Dynamics of the gravity field

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Abstract

We study the dynamics of the gravity field, on arbitrary spacetime x^μ, according to the quantum fields theory. Therefore, we suggest a canonical momentum π_I as a conjugate momentum for the canonical gravity field $\tilde{e}^I = ee_\mu^I n^\mu$. We derive both the canonical gravity field and its conjugate momentum from the holonomy $U(\gamma, A)$ of the complex selfdual connection A_a^i. The canonical momentum π_I is represented in the Lorentz group. We use it in deriving the path integral of the gravity field according to the quantum fields theory. We discuss the situation of the free gravity field (like the electromagnetic field). We find that this situation takes place in the background spacetime approximation, the situation of low matter density(weak gravity). We search for a theory, in which, the gravity field becomes a dynamical at any energy on arbitrary curved spacetime x^μ. For that purpose, we suggest a duality $e^I \leftrightarrow \Sigma^{JK}$, where the field $\Sigma^{IJ} = e^I \wedge e^J$ is the Area field. That duality allows us to treat both fields e^I and Σ^{IJ} as a dynamical on arbitrary curved spacetime. We find that the gravity field changes to the area field ($e^I \rightarrow \Sigma^{JK}$) in the spacelike region, while the area field changes to the gravity field ($\Sigma^{JK} \rightarrow e^I$) in the timelike region. We find that the tensor product of them, in selfdual representation, satisfies the reality condition. Finally, we derive the static potential of exchanging gravitons in scalar and spinor fields, the Newtonian gravitational potential.

key words: Conjugate momentum, path integral, free gravity propagator, gravity-area duality, area field.

1 The canonical conjugate momentum π^I and the path integral

We search for conditions to have a dynamical gravity field. The problem of the dynamics in the general relativity is that spacetime is itself a dynamical. It interacts with matter, it is an operator $d\hat{x}^\mu$. Therefore, we have to consider it as a quantum field like the other fields. But if spacetime is itself a dynamic, where do the fields exist? This problem is solved by considering that fields exist on each other, not on spacetime[1]. Both the dynamical curved spacetime x^μ and the gravity field e^I have the same entity, it is the gravity. Thus, we study only one of them as a gravity field: $\hat{e}^I$. We will see that it is substantially different in the background spacetime, the gravity field becomes like the usual quantum fields.

As usual in the quantum fields theory, we have to find the canonical conjugate momentum $\pi^I(x)$(represented in the Lorentz group). We find that it acts canonically on the local-Lorentz vectors $V^I(x)$ on a closed 3D surface δM immersed in arbitrary curved spacetime x^μ of a manifold M. The closed surface δM is parameterized by three parameters X^1, X^2 and X^3. In a certain gauge, we consider that they carry the spatial indices of the local-Lorentz frame $X^I : X^0, X^1, X^2, X^3$. This local-Lorentz frame is tangent-space on the curved spacetime x^μ. We find that the path integral of the gravity field is independent on this gauge.

Therefore, the exterior derivative operator, on the surface δM, leads to a change along the norm of that surface, so it causes the change in the time dX^0 direction. That allows the 3D surface δM to extend and have the 4D local-Lorentz frame X^I, which, in our gauge, parameterize the four dimensions x^μ coordinates of the curved spacetime in the manifold M. With that the gravity field propagates from one surface to another by the extension of those surfaces.

To study the gravity field propagation, we suggest canonical states $\left|\tilde{e}^I\right\rangle$ and $\left|\pi^I\right\rangle$ represented in Lorentz group. We use them in deriving the path integral. We find that there is no propagation on the dynamical spacetime x^μ. But in the background spacetime, we find that the gravity field propagates freely like the electromagnetic fields.

The holonomy of the complex connection A^i in the quantum gravity is[1, 2]

$$U(\gamma, A) = TrPe^{i\oint_\gamma A}, \tag{1.1}$$

where the path ordered P is defined in

$$Pe^{i\oint_\gamma A} = \sum_{n=0}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{n-1}} ds_n iA\left(\gamma\left(s_n\right)\right) ... iA\left(\gamma\left(s_1\right)\right) : \dot{\gamma}^\mu\left(s\right) = \frac{dx^\mu}{ds},$$

where $\gamma(s)$ is a closed path in the curved spacetime x^μ. In irreducible selfdual representation of the Lorentz group, we write $A = A^i\tau^i$, where τ^i are Pauli matrices. The element $U(\gamma, A)$ is invariant under local-Lorentz transformation $V^I \rightarrow L^I_J(x)V^J$ and under arbitrary changing of the spacetime coordinates $dx^\mu \rightarrow \Lambda^\mu{}_\nu(x)dx^\nu$, therefore the quantum gravity is studied using it[1].

The complex connection A^i is selfdual of the local-Lorentz spin connection $\omega(x)$[1]:

$$A^i_\mu\left(x\right) = \left(P^i\right)_{IJ}\omega^{IJ}_\mu\left(x\right),$$

where P^i are the selfdual projectors. We can write the holonomy $U(\gamma, A)$ using the real spin connection $\omega_\mu^{IJ} dx^\mu$ of the local-Lorentz frame, we get

$$U(\gamma, \omega) = TrPe^{i\oint_\gamma \omega^I{}_J}.$$

We expect that it has the same properties of $U(\gamma, A)$; satisfies the symmetries of GR.

For the free gravity field, we impose the relation:

$$(\omega_\mu)^{IJ} = \pi_K{}^{IJ} e_\mu^K,$$

where the conjugate momentum $\pi_K{}^{IJ}(x)$ is represented in the Lorentz group and acts on its vectors. Thus, we consider it as a dynamical operator. By inserting it in the holonomy $U(\gamma, \omega)$, we get

$$U(\gamma, \pi, e) = TrP \exp i \oint_\gamma \left(\pi_K{}^I{}_J\right) e_\mu^K dx^\mu.$$

In the free gravity field, we expect that the momentum π^{IJK} is antisymmetry. Thus, we can write it as

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK}.$$

This is our starting point in studying the dynamics of the quantum gravity. By that, the holonomy becomes

$$U(\gamma, \pi, e) = TrP \exp i \oint_\gamma \left(\pi^{KI}{}_J\right) e_{K\mu} dx^\mu = TrP \exp i \oint_\gamma \left(\varepsilon^{LKI}{}_J\right) \pi_L e_{K\mu} dx^\mu.$$

We write it as

$$\sum_{n=0}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{n-1}} ds_n \left(i\varepsilon^{LKI}{}_J \pi_L e_{K\mu} \dot{\gamma}^\mu\right)(s_n) \left(i\varepsilon^{L_1 K_1 J}{}_{J_1} \pi_{L_1} e_{K_1 \mu_1} \dot{\gamma}^{\mu_1}\right)(s_{n-1}) \\ ... \left(i\varepsilon^{L_{n-1} K_{n-1} J_{n-2}}{}_I \pi_{L_{n-1}} e_{K_{n-1}\mu_{n-1}} \dot{\gamma}^{\mu_{n-1}}\right)(s_1),$$

where $\left(i\varepsilon^{LKI}{}_J \pi_L e_{K\mu} \dot{\gamma}^\mu\right)(s_n) = i\varepsilon^{LKI}{}_J \pi_L(s_n) e_{K\mu}(s_n) \dot{\gamma}^\mu(s_n)$, with the tangent $\dot{\gamma}^\mu(s) = \frac{dx^\mu}{ds}$ on the closed path $\gamma(s)$ in the manifold M.

