau}{12} + \sum_{n\geq 1}\sum_{l\geq 1} n\cdot\Big(\frac{e^{2\pi\imath nl\tau}}{l\cdot n}\Big) \\
&= -\frac{\pi\imath\tau}{12} + \sum_{n\geq 1} a(m)\cdot\Big(\frac{e^{2\pi\imath m\tau}}{m}\Big)
\end{aligned}$$

where the coefficients $a(m)$ are $\sigma_1(n)$.
It doesn't take much from here to show the equality between the Fourier series.
Now, by twisting the coefficients by a character

$$\begin{aligned}
-\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\ln\big(\eta(\frac{\tau+l}{p})\big) &= \sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\cdot\Big(-\frac{\pi\imath}{12}\big(\frac{\tau+l}{p}\big)+\sum_{n\geq 1}\frac{\sigma_1(n)}{n}e^{2\pi\imath n(\frac{\tau+l}{p})}\Big) \\
&= -\Big(\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\frac{\pi\imath\tau}{12p}\Big) - \Big(\frac{\pi\imath}{12p}\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\cdot l\Big) + \sum_{n\geq 1}\frac{\sigma_1(n)}{n}e^{\frac{2\pi\imath n\tau}{p}}\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\zeta_p^{n\cdot l} \\
&= c + \sum_{n\geq 1}\tau(\bar{\chi}_p, n)\cdot\frac{\sigma_1(n)}{n}e^{\frac{2\pi\imath n\tau}{p}} \\
&= c + \tau(\bar{\chi}_p)\cdot\sum_{n\geq 1}\frac{\chi_p(n)\sigma_1(n)}{n}e^{\frac{2\pi\imath n\tau}{p}}
\end{aligned}$$

Now that a Fourier series is obtained for the particular sum of logarithms of η's, it is safe to come to the conclusion

$$\mathrm{G}_0(\chi_p,\tau) \equiv -\frac{4\pi\imath}{p}\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\ln\Big(\eta\big(\frac{\tau+l}{p}\big)\Big)$$

where the equivalence indicates the difference

$$\mathrm{G}_0(\chi_p,\tau) - \frac{-4\pi\imath}{p}\sum_{l\in\mathbb{Z}/p\mathbb{Z}}\bar{\chi}_p(l)\ln\Big(\eta\big(\frac{\tau+l}{p}\big)\Big) = \mathrm{C}$$

is a constant C.

3 Application to Dedekind Zeta function residue

3.1 Preface

A lot of the upcoming material found in this portion of this piece is going to utilise the previous portions. Particular statements, like how the Dedekind Zeta function of the numberfields considered look like in terms of products of L-functions, will be given without proof. The purpose of relating the previous portions is to conceive expressions for particular numerical quantities in terms of η-quotients and sums of η-quotients.

3.2 Biquadratic Fields

The types of biquadratic field considered here will be those of the form

$$K = \mathbb{Q}(\alpha, \sqrt{p^*})$$

where the quantity α is the quadratic irrationality defined is taken from the set

$$\mathrm{S} = \{\sqrt{-1}, \sqrt{-2}, \frac{-1+\sqrt{-3}}{2}, \frac{-1+\sqrt{-7}}{2}, \frac{-1+\sqrt{-11}}{2}, \frac{-1+\sqrt{-19}}{2}, \frac{-1+\sqrt{-43}}{2}, \frac{-1+\sqrt{-163}}{2}\}$$

and the quantity p^* is defined as

$$p^* = (-1)^{\frac{p-1}{2}} \cdot p$$

for p a prime where the minimal polynomial of the α in question is reducible in the finite field $\mathbb{F}_p$, but does not divide the discriminant of $\mathbb{Q}(\alpha)$. The set S is precisely the set of imaginary quadratics for which the ring $\mathbb{Z}(\alpha)$ is a **UFD**, which implies the Dedekind Zeta function of the field

$$k = \mathbb{Q}(\alpha)$$

can be written as

$$\begin{aligned}\zeta_k(s) &= \sum_{\mathrm{I} \subseteq \mathcal{O}_k} \mid \mathcal{O}_k/\mathrm{I} \mid^{-s} \\ &= w_k^{-1} \sum_{\alpha \in \mathcal{O}_k \setminus 0} \mid \alpha \mid^{-2s}\end{aligned}$$

where the former sum is precisely a sum over the non-zero ideals of the whole integer ring $\mathcal{O}_k$ and the latter sum is a sum over the non-zero elements α, and the former norm is the **Ideal norm** and the latter is the absolute value of the complex number α. The Dedekind Zeta functions, $\zeta_k(s)$ and $\zeta_K(s)$, can be written as a product of Dirichlet L-functions. The Dirichlet L-functions of a Dirichlet character χ are defined as

$$\begin{aligned} L(\chi, s) =& \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}} \\ =& \sum_{n \geq 1} \frac{\chi(n)}{n^s} \end{aligned}$$