By using the properties

$$\varepsilon_{IJKL}\varepsilon^{IJK_1L_1} = -2\left(\delta_K^{K_1}\delta_L^{L_1} - \delta_K^{L_1}\delta_L^{K_1}\right) \; and \; \varepsilon_{IJKL}\varepsilon^{I_1JKL} = -6\delta_I^{I_1},$$

the integrals of the holonomy $U(\gamma, \pi, e)$ become over terms like

$$...\pi_I(s_j)e_\mu^I(s_i)\dot{\gamma}^\mu(s_i)ds_i...\pi_J(s_i)e_\nu^J(s_k)\dot{\gamma}^\nu(s_k)ds_k... \; with \; i \neq j \; and \; i \neq k.$$

This holonomy satisfies the general relativity symmetries; invariance under local Lorentz transformation $V^I \to L^I{}_J(x)V^J$ and under arbitrary changing of the coordinates $dx^\mu \to \Lambda^\mu{}_\nu(x)dx^\nu$. Therefore, we use it in quantum gravity.

Let us suggest another term: $\oint_\gamma \pi_K e_\mu^K dx^\mu$. We expect that it satisfies the general relativity symmetries if it is integrated over a closed 3D surface δM instead of the closed path $\gamma(s)$. This is because

$$ed^4x = \frac{1}{4}d^3x_\mu \wedge dx^\mu = \frac{1}{4}e\varepsilon_{\mu\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\mu/3!$$

is invariant element, thus we can replace $\pi_K e_\mu^K dx^\mu$ with

$$\pi_K e^{K\mu}d^3x_\mu = \pi_K e^{K\mu}e\varepsilon_{\mu\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma/3!.$$

With integrating it over a three dimensions closed surface δM, it becomes invariant under GR transformations because in the free gravity there are no sources for the gravity field. As a sequence of that, the flux of the Lorentz vectors is invariant under arbitrary changing of the closed surface δM.
The determinant e of the gravity field e_μ^I is defined in $e = \sqrt{-g}$ with writing the metric $g_{\mu\nu}(x)$ on the curved spacetime x^μ as

$$g_{\mu\nu}(x) = \eta_{IJ}e_\mu^I e_\nu^J.$$

In arbitrary transformations, we have invariant element:

$$\sqrt{g}\varepsilon_{i_1...i_n} = \sqrt{g'}\varepsilon'_{i_1...i_n}.$$

Therefore,

$$e\varepsilon_{\mu\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma/3! = d^3x_\mu$$

is a co-vector, as ∂_μ. By that, the integral

$$U(\delta M, \pi, e) = \exp i \oint_{\delta M} \pi_I e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3! = \exp i \oint_{\delta M} \pi_I e^{I\mu} d^3 x_\mu$$

satisfies the same conditions of the holonomy $U(\gamma, A)$; invariant under local Lorentz transformation $V^I \to L^I_J(x) V^J$ and under arbitrary changing of the coordinates $dx^\mu \to \Lambda^\mu{}_\nu(x) dx^\nu$. That relates to the fact that the integrals of free vector fields over a closed surface δM, in a manifold M, are invariant if there are no sources for those fields. It is the conservation. The spin connection ω^μ and so $\pi_K e^{K\mu}$, as vectors, satisfy this fact in the free gravity.

The equation of motion of the gravity field e^I is

$$De^I = de^I + \omega^I{}_J \wedge e^J = 0.$$

With our imposing $(\omega_\mu)^{IJ} = \pi_K{}^{IJ} e^K_\mu$, it becomes

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J.$$

But the tensor

$$e^N \wedge e^J = e^N_\mu e^J_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}\left(e^N_\mu e^J_\nu - e^N_\nu e^J_\mu\right) dx^\mu \wedge dx^\nu$$

measures the area in the manifold M. Therefore, the changes of the gravity field around a closed path (rotation) relate to the flux of the momentum π through the area determined by that closed path. It is like the magnetic field generated by straight electric current. Therefore, we have

$$e^N \wedge e^J \to Area,$$

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J \to flux\ throw\ this\ Area.$$

For that reason, we suggested that the conjugate momentum π^{IJK} is antisymmetry.

Now, in the integral

$$\exp i \oint_{\delta M} \pi_I e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!,$$

we define the canonical gravity field $\tilde{e}^I$ as

$$\tilde{e}^I d^3X = \tilde{e}^I dX^1 dX^2 dX^3 \equiv e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!,$$

or

$$\tilde{e}^I = e e^I_\mu n^\mu (X^i),$$

where $n^\mu(X^i)$ is the norm to the surface δM. By that, the holonomy $U(\delta M, \pi, e)$ becomes

$$\tilde{U}_{\delta M}(\delta M, \pi, \tilde{e}) = \exp i \oint_{\delta M} \pi_I \tilde{e}^I d^3X,$$

where the parameters $X^I : I = 1, 2, 3$ parameterize the closed 3D surface δM in the manifold M. As mentioned before, in certain gauge, we consider that the indices $I = 1, 2, 3$ are the spatial indices of the local-Lorentz frame($I = 0, 1, 2, 3$). Therefore, the exterior derivative, on the surface δM, is along the time dX^0. The time dX^0 is the direction of the norm on the surface $\delta M(X^1, X^2, X^3)$. We will see that the result of the path integral is independent on this gauge.

As we suggested before, the integral $\exp i \oint_{\delta M} \pi_I \tilde{e}^I d^3X$ satisfies the same conditions of the holonomy $U(\gamma, A)$; invariant under local-Lorentz transformation and under arbitrary changing of the coordinates, thus we consider it as a canonical dynamical element.

Comparing it with

$$\langle \phi \mid \pi \rangle = \exp i \int d^3X \phi(X) \pi(X) / \hbar,$$

a canonical relation in the scalar field ϕ theory on flat spacetime. For $\hbar = 1$, we suggest canonical states $\left|\tilde{e}^I\right\rangle$ and $\left|\pi^I\right\rangle$ with

$$\left\langle \tilde{e}^I \mid \pi_I \right\rangle_{\delta M} = \exp i \int_{\delta M} \tilde{e}^I(X) \pi_I(X) d^3X,$$

where π_I is the canonical conjugate momentum of $\tilde{e}^I$. We can write this relation on the surface δM as

$$\left\langle \tilde{e}^I \mid \pi_I \right\rangle_{\delta M} = \prod_{n,I} \left\langle \tilde{e}^I \left(x_n + dx_n\right) \mid \pi_I \left(x_n\right) \right\rangle_{\delta M},$$

with

$$\left\langle \tilde{e}^I\left(x_n+dx_n\right) \mid \pi_I\left(x_n\right)\right\rangle_{\delta M}=\exp i\tilde{e}^I(x_n+dx_n)\pi_I(x_n)d^3X\to\exp i\tilde{e}^I(x_n)\pi_I(x_n)d^3X.$$

In general, for two points in adjacent surfaces δM_1 and δM_2, let us rewrite it as

$$\left\langle \tilde{e}^I\left(x_n+dx_n\right) \mid \pi_I\left(x_n\right)\right\rangle=\exp i\tilde{e}^I(x_n+dx_n)\pi_I(x_n)d^3X. \tag{1.2}$$

Here the variation

$$\tilde{e}^I(x_n+dx_n)-\tilde{e}^I(x_n)$$

is exterior derivative along the time dX^0 direction, the direction of the norm on the surface δM_1. It leads to the propagation. That allows the extension of the surface: $\delta M(X^1,X^2,X^3)\to M(X^0,X^1,X^2,X^3)$.

We need to make $\hat{e}d^4\hat{x}$ commutes with $\hat{\tilde{e}}^I d^3X$. For that we write

$$\begin{aligned}-\hat{e}d^4\hat{x}&=\hat{e}d\hat{x}^\mu\wedge\varepsilon_{\mu\nu\rho\sigma}d\hat{x}^\nu\wedge d\hat{x}^\rho\wedge d\hat{x}^\sigma/4!\\&=\hat{e}d\hat{x}^\mu\wedge\frac{\varepsilon_{\mu\nu\rho\sigma}}{4!}\frac{\partial\hat{x}^\nu}{\partial X^i}\frac{\partial\hat{x}^\rho}{\partial X^j}\frac{\partial\hat{x}^\sigma}{\partial X^k}\frac{\varepsilon^{ijk}}{3!}d^3X=\frac{1}{4}\hat{e}d\hat{x}^\mu\hat{n}_\mu d^3X.\end{aligned}$$

The indexes *i, j and k*, in our gauge, are the local-Lorentz indices for $I=1,2,3$. As we assumed before; $X^I: I=1,2,3$ parameterize the closed surface δM in the manifold M.

We can rewrite it(in our gauge) as:

$$-ed^4x=\frac{1}{4}edx^\mu n_\mu d^3X=\frac{1}{4}e\frac{\partial x^\mu}{\partial X^0}n_\mu d^3XdX^0=\frac{1}{4}ee_0^\mu n_\mu d^3XdX^0.$$

Comparing it with the term

$$\tilde{e}^Id^3X=e^{I\mu}e\varepsilon_{\mu\nu\rho\sigma}dx^\nu\wedge dx^\rho\wedge dx^\sigma/3!=ee^{I\mu}n_\mu d^3X,$$

we find that it commutes with it:

$$\left[\hat{e}\hat{e}^{I\mu}\hat{n}_\mu d^3X,\hat{e}\hat{e}_0^\mu\hat{n}_\mu d^3XdX^0\right]=0\to\left[\hat{\tilde{e}}^Id^3X,\hat{e}d^4\hat{x}\right]=0,$$

where $\left[\hat{e}_\mu^I,\hat{e}_\nu^J\right]=0$. Thus, the operator $\hat{e}d^4\hat{x}$ takes eigenvalues when it acts on the states $\left|\tilde{e}^I\right\rangle$.

The action of the gravity field is[1]

$$S(e,\omega)=\frac{1}{16\pi G}\int\varepsilon_{IJKL}\left(e^I\wedge e^J\wedge R^{KL}(\omega)+\lambda e^I\wedge e^J\wedge e^K\wedge e^L\right).$$

We consider only the first term:

$$S(e,\omega)=c\int\varepsilon_{IJKL}e^I\wedge e^J\wedge R^{KL}(\omega),$$

where C is constant. The Riemann curvature is

$$R^{KL}(\omega)=d\omega^{KL}+\omega^K{}_M\wedge\omega^{ML}.$$

By using the relation we imposed before:

$$(\omega)^{IJ}=\pi_K{}^{IJ}e^K,$$

the action becomes

$$S(e,\pi)=c\int\left[\varepsilon_{IJKL}e^I\wedge e^J\wedge d\left(\pi_M{}^{KL}e^M\right)+\varepsilon_{IJKL}e^I\wedge e^J\wedge\left(\pi_{K_1}{}^K{}_M\right)e^{K_1}\wedge\left(\pi_{K_2}{}^{ML}\right)e^{K_2}\right],$$

or

$$S(e,\pi)=c\int\left[\varepsilon_{IJKL}e^I\wedge e^J\wedge d\left(\pi_M{}^{KL}e^M\right)+\varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)e^I\wedge e^J\wedge e^{K_1}\wedge e^{K_2}\right]. \tag{1.3}$$

We get the term $d\left(\pi_M{}^{KL}e^M\right)$ from

$$\varepsilon_{IJKL}d\left(e^I\wedge e^J\wedge\pi_M{}^{KL}e^M\right),$$

by assuming that its integral is zero at the infinities. We have

$$\begin{aligned}\varepsilon_{IJKL}d\left(e^I\wedge e^J\wedge\pi_M{}^{KL}e^M\right)=&\varepsilon_{IJKL}\left(de^I\right)\wedge e^J\wedge\pi_M{}^{KL}e^M-\varepsilon_{IJKL}e^I\wedge\left(de^J\right)\wedge\pi_M{}^{KL}e^M\\&+\varepsilon_{IJKL}e^I\wedge e^J\wedge d\left(\pi_M{}^{KL}e^M\right).\end{aligned}$$

By the rearrangement:

$$\begin{aligned}-\varepsilon_{IJKL}e^I\wedge\left(de^J\right)\wedge\left(\pi_M{}^{KL}e^M\right)&=-\varepsilon_{IJKL}\left(de^J\right)\wedge e^I\wedge\left(\pi_M{}^{KL}e^M\right)\\&=\varepsilon_{JIKL}\left(de^J\right)\wedge e^I\wedge\pi_M{}^{KL}e^M,\end{aligned}$$

it becomes

$$\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \pi_M{}^{KL} e^M\right) = 2\varepsilon_{IJKL}\left(de^I\right)\wedge e^J \wedge \pi_M{}^{KL} e^M + \varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right).$$

Therefore, we can rewrite the action as

$$S(e,\pi) = c\int \left[-2\varepsilon_{IJKL}\left(de^I\right)\wedge e^J \wedge \left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}\right].$$

From the equation of motion of the gravity field:

$$0 = De^I = de^I + \omega^I{}_J \wedge e^J = de^I + \pi_N{}^I{}_J e^N \wedge e^J,$$

we get

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J.$$

Inserting it in the last action, it becomes

$$\begin{aligned} S(e,\pi) = c\int & 2\varepsilon_{IJKL}(\pi_N{}^I{}_B) e^N \wedge e^B \wedge e^J \wedge \left(\pi_M{}^{KL} e^M\right) \\ & + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}, \end{aligned}$$

or

$$\begin{aligned} S(e,\pi) = c\int & 2\varepsilon_{IJKL}\left(\pi_N{}^I{}_B\right)\left(\pi_M{}^{KL}\right) e^N \wedge e^B \wedge e^J \wedge e^M \\ & + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}. \end{aligned}$$

By rewriting it like

$$\begin{aligned} S(e,\pi) = c\int & 2\varepsilon_{IJKL}\left(\pi_N{}^I{}_B\right)\left(\pi_M{}^{KL}\right) e^B \wedge e^J \wedge e^N \wedge e^M \\ & + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}, \end{aligned}$$

and replacing $B \rightleftarrows I, N \to K_1$, and $M \to K_2$ in the first term, we get

$$\begin{aligned} S(e,\pi) = c\int & 2\varepsilon_{BJKL}\left(\pi_{K_1}{}^B{}_I\right)\left(\pi_{K_2}{}^{KL}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2} \\ & + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}. \end{aligned}$$

And by the replacing

$$e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2} \to \varepsilon^{IJK_1K_2} e^0 \wedge e^1 \wedge e^2 \wedge e^3,$$

we get

$$S(e,\pi) = c \int \left[2\varepsilon_{BJKL} \left(\pi_{K_1}{}^{B}{}_{I}\right) \left(\pi_{K_2}{}^{KL}\right) \varepsilon^{IJK_1K_2} + \varepsilon_{IJKL} \left(\pi_{K_1}{}^{K}{}_{M}\right) \left(\pi_{K_2}{}^{ML}\right) \varepsilon^{IJK_1K_2}\right] \times e^0 \wedge e^1 \wedge e^2 \wedge e^3.$$

By using the relation $\pi^{IJL} = \pi_K \varepsilon^{KIJL}$ we imposed before, the action

$$S(e,\pi) = c \int \left[2\varepsilon^{B}{}_{JKL} \left(\pi_{K_1BI}\right) \left(\pi_{K_2}{}^{KL}\right) \varepsilon^{IJK_1K_2} - 2\left(\pi_{K}{}^{K}{}_{M}\right) \left(\pi_{L}{}^{ML}\right) + 2\left(\pi_{L}{}^{K}{}_{M}\right) \left(\pi_{K}{}^{ML}\right)\right] \times e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

becomes

$$S(e,\pi) = c \int \left[2\varepsilon^{B}{}_{JKL} \pi^N \varepsilon_{NK_1BI} \left(\pi_{K_2}{}^{KL}\right) \varepsilon^{IJK_1K_2} + 2\left(\pi_{LKM}\right) \left(\pi^{KML}\right)\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3.$$