The explicit formula for the Dedekind Zeta functions are

$$\begin{aligned} \zeta_k(s) =& \zeta(s) \cdot L(\chi_d, s) \\ \zeta_K(s) =& \zeta(s) \cdot L(\chi_d, s) \cdot L(\chi_p, s) \cdot L(\chi_d \cdot \chi_p, s) \end{aligned}$$

for d the discriminant of k, and with χ_d and χ_p primitive, quadratic characters of modulus d and p respectively. Of interest is the value

$$L(\psi, 1)$$

for an arbitrary, primitive quadratic character ψ. Until further notice, assume the modulus of ψ is a prime; call it q. The quantity $L(\psi, 1)$ can be given by

$$L(\psi, 1) = -\frac{\ln(c)}{\tau(\psi)}$$

where the quantity $\tau(\psi)$ is the **Gauss sum** of the quadratic character ψ and the constant c is defined by the quotient of cyclotomic integers

$$c = \prod_{q-1 \geq l \geq 1} (1 - \zeta_q^l)^{\psi(l)}$$

where the quantity ζ_q is a q'th root of unity taking the value

$$\zeta_q = e^{\frac{2\pi\imath}{q}}$$

Proof

The value of the L-function at $s = 1$ is defined as

$$L(\psi, 1) = \sum_{n\geq 1} \frac{\psi(n)}{n}$$

The approach to show $L(\psi, 1)$ takes on the expression claimed, it is necessary to consider the function

$$-\ln(1 - X) = \sum_{n\geq 1} \frac{X^n}{n}$$

where the RHS is well defined for suitable X. Consider the properties

$$\begin{aligned}\tau(\psi, v) =& \bar{\psi}(v)\tau(\psi)\\ =& \psi(v)\tau(\psi)\end{aligned}$$

where the latter holds since the character ψ takes on purely real values; $\bar{\pm 1} = \pm 1$. These properties can be used to manipulate the McLaurin series of $-\ln(1 - X)$ to yield

$$\begin{aligned}\sum_{q-1\geq l\geq 1} -\psi(l)\ln(1 - \zeta_q^l \cdot X) =& \sum_{n\geq 1} \frac{X^n}{n} \sum_{q-1\geq l\geq 1} \psi(l) \cdot \zeta_q^{l\cdot n}\\ =& \tau(\psi) \cdot \sum_{n\geq 1} \psi(n) \cdot \frac{X^n}{n}\end{aligned}$$

The sum of the logarithms of the linear factors can be turned into the product

$$-\sum_{q-1\geq l\geq 1} \psi(l)\ln(1 - \zeta_q^l \cdot X) = -\ln\Big(\prod_{q-1\geq l\geq 1} \big(1 - \zeta_q^l \cdot X\big)^{\psi(l)}\Big)$$

By taking X to be 1 and normalizing the RHS by a factor of $\tau(\psi)$, this gives

$$\begin{aligned}-\frac{\ln\Big(\prod_{q-1\geq l\geq 1} \big(1 - \zeta_q^l\big)^{\psi(l)}\Big)}{\tau(\psi)} =& \sum_{n\geq 1} \frac{\psi(n)}{n}\\ =& L(\psi, 1)\end{aligned}$$

When the character ψ is an even quadratic character, the quantity c is a unit in the Quadratic integer ring $\mathcal{O}_F$ where F is the real quadratic field with discriminant Δ_F. If the character ψ is odd, then the quantity c is a q'th root of unity. The value $L(\psi, 1)$ contains information about the integers of F; namely **Class Number** and the generator of the unit group $\mathcal{O}_F^\times$, if F is a real field.

When F is an imaginary quadratic field with an integral basis

$$\mathcal{O}_F = \mathbb{Z}(\alpha)$$

then the following relationship holds if the class number h_F is equal to 1;

$$\mid (2\pi) \cdot \eta(\alpha)^2 \mid = \exp(-w_F \cdot \zeta_F'(0))$$

where $\eta(\tau)$ is the Dedekind Eta function mentioned in the section on Eisenstein Series. A refined expression can be given as

$$\mid \eta(\tau) \mid = (2\pi \mid \Delta_F \mid)^{-\frac{1}{4}} \cdot \Big(\prod_{\Delta_F \geq l \geq 1} \Gamma^{\psi(l)}(\frac{l}{\Delta_F}) \Big)^{\frac{w_F}{8}}$$

where $\Gamma(n)$ is the classical Gamma function. This can be found quite easily with a lack of rigor, but a proof can be established using the Kronecker Limit Formula.

By itself, this formula for $\eta(\alpha)$ only works when α is from the set S, up to an integer term. While it is not clear what can be done for all the other imaginary quadratic irrationalities, it is possible to coax the quantity

$$L(\chi_p, 1) \cdot L(\chi_p \cdot \chi_d, 1)$$

for particular p and d. Recall the expressions for the Dedekind Zeta function of the biquadratic field of K with restrictions on α and p^*; K is the field

$$K = \mathbb{Q}(\alpha, \sqrt{p^*})$$

The Dedekind Zeta function of K can be expressed in two ways; one which was given as

$$\zeta_K(s) = \zeta(s) L(\chi_d, s) L(\chi_p, s) L(\chi_p \cdot \chi_d, s)$$

The method of extracting a Kronecker Limit formula for $L(\chi_p, 1) \cdot L(\chi_p \cdot \chi_d, 1)$ relies on rewriting $\zeta_K(s)$ as

$$\zeta_K(s) = \zeta_k(s) \cdot L(k, \chi_p, s)$$

where the function $L(k, \chi_p, s)$ is defined as

$$\begin{aligned}
L(k, \chi_p, s) =& \prod_{\pi} \frac{1}{1 - \frac{\chi_p\left(|\pi|^2\right)}{\left(|\pi|\right)^{2s}}} \\
=& w_k^{-1} \cdot \sum_{\omega \in \mathbb{Z}(\alpha) \backslash 0} \frac{\chi_p(|\ \omega\ |^2)}{|\ \omega\ |^{2s}} \\
=& L(\chi_p, s) \cdot L(\chi_p \cdot \chi_d, s)
\end{aligned}$$

where the product in the first line runs over all the prime elements π of $\mathbb{Z}(\alpha)$ and the latter runs over all the non-zero elements ω of $\mathbb{Z}(\alpha)$. This is equivalent to using prime ideals $\mathfrak{p}$ and ideals $\mathfrak{o}$ in place of π and ω since

$$\mathrm{h}_k = 1$$

implies the ring $\mathbb{Z}(\alpha)$ is a **PID** . By taking s to be 1, the product of the L-values admits the sum

$$w_k L(\chi_p, 1) \cdot L(\chi_p \cdot \chi_d, 1) = \sum_{\omega \in \mathbb{Z}(\alpha) \backslash 0} \frac{\chi_p(\omega \cdot \bar{\omega})}{\omega \cdot \bar{\omega}}$$