By using $\varepsilon_{NK_1BI} = -\varepsilon_{IK_1BN} = \varepsilon_{IK_1NB}$, $\varepsilon_{ILKM} = -\varepsilon_{ILMK}$ and $\varepsilon^{JKML} = -\varepsilon^{JLMK}$, that action becomes

$$S(e,\pi) = c \int \left[2\varepsilon^{B}{}_{JKL} \pi^N \varepsilon_{IK_1NB} \left(\pi_{K_2}{}^{KL}\right) \left(-\varepsilon^{IK_1JK_2}\right) + 2\pi^I \varepsilon_{ILMK} \pi_J \varepsilon^{JLMK}\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3,$$

and by setting

$$\varepsilon_{IK_1NB} \varepsilon^{IK_1JK_2} = -2\left(\delta^J_N \delta^{K_2}_B - \delta^J_B \delta^{K_2}_N\right) \text{ and } \varepsilon_{ILMK} \varepsilon^{JLMK} = -6\delta^J_I,$$

it becomes

$$S(e,\pi) = c \int \left[4\varepsilon^{K_2}{}_{JKL} \pi^J \left(\pi_{K_2}{}^{KL}\right) - 12\pi_I \pi^I\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3.$$

Or

$$S(e,\pi) = c \int \left[4\varepsilon_{K_2JKL} \pi^J \left(\pi^{K_2KL}\right) - 12\pi^2\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3,$$

then

$$S(e,\pi) = c \int \left[-4\varepsilon_{JK_2KL} \pi^J \pi_I \varepsilon^{IK_2KL} - 12\pi^2\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3.$$

Finally, the action becomes

$$S_0(e,\pi) = c\int \left[24\pi^2 - 12\pi^2\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3 = c\int 12\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3$$
$$= c\int 12\pi^2 e d^4x.$$

In the background spacetime, we have $e \to 1 + \delta e$, therefore this action becomes

$$S_0(\delta e, \pi) \to \int 12c\pi^2 d^4x + ...$$

To find its meaning, we compare it with the scalar field Lagrange in the flat spacetime:

$$Ld^4x = (\pi\partial_0\phi - H(\phi,\pi))\, d^4x \text{ with } H(\phi,\pi)d^4x = \left(\frac{1}{2}\pi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2\right) d^4x.$$

For $\hbar = 1$, we conclude that the term

$$\int 12c\pi^2 d^4x \succ 0$$

is the energy of the gravity field in the background spacetime. As we will find in the result of the path integral, in the background spacetime limit, we have to replace c with $-c$ when we compare our results with the electromagnetic field. Thus, in the background spacetime, we expect the replacement:

$$S(e,\pi) \to -\int 12c\pi^2 d^4x = -\int Hd^4x.$$

This is not surprise, because the general relativity equation (Einstein field equation) is derived to satisfy the energy-momentum conservation on arbitrary curved spacetime, that equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

It satisfies the energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$. But, as we know, in the quantum field theory in the background spacetime limit, we have to write the canonical law of the conservation as

$$\partial_\mu \left(T^{\mu\nu}_{matter} + T^{\mu\nu}_{gravity}\right) = 0,$$

therefore we write

$$T_{\mu\nu} + \frac{-1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = T_{\mu\nu}\,(matter) + T_{\mu\nu}\,(gravity) = constant.$$

Thus, we conclude the relation:

$$T_{\mu\nu}\,(gravity) = -\frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right).$$

Therefore, we have to replace c with $-c$. We will see this, when we compare the result of the gravity path integral with the electromagnetic field on the background spacetime.

Now, we derive the path integral as usual in the quantum fields theory. As we saw before, in our gauge, the operator $\hat{e}d^4\hat{x}$ takes eigenvalues when it acts on the states $\left|\tilde{e}^I\right\rangle$. By using (1.2), we get the amplitude

$$\left\langle \tilde{e}^I\,(x+dx)\right| e^{iS}\left|\pi_I\,(x)\right\rangle \to \left\langle \tilde{e}^I\,(x+dx)\right| e^{i12c\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\pi_I\,(x)\right\rangle$$

$$= \exp\left(i12c\pi^2\,(x)\,e\,(x+dx)\,d^4x + i\tilde{e}^I\,(x+dx)\,\pi_I(x)d^3X\right)$$

$$\to \exp\left(i12c\pi^2\,(x)\,e\,(x)\,d^4x + i\tilde{e}^I\,(x+dx)\,\pi_I(x)d^3X\right),$$

where we let the momentum π_I acts on the left. The amplitude of the propagation between two points x and $x+dx$ of adjacent surfaces δM_1 and δM_2 is

$$\left\langle \tilde{e}_I\,(x+dx)\right| e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\tilde{e}^I\,(x)\right\rangle_{\delta M_1\to\delta M_2}$$

$$= \int\prod_I d\pi^I \left\langle \tilde{e}_I\,(x+dx)\right| e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\pi^I\,(x)\right\rangle_{\delta M_1\to\delta M_2}\left\langle \pi_I\,(x)\;\middle|\;\tilde{e}^I\,(x)\right\rangle_{\delta M_1}$$

$$= \int\prod_I d\pi^I \exp\left[i12c\pi^2\,(x)\,e\,(x+dx)\,d^4x + i\tilde{e}^I(x+dx)\pi_I(x)d^3X\right]\exp\left(-i\tilde{e}^I(x)\pi_I(x)d^3X\right)$$

$$\to \int\prod_I d\pi^I \exp\left[i12c\pi^2\,(x)\,e\,(x)\,d^4x + i\left(\tilde{e}^I(x+dx) - \tilde{e}^I(x)\right)\pi_I(x)d^3X\right].$$

The exterior derivative

$$\left(\tilde{e}^I(x+dx)-\tilde{e}^I(x)\right)d^3X=\frac{\partial\tilde{e}^I(x)}{\partial X^0}d^3XdX^0=d\tilde{e}^I(x)d^3X$$

is along the time dX^0 direction, the direction of the norm on the surface $\delta M(X^1,X^2,X^3)$, therefore it leads to the propagation from one surface to another.

Thus, we write the amplitude as

$$\left\langle\tilde{e}_I\left(x+dx\right)\right|e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\tilde{e}^I\left(x\right)\right\rangle_{\delta M_1\to\delta M_2}=\int\prod_I d\pi^I\exp\left[i12c\pi^2\left(x\right)e\left(x\right)d^4x+i\pi_I(x)d\tilde{e}^I(x)d^3X\right].$$

The path integral is the integral of ordered product of those amplitudes on all spacetime points(over all ordered 3D surfaces), thus we write it as

$$\begin{aligned}W_{ST}&=\int\prod_I D\tilde{e}^I D\pi_I\exp i\int\left(12c\pi^2ed^4x+\pi_Id\tilde{e}^Id^3X\right)\\&=\int\prod_I D\tilde{e}^I D\pi_I\exp i\int\left(12c\pi^2e^0\wedge e^1\wedge e^2\wedge e^3+\pi_Id\tilde{e}^Id^3X\right).\end{aligned}$$

In selfdual representation, we consider that the propagation is in the direction of expanding of the surface δM(positive direction).

There is no problem with Lorentz non-invariance in $\frac{\partial\tilde{e}^I(x)}{\partial X^0}d^3XdX^0$, because the equation of motion we find in the result of the path integral is

$$\frac{\partial\tilde{e}^I(x)}{\partial X^0}\propto-\pi^I,$$

thus we have

$$\frac{\partial\tilde{e}^I(x)}{\partial X^0}\pi_Id^3XdX^0\propto-\pi_I\pi^Id^3XdX^0.$$

This is Lorentz invariant. This is like the equation of motion of the scalar field ϕ; $\pi=\partial_0\phi$ which solves the same problem.

In our gauge, we have

$$\begin{aligned}\pi_I\pi^Id^3XdX^0\to\pi^2dX^0\wedge dX^1\wedge dX^2\wedge dX^3&=\pi^2e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma\\&=\pi^2e^0_\mu e^1_\nu e^2_\rho e^3_\sigma\varepsilon^{\mu\nu\rho\sigma}d^4x=\pi^2ed^4x.\end{aligned}$$

It is invariant element, we find it in the path integral.

The path integral

$$W_{ST} = \int \prod_I D\tilde{e}^I D\pi_I \exp i \int \left(12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I d\tilde{e}^I d^3X\right)$$

vanishes unless

$$\frac{\delta}{\delta\pi_I}\left(12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I d\tilde{e}^I d^3X\right) = 24c\pi^I e^0 \wedge e^1 \wedge e^2 \wedge e^3 + d\tilde{e}^I d^3X = 0.$$

Thus we get the path(equation of motion):

$$\hat{\pi}^I = \frac{-1}{24c}\left(\hat{e}^0 \wedge \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3\right)^{-1} d\hat{\tilde{e}}^I d^3X, \tag{1.4}$$

or

$$\pi^I \pi^J = \frac{1}{(24c)^2}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-2} d\tilde{e}^I d^3X d\tilde{e}^J d^3X. \tag{1.5}$$

By using it we get

$$\begin{aligned} 12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I d\tilde{e}^I d^3X &= \frac{1}{48c}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right) \\ &- \frac{1}{24c}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right). \end{aligned}$$

By setting it in the path integral, it becomes

$$W_{ST} = \int \prod_I D\tilde{e}^I Exp \frac{-i}{48c} \int \left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right).$$

The canonical field $\tilde{e}^I$ is defined in

$$\tilde{e}^K d^3X = e^{K\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!.$$

By applying the exterior derivative, we get

$$\left(d\hat{\tilde{e}}^K\right) d^3X = \left(\hat{D}_{\mu_1} \hat{e}^{K\mu}\right) \hat{e} \varepsilon_{\mu\nu\rho\sigma} d\hat{x}^{\mu_1} \wedge d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma / 3!,$$

where D is the co-variant derivative defined in

$$DV^I = dV^I + \omega^I{}_J \wedge V^J.$$

Thus, the term

$$\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1} \left(d\tilde{e}_I d^3 X\right) \left(d\tilde{e}^I d^3 X\right) = \frac{\left(d\tilde{e}_I d^3 X\right) \left(d\tilde{e}^I d^3 X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3},$$

in the path integral, becomes

$$\frac{\left(\hat{D}_{\mu_1} \hat{e}^{\mu}_I\right) \hat{e}\varepsilon_{\mu\nu\rho\sigma} d\hat{x}^{\mu_1} \wedge d\hat{x}^{\nu} \wedge d\hat{x}^{\rho} \wedge d\hat{x}^{\sigma} \left(\hat{D}_{\mu_2} \hat{e}^{I\mu'}\right) \hat{e}\varepsilon_{\mu'\nu'\rho'\sigma'} d\hat{x}^{\mu_2} \wedge d\hat{x}^{\nu'} \wedge d\hat{x}^{\rho'} \wedge d\hat{x}^{\sigma'}}{3!3! \hat{e}^0_{\mu_3} \hat{e}^1_{\nu_3} \hat{e}^2_{\rho_3} \hat{e}^3_{\sigma_3} d\hat{x}^{\mu_3} \wedge d\hat{x}^{\nu_3} \wedge d\hat{x}^{\rho_3} \wedge d\hat{x}^{\sigma_3}}.$$

Let us define the inverse:

$$\left(e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\right)^{-1} = E^{\mu'}_0 E^{\nu'}_1 E^{\rho'}_2 E^{\sigma'}_3 \frac{\partial}{\partial x^{\sigma'}} \wedge \frac{\partial}{\partial x^{\rho'}} \wedge \frac{\partial}{\partial x^{\nu'}} \wedge \frac{\partial}{\partial x^{\mu'}}.$$

We write it in the form

$$e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \frac{1}{4} e d^3 x_\mu \wedge dx^\mu.$$

(Actually, we have to write the tensors $\varepsilon^{\mu\nu\rho\sigma}$ and $\varepsilon_{\mu\nu\rho\sigma}$ like $e^{-1}\varepsilon^{\mu\nu\rho\sigma}$ and $e\varepsilon_{\mu\nu\rho\sigma}$ but here we neglect that, because it gives the same results).

Also, we can write

$$E^{\mu'}_0 E^{\nu'}_1 E^{\rho'}_2 E^{\sigma'}_3 \partial_{\sigma'} \wedge \partial_{\rho'} \wedge \partial_{\nu'} \wedge \partial_{\mu'} = E \partial_\nu \wedge \partial^{3\nu},$$

with inner product like

$$\left(E\partial_\nu \wedge \partial^{3\nu}\right) \left(\frac{1}{4} e d^3 x_\mu \wedge dx^\mu\right) = \frac{1}{4} E e \partial_\nu \wedge \partial^{3\nu} d^3 x_\mu \wedge dx^\mu = \frac{1}{4} E e \left(\delta^\nu_\mu\right) \partial_\nu dx^\mu = Ee = 1.$$

In general, we can write it like

$$\left(E\partial_\nu \wedge \partial^{3\nu}\right) \left(e d^3 x_{\mu'} \wedge dx^\mu\right) = E e \partial_\nu \wedge \partial^{3\nu} d^3 x_{\mu'} \wedge dx^\mu = E e \delta^\nu_{\mu'} \partial_\nu dx^\mu = \delta^\mu_{\mu'}.$$

In the path integral, we set the replacement:

$$\left(D_{\mu_1} e^\mu_I\right) e\varepsilon_{\mu\nu\rho\sigma} dx^{\mu_1} \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3! \to \left(D_{\mu_1} e^\mu_I\right) e dx^{\mu_1} \wedge d^3 x_\mu = -\left(D_{\mu_1} e^\mu_I\right) e d^3 x_\mu \wedge dx^{\mu_1},$$

and

$$\left(D_{\mu_2}e^{I\mu'}\right)e\varepsilon_{\mu'\nu'\rho'\sigma'}dx^{\mu_2}\wedge dx^{\nu'}\wedge dx^{\rho'}\wedge dx^{\sigma'}/3!\rightarrow-\left(D_{\mu_2}e^{I\mu'}\right)ed^3x_{\mu'}\wedge dx^{\mu_2}.$$

Let us assume the following replacing:

$$d^3x_\mu\wedge dx^\mu=-dx_\mu\wedge d^3x^\mu\rightarrow d^3x_\mu\wedge dx^{\mu_1}=-dx_\mu\wedge d^3x^{\mu_1}.$$

There is no problem with this trick because in any 4D spacetime we have the contraction $(d^3x_\mu\wedge dx^\nu)=\delta^\nu_\mu d^4x$.
Therefore, we set the replacement:

$$-\left(D_{\mu_1}e^\mu_I\right)ed^3x_\mu\wedge dx^{\mu_1}\rightarrow\left(D_{\mu_1}e^\mu_I\right)edx_\mu\wedge d^3x^{\mu_1}.$$