Alternatively, this quantity can be expressed as the following;

$$w_k L(\chi_p, 1) \cdot L(\chi_p \cdot \chi_d, 1) = \Big(\frac{-4\pi\chi_p(-\sqrt{\Delta_k})}{p \cdot \sqrt{|\ \Delta_k\ |}}\Big) \cdot \ln\Big(\prod_{p \geq l \geq 1} \eta^{2\chi_p(l)}\big(\frac{\gamma^{-1} \circ \alpha + l}{p}\big)\Big)$$

For $\eta(\tau)$ the Dedekind Eta function, the quantity $\gamma^{-1} \circ \alpha$ a linear fractional transformation γ^{-1} on the quantity α, and the instance of $\sqrt{\Delta_k}$ appearing within the character χ_p is taken to be a solution in $\mathbb{F}_p$ to the equation

$$x^2 = \Delta_k$$

There is some ambiguity for the η quotient when the character χ_p is odd; this can be attributed to the non-vanishing quantity;

$$\begin{aligned}
c =& \frac{-1}{p} \sum_{p \geq l \geq 1} \chi_p(l) \cdot l \\
\neq& 0
\end{aligned}$$

More will be elaborated in the proof

Proof

A quick recap on the criteria for the primes p considered for an α in S; it doesn't divide the discriminant Δ_k for k the quadratic field $\mathbb{Q}(\alpha)$ and the minimal polynomial of α must be reducible in $\mathbb{F}_p$. Consider the weight two Eisenstein series of a real, non-trivial character ψ of modulus 'π'

$$\mathrm{G}_2(\psi_\pi, \mathbb{Z}(\alpha), \tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{\psi_\pi(m+n\alpha)\cdot\psi(m+n\bar{\alpha})}{(m+n\tau)^2}$$

for π an element of $\mathbb{Z}(\alpha)$ with field norm of p. The character ψ_π is 'the same' as the character quadratic character χ_p related by

$$\psi_\pi(m+n\alpha) = \chi_p\big(m - n(a\cdot b^{-1})\big)$$

given that π is of the form

$$\pi = a + b\alpha$$

and the quantity b^{-1} is the multiplicative inverse of b in the ring $\mathbb{Z}/p\mathbb{Z}$. The product $-(a\cdot b^{-1})$ is the root chosen within the characters for α, taken from the solution to the equation

$$X^2 - \mathrm{Tr}(\alpha)\cdot X + \mid \alpha \mid^2 \equiv 0 \pmod p$$

with the quantity $\bar{\alpha}$ appearing within the characters is taken as the other solution of the equation.

The integral of $\mathrm{G}_2(\psi_\pi, \mathbb{Z}(\alpha), \tau)$ has the following Eisenstein series;

$$\begin{aligned}\int \mathrm{G}_2(\psi_\pi, \mathbb{Z}(\alpha), \tau)d\tau &= \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} -\frac{\psi_\pi(m+n\alpha)\cdot\psi(m+n\bar{\alpha})}{n\cdot(m+n\tau)} \\ &= \mathrm{G}_0(\psi_\pi, \mathbb{Z}(\alpha), \tau)\end{aligned}$$

Evaluating at τ equal to α and refining the lattice sum gives

$$\begin{aligned}
\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} -\frac{\psi_\pi(m+n\alpha)\cdot\psi(m+n\bar{\alpha})}{n\cdot(m+n\alpha)} =&(\bar{\alpha}-\alpha)\cdot\sum_{\omega\in\mathbb{Z}(\alpha)\setminus 0}\frac{\psi_\pi(|\,\omega\,|^2)}{(\omega-\bar{\omega})\cdot\omega}\\
=&-\sqrt{\Delta_k}\cdot\sum_{\omega\in\mathbb{Z}(\alpha)\setminus 0}\frac{\psi_\pi(|\,\omega\,|^2)}{(\omega-\bar{\omega})\cdot\omega}\\
=&-\sqrt{\Delta_k}\cdot\sum_{\omega\in\mathbb{Z}(\alpha)\setminus 0}\frac{\psi_\pi(|\,\bar{\omega}\,|^2)}{(\bar{\omega}-\omega)\cdot\bar{\omega}}\\
=&-\frac{\sqrt{\Delta_k}}{2}\cdot\sum_{\omega\in\mathbb{Z}(\alpha)\setminus 0}\Big(\frac{\psi_\pi(|\,\bar{\omega}\,|^2)}{(\bar{\omega}-\omega)\cdot\bar{\omega}}+\frac{\psi_\pi(|\,\omega\,|^2)}{(\omega-\bar{\omega})\cdot\omega}\Big)\\
=&-\frac{\sqrt{\Delta_k}}{2}\cdot\sum_{\omega\in\mathbb{Z}(\alpha)\setminus 0}\Big(\frac{\psi_\pi(|\,\omega\,|^2)}{\omega-\bar{\omega}}\Big)\cdot\Big(\frac{1}{\omega}-\frac{1}{\bar{\omega}}\Big)\\
=&-\frac{\sqrt{\Delta_k}}{2}\cdot\sum_{\omega\in\mathbb{Z}(\alpha)\setminus 0}\Big(\frac{\bar{\omega}-\omega}{\omega-\bar{\omega}}\Big)\cdot\Big(\frac{\psi_\pi(|\,\omega\,|^2)}{|\,\omega\,|^2}\Big)\\
=&\frac{\sqrt{\Delta_k}}{2}\cdot\sum_{\omega\in\mathbb{Z}(\alpha)\setminus 0}\frac{\psi_\pi(|\,\omega\,|^2)}{|\,\omega\,|^2}
\end{aligned}$$