By that, the term

$$\frac{\left(\hat{D}_{\mu_1}\hat{e}^\mu_I\right)\hat{e}\varepsilon_{\mu\nu\rho\sigma}d\hat{x}^{\mu_1}\wedge d\hat{x}^\nu\wedge d\hat{x}^\rho\wedge d\hat{x}^\sigma\left(\hat{D}_{\mu_2}\hat{e}^{I\mu'}\right)\hat{e}\varepsilon_{\mu'\nu'\rho'\sigma'}d\hat{x}^{\mu_2}\wedge d\hat{x}^{\nu'}\wedge d\hat{x}^{\rho'}\wedge d\hat{x}^{\sigma'}}{3!3!\hat{e}^0_{\mu_3}\hat{e}^1_{\nu_3}\hat{e}^2_{\rho_3}\hat{e}^3_{\sigma_3}d\hat{x}^{\mu_3}\wedge d\hat{x}^{\nu_3}\wedge d\hat{x}^{\rho_3}\wedge d\hat{x}^{\sigma_3}},$$

in the path integral, becomes

$$-\left(E\partial_\nu\wedge\partial^{3\nu}\right)\left(\left(D_{\mu_1}e^\mu_I\right)edx_\mu\wedge d^3x^{\mu_1}\right)\left(\left(D_{\mu_2}e^{I\mu'}\right)ed^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

$$=\left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right)e\left(\partial_\nu\wedge\partial^{3\nu}\right)\left(d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right),$$

where we used

$$-dx_\mu\wedge d^3x^{\mu_1}=d^3x^{\mu_1}\wedge dx_\mu\ then\ d^3x_{\mu_1}\wedge dx^\mu.$$

Thus we can write

$$\frac{\left(d\tilde{e}_Id^3X\right)\left(d\tilde{e}^Id^3X\right)}{e^0\wedge e^1\wedge e^2\wedge e^3}\rightarrow\left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right)e\left(\partial_\nu\wedge\partial^{3\nu}\right)\left(d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right).$$

We can choose the contraction:

$$\left(\partial_\nu\wedge\partial^{3\nu}\right)\left(d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)=\left(\partial_\nu\wedge\partial^{3\nu}d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

$$=\delta^\nu_{\mu_1}\left(\partial_\nu\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)=\delta^\nu_{\mu_1}\left(-dx^\mu\wedge\partial_\nu\right)\left(-dx^{\mu_2}\wedge d^3x_{\mu'}\right)$$

$$=\delta^\nu_{\mu_1}dx^\mu\wedge\partial_\nu dx^{\mu_2}\wedge d^3x_{\mu'}=\delta^\nu_{\mu_1}\delta^{\mu_2}_\nu dx^\mu\wedge d^3x_{\mu'}.$$

Thus we can write the term in the path integral as

$$\frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0\wedge e^1\wedge e^2\wedge e^3}\to\left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right)e\delta^{\nu}_{\mu_1}\delta^{\mu_2}_{\nu}dx^{\mu}\wedge d^3x_{\mu'}$$

$$=\left(D_{\nu}e_{I\mu}\right)\left(D^{\nu}e^{I\mu'}\right)edx^{\mu}\wedge d^3x_{\mu'}=-\left(D_{\nu}e_{I\mu}\right)\left(D^{\nu}e^{I\mu'}\right)ed^3x_{\mu'}\wedge dx^{\mu}$$

$$=-\left(D_{\nu}e_{I\mu}\right)\left(D^{\nu}e^{I\mu'}\right)e\delta^{\mu}_{\mu'}d^4x=-\left(D_{\nu}e_{I\mu}\right)\left(D^{\nu}e^{I\mu}\right)ed^4x.$$

We can also choose another contraction:

$$\left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right)e\left(\partial_{\nu}\wedge\partial^{3\nu}\right)\left(d^3x_{\mu_1}\wedge dx^{\mu}\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)\to$$

$$\left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right)e\left(\partial_{\nu}\wedge\partial^{3\nu}d^3x_{\mu_1}\wedge dx^{\mu}\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

$$=\left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right)e\left(\delta^{\nu}_{\mu_1}\partial_{\nu}\wedge dx^{\mu}\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

$$=\delta^{\nu}_{\mu_1}\delta^{\mu}_{\nu}\left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right)e\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right).$$

Thus, we get

$$\frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0\wedge e^1\wedge e^2\wedge e^3}\to\left(D^{\mu}e_{I\mu}\right)\left(D_{\mu'}e^{I\mu'}\right)ed^4x.$$

By the two possible contractions, we can write the final result as

$$\left(e^0\wedge e^1\wedge e^2\wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)=\frac{-1}{2}\left(D_{\mu}e^{\nu}_{I}D^{\mu}e^{I}_{\nu}-D_{\mu}e^{\nu}_{I}D_{\nu}e^{I\mu}\right)ed^4x.$$

This Lagrange is like the Lagrange of electromagnetic field, but with opposite sign. It is also independent on the gauge we chose for the surface δM. It is invariant under local Lorentz transformation $V^I\to L^I{}_J(x)V^J$ and under any coordinate transformation $V^{\mu}\to\frac{\partial x^{\mu}}{\partial x'^{\nu}}V'^{\nu}$.

The path integral of the gravity field becomes, after the replacement $c\to -c$:

$$W_{ST}=\int\prod_I De^I\exp\frac{i}{48c}\frac{1}{2}\left(-D_{\mu}e^{\nu}_{I}D^{\mu}e^{I}_{\nu}+D_{\mu}e^{\nu}_{I}D_{\nu}e^{I\mu}\right)ed^4x,$$

with the free gravity field Lagrange

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x. \tag{1.6}$$

We determine the constant c in the Newtonian gravitational potential $c \succ 0$.

In the background spacetime, weak gravity; $D_\mu \to \partial_\mu$ and $e \to 1 + \delta e$, we have

$$L \to \frac{1}{48c}\frac{1}{2}\left(-\partial_\mu e_I^\nu \partial^\mu e_\nu^I + \partial_\mu e_I^\nu \partial_\nu e^{I\mu}\right),$$

or

$$L_0 = \frac{1}{48c}\frac{1}{2}\eta_{IJ} e_\mu^I \left(g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu\right) e_\nu^J.$$

Without background spacetime approximation, in strong gravity field, we have a problem with the determinant e in the path integral:

$$W_{ST} = \int \prod_I De^I exp\frac{i}{48c}\int \frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) e_{\mu_1}^0 e_{\nu_1}^1 e_\rho^2 e_\sigma^3 \varepsilon^{\mu_1\nu_1\rho\sigma} d^4x,$$

with $\eta_{0123} = -1$, we rewrite

$$W_{ST} = \int \prod_I De^I exp\frac{i}{48c}\int \frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right)\left(-\eta_{I_1JKL}\right) e_{\mu_1}^{I_1} e_{\nu_1}^J e_\rho^K e_\sigma^L \varepsilon^{\mu_1\nu_1\rho\sigma} d^4x/4!.$$

Always there is a gravity field e_ρ^K which is different from e_μ^I and e_ν^I, thus the integral over it yields delta Dirac:

$$\int \prod_I De^I exp\frac{i}{48c}\int \frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right)\left(-\eta_{I_1JKL}\right) e_{\mu_1}^{I_1} e_{\nu_1}^J e_\rho^K e_\sigma^L \varepsilon^{\mu_1\nu_1\rho\sigma} d^4x/4!,$$

$$\to \delta\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right),$$

$$\to -D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu} = 0,$$

it yields

$$\pi^2 = 0 \;\to\; S(\pi, e) = c\int 12\pi^2 ed^4x = 0 \;\to\; H(\pi, e) = 0.$$

This path integral is trivial, there is no propagation, because there is no gravity energy $H(\pi, e) = 0$. Like Wheeler-DeWitt equation $\hat{H}\psi = 0$. The reason of that is because the gravity field e^I_μ has the entity of spacetime. It is impossible for spacetime to be a dynamical on itself, to propagate over itself.