To tidy things up, the evaluation at α should be given by

$$\begin{aligned}
\Big(\frac{2}{\sqrt{\Delta_k}}\Big)\cdot \mathrm{G}_0(\psi_\pi,\mathbb{Z}(\alpha),\alpha)=&\sum_{\omega\in\mathbb{Z}(\alpha)\setminus 0}\frac{\psi_\pi(|\,\omega\,|^2)}{|\,\omega\,|^2}\\
=&w_k\cdot L(\chi_p,1)\cdot L(\chi_p\cdot\chi_d,1)
\end{aligned}$$

This equality is not entirely true; there is an extra step to be taken when χ_p is odd, which will be elaborated on further down the line.

For now, recall from the portion on Eisenstein series where the existence of a single linear fractional transformation γ was shown to relate $\mathrm{G}_{2k}(\psi_\pi,\mathbb{Z}(\alpha),\tau)$ to $\mathrm{G}_{2k}(\chi_p,\tau)$ for all positive integer k and fixed triplet (ψ_π,χ_p,α), with restrictions on the modulus p not dividing the discriminant Δ_k and such. The relationship was given, in an equivalent form, as

$$(\lambda\tau+1)^{-2k}\cdot \mathrm{G}_{2k}(\psi_\pi,\mathbb{Z}(\alpha),\gamma\circ\tau)=\chi_p(\lambda)\cdot \mathrm{G}_{2k}(\chi_p,\tau)$$

where γ was the linear fractional transformation

$$\gamma=\begin{pmatrix}\lambda\alpha+1 & \alpha\\ \lambda & 1\end{pmatrix}$$

where λ is the quantity

$$\begin{aligned}\lambda \equiv & \big(\bar{\alpha}-\alpha\big)^{-1} \pmod p \\ \equiv & (-\sqrt{\Delta_k})^{-1} \pmod \pi\end{aligned}$$

and the α's/λ's appearing within the entries of γ are integers defined using the modulus π of ψ. By choosing the weight k to be 2 and integrating both sides of the relationship on the Eisenstein series, this gives

$$\begin{aligned}\int \mathrm{G}_2(\psi_\pi, \mathbb{Z}(\alpha), \gamma \circ \tau)\frac{d\tau}{(\lambda\tau+1)^2} =& \int \mathrm{G}_2(\psi_\pi, \mathbb{Z}(\alpha), z)dz \\ =& \mathrm{G}_0(\psi_\pi, \mathbb{Z}(\alpha), z) + \mathrm{C} \\ =& \chi_p(\lambda)\int \mathrm{G}_2(\chi_p, \tau)d\tau \\ =& \chi_p(\lambda)\mathrm{G}_0(\chi_p, \tau) + \mathrm{C}\end{aligned}$$

where in the first line, the substitutions

$$\begin{aligned}z =& \gamma \circ \tau \\ \frac{dz}{d\tau} =& \frac{1}{(\lambda\tau+1)^2}\end{aligned}$$

This allows the weight 0 integrals to be related as

$$\begin{aligned}\mathrm{G}_0(\psi_\pi, \mathbb{Z}(\alpha), \tau) =& \bar{\chi}_p(\lambda)\mathrm{G}_0(\chi_p, \gamma^{-1} \circ \tau) \\ =& -\frac{4\pi\imath}{p}\sum_{p\geq l\geq 1} \bar{\chi}_p(\lambda \cdot l)\ln\Big(\eta\big(\frac{\gamma^{-1}\circ\tau + l}{p}\big)\Big) + \mathrm{C}\end{aligned}$$

This works in general for a character χ. When χ is even, the constant of integration C is to be set to zero. If it is odd, then the constant should be set to zero but the η quotient in the logarithm should be modified. This gives a way to evaluate the η quotient in terms of powers of units in some ring of integers and vice-versa.

3.3 Other units

In the cyclotomic fields, there is an advantage of using the Galois Group and the **Cyclotomic units**

$$\frac{1-\zeta^l}{1-\zeta}$$

to conjure the units in the intermediate abelian extensions of said cyclotomic fields. But what about the generators of the unit group

$$\mathcal{O}_K^{\times}$$

where K is a pure cubic field? Not only are there no roots of unity present in K, but the automorphism group

$$\mathrm{Gal}(K/\mathbb{Q})$$

is the trivial group. A method like done for real quadratic fields can be used.

3.3.1 Pure Cubic Fields

Let K be the pure cubic field

$$K = \mathbb{Q}(\sqrt[3]{p})$$

where p is a prime of the form $3k+1$. The method of $9k+1$ will be worked out first, and afterwards the method of $3k+1$ not congruent to 1 modulo 9 will be given.
Let the Dedekind Zeta function of K be defined as follows

$$\begin{aligned}\zeta_K(s) =& \prod_{\mathfrak{p}} \big(1 - \frac{1}{\mid \mathfrak{p} \mid^s}\big)^{-1} \\ =& \zeta(s) \cdot L(\rho_K, s)\end{aligned}$$

where the product runs over the prime ideals $\mathfrak{p}$ of the ring of integers of K, $\zeta(s)$ is the classical Zeta function and $L(\rho_K, s)$ is defined as

$$L(\rho_K, s) = \prod_{\pi} \big(1 - \frac{\bar{\psi}_\beta(\pi) \cdot \psi_\beta(\bar{\pi})}{(\pi \cdot \bar{\pi})^s}\big)^{-1}$$

where the product runs over the **Primary primes** π of $\mathbb{Z}(\frac{-1+\sqrt{-3}}{2})$ and ψ is the cubic character of a prime modulus β with field norm p. Until further notice, ω will be used in place of $\frac{-1+\sqrt{-3}}{2}$. A prime in $\mathbb{Z}(\omega)$ is primary if it is of the form $3k+2$, not a unit multiple of $1-\omega$, or if it is of the form

$$\pi = m + 3n \cdot \omega$$

It can be shown every prime π and its conjugate $\bar{\pi}$ can be written as $m + 3n\omega$ for integers m, n, and any product of primary primes is of the form $m + 3n\omega$.

This is because of the explicit representation the cubic gauss sum of a prime modulus p takes when cubed in characteristic 3:

$$\begin{aligned}\tau^3(\chi_p) =& p \cdot \pi \\ =& pm + pn\omega \\ \equiv& \sum_{p \geq l \geq 1} \chi_p^3(l) \cdot \zeta_p^{3l} \pmod 3 \\ \equiv& \sum_{p \geq l \geq 1} \zeta_p^{3l} \pmod 3 \\ \equiv& -1 \pmod 3\end{aligned}$$

which was used for cubic reciprocity. It is straightforward to check that If p is of the form $9k+1$, then any unit multiple of π works within the character $\bar{\psi}_\beta(\pi)\psi_\beta(\bar{\pi})$ and there is no need to talk about primary primes. By the **Class Number Formula** , the value $L(\rho_K, 1)$ can be given by

$$\begin{aligned}\mathrm{Res}_{s=1}\zeta_K(s) =& \frac{4\pi \mathrm{h}_K \ln(\epsilon_K)}{2\sqrt{\mid \Delta_K \mid}} \\ =& L(\rho_K, 1)\end{aligned}$$

where h_K is the class number of K, ϵ_K is the smallest generator of $\mathcal{O}_K^\times$ that is greater than 1 in the real embedding of K, aka the fundamental unit, and Δ_K is the discriminant of K. This can be simplified as

$$\begin{aligned}L(\rho_K, 1) =& \frac{2\pi \mathrm{h}_K \ln(\epsilon_K)}{p\sqrt{27}} && \text{if } p = 9k+4, 9k+7 \\ L(\rho_K, 1) =& \frac{2\pi \mathrm{h}_K \ln(\epsilon_K)}{p\sqrt{3}} && \text{if } p = 9k+1\end{aligned}$$

which can be rewritten as the sum

$$\begin{aligned}2L(\rho_K, 1) =& \sum_{\alpha \in \mathbb{Z}(3\omega)\setminus 0} \frac{\bar{\psi}_\beta(\alpha) \cdot \psi_\beta(\bar{\alpha})}{\alpha \cdot \bar{\alpha}} && \text{if } p = 9k+4, 9k+7 \\ 6L(\rho_K, 1) =& \sum_{\alpha \in \mathbb{Z}(\omega)\setminus 0} \frac{\bar{\psi}_\beta(\alpha) \cdot \psi_\beta(\bar{\alpha})}{\alpha \cdot \bar{\alpha}} && \text{if } p = 9k+1\end{aligned}$$

where the presence of the scalars 6 and 2 is attributed to the number of units in $\mathbb{Z}(\omega)$ and $\mathbb{Z}(3\omega)$, respectively. Like done with real quadratic fields, the unit will be found by taking advantage of the Eisenstein Series

$$\mathrm{G}_0(\psi, \mathbb{Z}(\omega), \tau) = -\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{\bar{\psi}_\beta(m+n\omega)\cdot\psi_\beta(m+n\bar{\omega})}{n\cdot(m+n\tau)}$$

in the $9k+1$, and something else in the other $3k+1$ cases. For the $9k+1$ primes, the fundamental unit ϵ_K can be written as

$$\epsilon^{3\mathrm{h}_K} = \prod_{p\geq l\geq 1} \eta^{a(l)}\Big(\frac{\gamma^{-1}\circ\omega + l}{p}\Big)$$

where the exponents $a(l)$ is

$$\begin{aligned} a(l) =& -\operatorname{Tr}\big(\chi_p(\lambda\cdot l)\big) \\ =& -\big(\chi_p(\lambda\cdot l) + \bar{\chi}_p(\lambda\cdot l)\big) \end{aligned}$$

and γ is the linear transformation with entries λ and ω defined by

$$\begin{aligned} \omega^2 =& -\omega - 1 \pmod p \\ \lambda =& (\bar{\omega}-\omega)^{-1} \pmod p \end{aligned}$$

For the other $3k+1$ cases, it is not as straightforward. It is necessary to take λ to be divisble by 3 so that the following matrix, which is not the γ relating the Eisenstein Series, satisfies

$$\begin{pmatrix} \frac{\lambda}{3} & \lambda\omega + 1 \\ 1 & 3\omega \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

Once the criteria on λ has been satisfied, it is then possible to express the fundamental unit in K for all the other prime $3k+1$ cases as

$$\epsilon_K^h = \prod_{p\geq l\geq 1} \eta^{b(l)}\Big(\frac{3\cdot(\gamma^{-1}\circ\omega) + l}{p}\Big)$$

where the coefficients $b(l)$ are defined as

$$\begin{aligned} b(l) =& -\operatorname{Tr}(\chi_p(l)) \\ =& -(\chi_p(l) + \bar{\chi}_p(l)) \end{aligned}$$

Unlike with the $9k+1$ primes, the presence of λ within the characters of $b(l)$ is omitted for all the other $3k+1$ primes. The usage of 3ω rather than ω forces $b(l)$ to be

$$\begin{aligned} b(l) &= -\operatorname{Tr}\big(\chi_p(3)\cdot\bar{\chi}_p(\lambda\cdot l)\big) \\ &= -\operatorname{Tr}(\bar{\chi}_p(\frac{\lambda}{3}\cdot l)) \\ &= -\operatorname{Tr}(\chi_p(-\lambda^3\cdot l)) \end{aligned}$$

Since χ_p is the non-trivial cubic character and λ^3 is a perfect, non-zero cube in $\mathbb{Z}/p\mathbb{Z}$, the factor of 3λ can be disregarded completely. Also, since every prime p of the form $3k+1$ is also of the form $2n+1$, the quantity $(-1)^{\frac{p-1}{3}}$ is always an even power of -1; hence the factor of -1 can also be disregarded.