But if we write $e^I_\mu(x) \to \delta^I_\mu + h^I_\mu(x)$, the path integral exists. The propagation is possible. Thus, the dynamics of the gravity is takes place only on the background spacetime. This is the situation of the weak gravity (low energy densities). In this situation, the gravity field becomes dynamical as the other fields.

Latter, we will search for conditions allow the gravity field to propagate over curved spacetime x^μ, for that purpose we impose the duality; Gravity-Area.

The path integral of weak gravity field in the background spacetime is

$$w = \int \prod_I De^I \exp i \int \frac{1}{48c}\frac{1}{2} e^I_\mu \left(\eta_{IJ} g^{\mu\nu}\partial^2 - \eta_{IJ}\partial^\mu\partial^\nu\right) e^J_\nu d^4x. \tag{1.7}$$

Thus, the gravity field propagator, $g = \eta$ and $k_\mu e^{\mu I} = 0$, is

$$\Delta^{\mu\nu}_{IJ}(x_2 - x_1) = 48c \int \frac{d^4k}{(2\pi)^4} \frac{\eta_{IJ} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon},$$

or

$$\Delta^{\mu\nu}_{\rho\sigma}(x_2 - x_1) = 48c \int \frac{d^4k}{(2\pi)^4} \frac{g_{\rho\sigma} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon}. \tag{1.8}$$

We will use this propagation in the gravity interaction with the scalar and spinor fields.

2 The need for the duality Gravity-Area

We search for conditions to have a dynamical gravity field in arbitrary curved spacetime without spacetime background approximation. We found that the curved spacetime path integral W_{ST} is trivial. There is no propagation without spacetime background. We can solve this problem by assuming that the fields exist on themselves, not on spacetime[1].

As we saw in the path integral of the gravity field on curved spacetime, we have a problem in the gravity fields $e^0 \wedge e^1 \wedge e^2 \wedge e^3$. All of them must be different, the integral over one of them yields delta Dirac. This is trivial path integral W_{ST}. Therefore, there must be a new field, it is the area field $\Sigma^{KJ} = e^K \wedge e^J$, by that, the path integral of the gravity field takes place. This means that the gravity field becomes a dynamical on the area field, not on spacetime.

According to the general relativity, the length, the area and the volume are another form of the gravity. We can illustrate that, by the duality *gravity* $\leftrightarrow$ *areas* and volumes. We try to find this duality using the trivial path integral W_{ST}, by suggesting conditions allow the gravity field(the dynamical spacetime) to propagate on arbitrary spacetime. That propagation is $e^I \leftrightarrow \Sigma^{JK}$, it means that they propagate when they change to each other. Also we find that the tensor product of them $\left|e^I\right\rangle \otimes \left|\Sigma^{JK}\right\rangle$, in selfdual representation, satisfies the reality condition.

Starting from the full Lagrange (1.6):

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^{I\mu}\right) ed^4x,$$

where the covariant derivative D is defined in

$$De^I = de^I + \omega^I{}_J \wedge e^J.$$

By using our assuming:

$$\omega^{IJ} = \pi_K{}^{IJ} e^K,$$

the covariant derivative becomes

$$De^I = de^I + \left(\pi_K{}^I{}_J\right) e^K \wedge e^J.$$

The Area field is anti-symmetry field:

$$\Sigma^{IJ}_{\mu\nu} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right).$$

By inserting it in the covariant derivative, it becomes

$$De^I = de^I + \left(\pi_K{}^I{}_J\right)\Sigma^{KJ} = de^I + \pi^{KIJ}\Sigma_{KJ}.$$

And by using our assumption

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK},$$

the derivative becomes

$$De^I = de^I + \pi^{KIJ}\Sigma_{KJ} = de^I + \pi_L \varepsilon^{LKIJ}\Sigma_{KJ} = de^I + \varepsilon^{ILKJ}\pi_L \Sigma_{KJ}.$$

By that, we have two fields e^I and Σ^{KJ} in the Lagrange. They are inseparable, therefore we suggest a duality $e^I \leftrightarrow \Sigma^{KJ}$.

The full Lagrange of the gravity field is

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x,$$

where

$$-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu} = -D_\mu e_I^\nu \left(D^\mu e_\nu^I - D_\nu e^{I\mu}\right).$$

It becomes

$$-\left(\partial_\mu e_I^\nu + \varepsilon_{IJKL}\pi^J \Sigma_\mu^{KL\nu}\right)\left(\partial^\mu e_\nu^I + \varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma^\mu_{K_1L_1\nu} - \partial_\nu e^{I\mu} - \varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma_{K_1L_1\nu}{}^\mu\right),$$

or

$$-\left(\partial_\mu e_I^\nu + \varepsilon_{IJKL}\pi^J \Sigma_\mu^{KL\nu}\right)\left(\partial^\mu e_\nu^I - \partial_\nu e^{I\mu} + 2\varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma^\mu_{K_1L_1\nu}\right).$$

We write it as

$$\begin{aligned}&-\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I - \partial_\nu e^{I\mu}\right) - 2\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e_I^\nu\right)\pi_{J_1}\Sigma^\mu_{K_1L_1\nu}\\ &\quad - 2\varepsilon_{IJKL}\Sigma_\mu^{KL\nu}\pi^J\left(\frac{\partial^\mu e_\nu^I - \partial_\nu e^{I\mu}}{2}\right) - 2\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\Sigma_\mu^{KL\nu}\pi^J\pi_{J_1}\Sigma^\mu_{K_1L_1\nu}.\end{aligned}$$

To complete it, we need to replace the momentum π^I with its value. We had before (1.4) and (1.5):

$$\pi^I\pi^J = \frac{-1}{(48c)^2}\frac{1}{2}\left(D_\mu e^{I\nu}D^\mu e_\nu^J - D_\mu e^{I\nu}D_\nu e^{J\mu}\right).$$

We consider only the term:

$$\pi^I \pi^J = \frac{-1}{(48c)^2}\frac{1}{2}\left(\partial_\mu e^{I\nu}\partial^\mu e^J_\nu\right).$$

Let us expect the contraction:

$$2\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\pi_{J_1}\Sigma^\mu_{K_1L_1\nu} \rightarrow \frac{-1}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu}.$$

By inserting it in the Lagrange terms, we get

$$-\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu - \partial_\nu e^{I\mu}\right) + \frac{2}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu}$$

$$+\frac{1}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\Sigma^{KL\nu}_\mu\left(\partial_\sigma e^J_\rho\right)\left(\partial^\sigma e^\rho_{J_1}\right)\Sigma^\mu_{K_1L_1\nu}.$$

Thus, the Lagrange

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^{I\mu}\right)ed^4x,$$

becomes

$$Ld^4x \rightarrow \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu\right)ed^4x + \frac{1}{(48c)^3}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu}ed^4x$$
$$+\frac{1}{2*(48c)^3}\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\left(\partial_\sigma e^J_\rho\right)\left(\partial^\sigma e^\rho_{J_1}\right)\Sigma^{KL}_{\mu\nu}\Sigma^{\mu\nu}_{K_1L_1}ed^4x,$$

where we used the gauge $\partial_\mu e^{I\mu} = 0$.

Now, we use the selfdual projection. We can write any real anti-symmetry tensor T^{IJ} in two unmixed representations, selfdual and anti-selfdual. In general relativity the selfdual is chosen. Its projector is[1]

$$\left(P^i\right)_{jk} = \frac{1}{2}\varepsilon^i{}_{jk}\ ,\ \left(P^i\right)_{0j} = \frac{i}{2}\delta^i_j : i = I \text{ for } I = 1,2,3.$$

We see that these projectors satisfy

$$2i\left(P^i\right)^{IJ}\left(P_i\right)^{KL} - 2i\left(\bar{P}^i\right)^{IJ}\left(\bar{P}_i\right)^{KL} \rightarrow \varepsilon^{IJKL}.$$

It is a projection from $I \neq J$ and $K \neq L$ in the left to $I \neq J \neq K \neq L$ in the right.