3.3.2 Pure Quartic Fields

Consider the field

$$K = \mathbb{Q}(\sqrt[4]{p})$$

for p a positive prime of the form $4k+1$. Like with the cubic cases, it will be necessary to split them into $8k+1$ primes and $8k+5$ primes. The Dedekind Zeta function of K is of the form

$$\begin{aligned} \zeta_K(s) &= \prod_{\mathfrak{p}}\Big(1-\frac{1}{|\,\mathfrak{p}\,|^s}\Big)^{-1} \\ &= \zeta(s)\cdot L(\chi_p,s)\cdot L(\rho_K,s) \end{aligned}$$

where $L(\rho_K,s)$ is the L-function defined by the Euler Product

$$L(\rho_K,s) = \prod_{\pi}\big(1-\frac{\bar{\psi}_\beta(\pi)\cdot\psi_\beta(\bar{\pi})}{(\pi\cdot\bar{\pi})^s}\big)^{-1}$$

where the product runs over all of the primary primes π in the ring $\mathbb{Z}(\imath)$, β is a Gaussian prime with norm p, and the character ψ_β is the quartic, multiplicative character on $\mathcal{O}/(\beta)$. If p is of the form $8k+5$, then the product excludes the ramified prime $1+\imath$. In the $8k+1$ case, the ramified prime is included. Let the group $\mathcal{O}^\times$ be generated by the units

$$\mathcal{O}^\times = <\epsilon_k, \epsilon_K>$$

where ϵ_k is the fundamental unit in the integer ring of the intermediate quadratic field

$$k = \mathbb{Q}(\sqrt{p})$$

and ϵ_K is a unit in the integers of $\mathcal{O}_K$ such that the unit group is indeed spanned by the two chosen generators. It is true that any quantity $\epsilon^{'}$ of the form

$$\epsilon_K^{'} = \epsilon_k^n \cdot \epsilon_K$$

will make do in place of ϵ_K. This will not matter, since ϵ_k is invariant under either choice of real embedding of K into $\mathbb{R}$. Denote the quantity ϵ_K^* as

$$\epsilon_K^* = \frac{\epsilon_{K,+}}{\epsilon_{K,-}}$$

where $\epsilon_{K,\pm}$ is ϵ_K in one of the two embeddings of K into $\mathbb{R}$ such that ϵ^* is positive and greater than 1. Regardless of choice of $\epsilon_K^{'}$, this particular quotient is fixed! Alternatively, it can be expressed as the quotient

$$\epsilon_K^* = \exp\left(\frac{\mathrm{Reg}_K}{\mathrm{Reg}_k}\right)$$

where Reg_E is the **Regulator** of the field E.

Using the **Class Number Formula**, this invariant quantity can be expressed as

$$(\epsilon_K^*)^{\mathrm{h}_{K/k}} = \zeta_m^n \cdot \prod_{p \geq l \geq 1} \eta^{c(l)}\Big(\frac{2(\gamma^{-1} \circ \imath) + l}{p}\Big) \qquad \text{if } p \equiv 5 \pmod 8$$

$$(\epsilon_K^*)^{2\mathrm{h}_{K/k}} = \prod_{p \geq l \geq 1} \eta^{d(l)}\Big(\frac{\gamma^{-1} \circ \imath + l}{p}\Big) \qquad \text{if } p \equiv 1 \pmod 8$$

where the ζ_m^n is some root of unity, the coefficients $c(l)$, $d(l)$ are given by

$$c(l) = -\operatorname{Tr}(\chi_p(8\lambda \cdot l)$$
$$d(l) = -\operatorname{Tr}(\chi_p(\lambda \cdot l)$$

and the numerical value $\mathrm{h}_{K/k}$ is the quotient

$$\mathrm{h}_{K/k} = \frac{\mathrm{h}_K}{\mathrm{h}_k}$$

where again, h_E is the class number of a number field E.

3.3.3 Pure Sextic Fields

Let K be the field

$$K = \mathbb{Q}(\sqrt[6]{(-1)^{\frac{p-1}{2}} p})$$

where p is a prime of the form $3k+1$. If p is of the form $12k+7$ or $12k+1$, then K has exacly 3 pairs of complex embeddings or 2 real embeddings and 2 pairs of complex embeddings, respectively. Let the unit group of K be defined by the generators

$$\mathcal{O}^\times = < \epsilon_{k_2}, \epsilon_K > \qquad \text{if } p \equiv 7 \pmod{12}$$
$$\mathcal{O}^\times = < \epsilon_{k_1}, \epsilon_{k_2}, \epsilon_K > \qquad \text{if } p \equiv 1 \pmod{12}$$

where the units ϵ_{k_1} and ϵ_{k_2} generate the unit groups of the intermediate quadratic and cubic unit groups respectively. For p of the form $12k+7$, the intermediate quadratic extension is imaginary and has a unit group of rank 0; hence there is no k_1 for the corresponding $\mathcal{O}^\times$.

The regulators of these fields were computed to be

$$\begin{aligned} \mathrm{Reg}_K =& \mathrm{Reg}_{k_2} \cdot \mathrm{Reg}_{K/k_2} && \text{if } p \equiv 7 \pmod{12} \\ \mathrm{Reg}_K =& \mathrm{Reg}_{k_1} \cdot \mathrm{Reg}_{k_2} \cdot \mathrm{Reg}_{K/k_1 \cdot k_2} && \text{if } p \equiv 1 \pmod{12} \end{aligned}$$

where the quantities Reg_{K/k_2} and $\mathrm{Reg}_{K/k_1 \cdot k_2}$ are of the form $\ln(\epsilon_K^*)$. Similar to the quartic case, the quantity ϵ_K^* is a positive quantity greater than 1 which involves a quotient of the non-trivial generator ϵ_K in the different embeddings of K into $\mathbb{C}$. This can be given as

$$\begin{aligned} \epsilon_K^* =& \frac{\epsilon_K^{+\rho} \cdot \epsilon_K^{+\bar{\rho}}}{\epsilon_K^{-\rho} \cdot \epsilon_K^{-\bar{\rho}}} && \text{if } p \equiv 7 \pmod{12} \\ \epsilon_K^* =& \Big(\frac{\epsilon_K^{-}}{\epsilon_K^{+}}\Big)^3 \cdot \Big(\frac{\epsilon_K^{+} \cdot \epsilon_K^{+\omega} \cdot \epsilon_K^{+\bar{\omega}}}{\epsilon_K^{-} \cdot \epsilon_K^{-\omega} \cdot \epsilon_K^{-\bar{\omega}}}\Big) && \text{if } p \equiv 1 \pmod{12} \end{aligned}$$

where the quantities $\pm\omega$, $\pm\rho$, and their conjugate counterparts are given as

$$\begin{aligned} \pm\omega &= \pm \frac{-1 + \sqrt{-3}}{2} \\ \rho &= \mp \imath \cdot \omega \end{aligned}$$

and the quantities ϵ_K^σ is the generator ϵ_K embedded into $\mathbb{C}$ using the σ embedding. Each ϵ_K^* can be expressed in terms of η quotients as well;

$$\begin{aligned} \big(\epsilon_K^*\big)^{3\mathrm{h}_K^*} =& c \prod_{p \geq l \geq 1} \eta^{a(l)} \Big(\frac{\gamma^{-1} \circ \omega + l}{p}\Big) && \text{if } p \equiv 1 \pmod{9} \\ \big(\epsilon_K^*\big)^{\mathrm{h}_K^*} =& c \prod_{p \geq l \geq 1} \eta^{b(l)} \Big(\frac{3(\gamma^{-1} \circ \omega) + l}{p}\Big) && \text{if } p \equiv 4, 7 \pmod{9} \end{aligned}$$

where the coefficients $a(l)$ and $b(l)$ are given by

$$\begin{aligned} a(l) &= -\,\mathrm{Tr}\big(\bar{\chi}_p(\lambda \cdot l)\big) \\ b(l) &= -\,\mathrm{Tr}\big(\bar{\chi}_p(243\lambda \cdot l)\big) \end{aligned}$$

for χ_p the sextic character of modulus p, the constant c is 1 if p is of the form $4k + 1$ and equal to some root of unity ζ_m^n if p is of the form $4k + 3$, and the quantity h_K^* is the quotient

$$\mathrm{h}_K^* = \frac{\mathrm{h}_K}{\mathrm{h}_{k_1} \cdot \mathrm{h}_{k_2}}$$

where h_F is the Class number of an algebraic number field F.

3.4 Computations of η quotients

Here is a list of example η quotients organized by type of field/character. The regulators for the respective quadratic fields were computed using Mathematica.

3.4.1 Quadratic Characters

$$\frac{\eta^4\left(\frac{13+\imath}{50}\right)\cdot\eta^4\left(\frac{23+\imath}{50}\right)}{\eta^4\left(\frac{3+\imath}{50}\right)\cdot\eta^4\left(\frac{33+\imath}{50}\right)}=\left(\frac{1+\sqrt{5}}{2}\right)^8$$

$$\prod_{13\geq l\geq 1}\eta^{-2\chi_{13}(-4\cdot l)}\left(\frac{-99+26l+\imath}{338}\right)=\left(\frac{3+\sqrt{13}}{2}\right)^4$$

$$\prod_{13\geq l\geq 1}\eta^{-2\chi_{13}(-2\cdot l)}\left(\frac{-45+26l+\sqrt{-3}}{338}\right)=\left(\frac{3+\sqrt{13}}{2}\right)^{12}$$

$$\left|\prod_{7\geq l\geq 1}\eta^{-2\chi_7(l)}\left(\frac{-61+266l+\sqrt{-3}}{1862}\right)\right|=\left(\frac{5+\sqrt{21}}{2}\right)^3$$

$$\left|\frac{\eta^6\left(\frac{8+\sqrt{-2}}{18}\right)}{\eta^6\left(\frac{2+\sqrt{-2}}{18}\right)}\right|=5+2\sqrt{6}$$

$$\left|\frac{\eta^4\left(\frac{5+\sqrt{-7}}{64}\right)}{\eta^4\left(\frac{37+\sqrt{-7}}{64}\right)}\right|=8-3\sqrt{7}$$

$$\left|\frac{\eta^6\left(\frac{7+\sqrt{-11}}{18}\right)}{\eta^6\left(\frac{1+\sqrt{-11}}{18}\right)}\right|=23+4\sqrt{33}$$

$$\frac{\eta^2\left(\frac{-7+\sqrt{-11}}{50}\right)\cdot\eta^2\left(\frac{23+\sqrt{-11}}{50}\right)}{\eta^2\left(\frac{3+\sqrt{-11}}{50}\right)\cdot\eta^2\left(\frac{13+\sqrt{-11}}{50}\right)}=\left(\frac{1+\sqrt{5}}{2}\right)^4$$

3.4.2 Cubic Character

Here, χ_p is the non-trivial cubic character, or its conjugate and ω is the third root of unity $\dfrac{-1+\sqrt{-3}}{2}$. The matrix γ is rather tedious to calculate and so only the first three $3k+1$ primes were used.

$$\prod_{7\geq l\geq 1} \eta^{-\mathrm{Tr}\left(\chi_7(l)\right)}\left(\frac{57+49l+3\omega}{343}\right) = \left(4+2\cdot\sqrt[3]{7}+\sqrt[3]{7^2}\right)^3$$

$$\prod_{13\geq l\geq 1} \eta^{-\mathrm{Tr}\left(\chi_{13}(l)\right)}\left(\frac{576+2821l+3\omega}{36673}\right) = \left(94+40\cdot\sqrt[3]{13}+17\cdot\sqrt[3]{13^2}\right)^3$$

$$\prod_{19\geq l\geq 1} \eta^{-\mathrm{Tr}\left(\chi_{19}(-5\cdot l)\right)}\left(\frac{293+1501l+\omega}{28519}\right) = \left(\frac{14+5\cdot\sqrt[3]{19}+2\cdot\sqrt[3]{19^2}}{3}\right)^3$$

3.5 Quartic Characters

$$\left|\frac{\eta^2\left(\frac{231+2\imath}{325}\right)}{\eta^2\left(\frac{166+2\imath}{325}\right)}\right| = 6+3\sqrt{5}+2\sqrt{20+9\sqrt{5}}$$

$$\prod_{13\geq l\geq 1} \eta^{-\mathrm{Tr}\left(\chi_{13}(7\cdot l)\right)}\left(\frac{198+377l+2\imath}{4901}\right) = 3862+1071\sqrt{13}+6\sqrt{828516+229789\sqrt{13}}$$

$$\prod_{17\geq l\geq 1} \eta^{-\mathrm{Tr}\left(\chi_{17}(2\cdot l)\right)}\left(\frac{-38+85l+\imath}{1445}\right) = \left(33+8\sqrt{17}+4\sqrt{136+33\sqrt{17}}\right)^4$$

3.5.1 Sextic Characters

The products remain the same as the cubic case, except for when p is also a $4k+3$ prime, the absolute value of the product must be taken to account for the n^{th} root of unity factor. Technically, one could multiply each $\eta(w)$ factor in the quotient by $\eta(-\bar{w})$ and square the expected unit. This will be relevant later on. Also, there is a need to replace the cubic character with the sextic character.

$$\Big|\prod_{7\geq l\geq 1} \eta^{-\mathrm{Tr}\left(\chi_7(6\cdot l)\right)}\left(w_l\right)\Big| =$$

$$\Big(\frac{527+300\cdot\sqrt[3]{7}+150\cdot\sqrt[3]{7^2}}{2}\Big)+\sqrt[6]{189}\cdot\Big(\frac{230+125\cdot\sqrt[3]{7}+60\sqrt[3]{7^2}}{2}\Big)$$

$$\prod_{13\geq l\geq 1} \eta^{-\mathrm{Tr}\left(\chi_{13}(8\cdot l)\right)}\left(w_l\right) =$$

$$36973+15840\cdot\sqrt[3]{13}+6714\cdot\sqrt[3]{13^2}+6\sqrt[6]{13}\cdot\left(4033+1722\cdot\sqrt[3]{13}+727\cdot\sqrt[3]{13^2}\right)$$

$$\Big|\prod_{19\geq l\geq 1} \eta^{-\mathrm{Tr}\left(\chi_{19}(-5\cdot l)\right)}\left(w_l\right)\Big| =$$

$$\Big(\Big(\frac{79+22\cdot\sqrt[3]{19}+10\cdot\sqrt[3]{19^2}}{3}\Big)+4\sqrt[6]{513}\cdot\Big(\frac{6+2\cdot\sqrt[3]{19}+\sqrt[3]{19^2}}{3}\Big)\Big)^3$$

where the sequence w_l is the same as the entries in the l^{th} η factor found for the cubic characters. The only difference from the cubic case to the sextic case is the factor c appearing within $-\mathrm{Tr}\big(\chi_p(c\cdot l)\big)$ of the exponent and the cubic character being substituted with the sextic character.

4 Closing remarks

In the not so distant future, it is expected to apply these methods using weight 1 Eisenstein Series to find the closed form of L-functions of particular elliptic curves at $s = 1$. Also, there are conjectures as the behavior of the η-quotient affiliated to the unit $\left(\epsilon_K^*\right)^{\mathrm{h}_K^*}$ when the coefficients in the exponent are changed from

$$-\mathrm{Tr}\big(\chi_p(c \cdot l)\big)$$

to

$$-\mathrm{Tr}\big(\chi_p(n \cdot c \cdot l)\big)$$

for some n relatively prime to p. The conjectured behavior involves which embedding the unit sits in after the transformation of the exponential coefficient, up to some root of unity in the situation where the character χ is odd. Under this conjectures as well as more knowledge of the L-functions of characters of the form $\bar{\chi}(\pi) \cdot \chi(\bar{\pi})$, it would allow one to rewrite particular η values in a neat fashion.

Any feedback or criticism is welcome, as well as pointing out any mistakes. My email is listed at the top of this submission for the diposal of the reader.

References

[1] Wikipedia contributors. "Class number formula." Wikipedia, The Free Encyclopedia. 10 Jan. 2017.

[2] Weisstein, Eric W. "Heegner Number." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/HeegnerNumber.html