The second term is for the anti-selfdual. We consider only the selfdual representation. Latter, we discuss the reason of that. Now, we consider only the first term, thus

$$\varepsilon^{IJKL} \to 2i \left(P^i\right)^{IJ} \left(P_i\right)^{KL}.$$

We use it in the determinant e:

$$e = e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \varepsilon^{\mu\nu\rho\sigma} \to -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! : \varepsilon_{0123} = -1.$$

With the selfdual projection, we have

$$e = -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! \to -2i \left(P^i\right)_{IJ} \left(P_i\right)_{KL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4!.$$

We can rewrite

$$e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right) e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right)\frac{1}{2}\left(e^K_\rho e^L_\sigma - e^K_\sigma e^L_\rho\right)\varepsilon^{\mu\nu\rho\sigma}.$$

Thus, we can rewrite it using the area field Σ^{IJ}:

$$e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \Sigma^{IJ}_{\mu\nu} \Sigma^{KL}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma},$$

therefore the determinant e becomes

$$e = -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! \to -\frac{2i}{4!}\left(P^i\right)_{IJ} \left(P_i\right)_{KL} \Sigma^{IJ}_{\mu\nu} \Sigma^{KL}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}. \quad (2.1)$$

Now, we can write the area field as a vector $i = 1, 2, 3$ in the selfdual representation:

$$\Sigma^i_{\mu\nu} = \left(P^i\right)_{IJ} \Sigma^{IJ}_{\mu\nu}.$$

Thus, the determinant e becomes

$$e \to -\frac{2i}{4!}\left(\Sigma^i\right)_{\mu\nu} \left(\Sigma_i\right)_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} \quad \text{or} \quad -\frac{2i}{4!}\Sigma^i_{\mu\nu} \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}. \quad (2.2)$$

We wrote it in this form to get rid of the gravity fields in the path integral. As we saw before, the integral over one of them yields delta Dirac. It cancels the propagation.

By that, the full Lagrange of the gravity field

$$Ld^4x \to \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right) ed^4x + \frac{1}{(48c)^3}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu} ed^4x$$
$$+\frac{1}{2*(48c)^3}\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\left(\partial_\sigma e_\rho^J\right)\left(\partial^\sigma e_{J_1}^\rho\right)\Sigma^{KL}_{\mu\nu}\Sigma^{\mu\nu}_{K_1L_1} ed^4x,$$

becomes

$$Ld^4x \to \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right)\left(-\frac{2i}{4!}\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x$$
$$+\frac{1}{(48c)^3}\left(2ip_i^{IJ_1}\right)\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^{i\rho}_\nu\left(-\frac{2i}{4!}\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x$$
$$-\frac{2}{(48c)^3}\left(p_i\right)^{IJ_1}\left(p_j\right)_{IJ}\left(\partial_\sigma e_\rho^J\right)\left(\partial^\sigma e_{J_1}^\rho\right)\Sigma^j_{\mu\nu}\Sigma^{i\mu\nu}\left(-\frac{2i}{4!}\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x,$$

or

$$Ld^4x \to \frac{2i}{48c}\frac{1}{2}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right)\left(\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x$$
$$+\frac{4}{(48c)^3}\left(p_i\right)^{IJ_1}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^{i\rho}_\nu\left(\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x$$
$$+\frac{4i}{(48c)^3}\left(p_i\right)^{IJ_1}\left(p_j\right)_{IJ}\left(\partial_\sigma e_\rho^J\right)\left(\partial^\sigma e_{J_1}^\rho\right)\Sigma^j_{\mu\nu}\Sigma^{i\mu\nu}\left(\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x.$$

It is quadratic in e^I, therefore its integral is not trivial. Here we can consider the area field Σ^i as a background field that the gravity field propagates over it. Or suggesting the duality $e^I \leftrightarrow \Sigma^i$, by it the amplitude of propagation of e^I between x and $x+dx$ is $\left\langle e^I(x+dx) \,\middle|\, \Sigma^i(x)\right\rangle$.

If we consider the first term, we can discover its behavior by testing one wave; $\cos\left(k_\mu x^\mu\right)$. We have

$$\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right) \to -e_I^\nu\partial_\mu\partial^\mu e_\nu^I \to -\partial_\mu\partial^\mu\cos\left(k_\mu x^\mu\right) = k_\mu k^\mu\cos\left(k_\mu x^\mu\right),$$

it yields

$$e^{i\int Ld^4x} \to \exp\int\frac{2}{48c}\frac{i^2}{2}\left(k_\mu k^\mu e_I^\nu e_\nu^I\right)\left(\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x + ...$$
$$\to \exp\int\frac{2}{48c}\frac{1}{2}\left(-k_\mu k^\mu\right)\left(e_I^\nu e_\nu^I\right)\left(\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x + ...,$$

then

$$e^{iS} \to \exp \int \frac{2}{48c}\frac{1}{2}\left(k_0^2 - \vec{k}^2\right)\left(e_I^\nu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i \Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right) d^4x + ... \quad (2.3)$$

We consider that the area field is in the positive direction $Re\left(\Sigma_{\mu\nu}^i dx^\mu \wedge dx^\nu\right) \succ 0$, the direction of the expanding, then $Re\left(\Sigma_{\mu\nu}^i \Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right) \succ 0$.

We find that in time-like region $k_0^2 - \vec{k}^2 \succ 0$, the gravity field is created, while in the space-like region $k_0^2 - \vec{k}^2 \prec 0$, it is annihilated $e_\mu^I \to \Sigma_{\nu\rho}^i$. Oppositely to the area field, as we will see. This is the duality $e_\mu^I \leftrightarrow \Sigma_{\nu\rho}^i$. It is like to say, in the time-like region we find the gravity field and in the space-like region we find the area field.

The time-like region is the region of exchanging energies (interactions), while the space-like region is the region of the static fields, the situation of located matter. Therefore spacetime, in which matter is located, is consisted of quanta of area and volume. The duality $e_\mu^I \leftrightarrow \Sigma_{\nu\rho}^i$, as we will see, satisfies the reality condition. It is like the right and left spinor fields.

3 The Lagrange of the Area field

We derive here the Lagrange of the area field. We find that in the background spacetime it is like the electromagnetic field, but with opposite sign in the Lagrange. We can get rid of that opposite sign by the replacement $\partial_\mu \to i\partial_\mu$, it is equivalent to replace k_μ with ik_μ in the free solutions: $e^{ikx} \to e^{-kx}$ *or* e^{kx}. We find that the behavior of the area field, in selfdual representation, is opposite to the behavior of the gravity field. For that reason, we suggest the gravity-area duality, which satisfies the reality condition.

The area field is defined in

$$\Sigma^{IJ} = e^I \wedge e^J \ , \ \text{ with } \quad \Sigma_{\mu\nu}^{IJ} = \frac{1}{2}\left(e_\mu^I e_\nu^I - e_\nu^I e_\mu^I\right).$$

We start with the Lagrange (1.3):

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(\pi_{K_1}{}^K{}_M\right) e^{K_1} \wedge \left(\pi_{K_2}{}^{ML}\right) e^{K_2}\right].$$

As done before, we assume that the integral of

$$\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \left(\pi_M{}^{KL} e^M\right)\right) = \varepsilon_{IJKL} d\left(\Sigma^{IJ} \wedge \left(\pi_M{}^{KL} e^M\right)\right)$$

is zero at the infinities, thus we get

$$d\Sigma^{IJ} \wedge \left(\pi_M{}^{KL}\right) e^M + e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) = -\left(\pi_M{}^{KL}\right) e^M \wedge d\Sigma^{IJ} + e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right).$$

By using it, the Action becomes

$$S(e,\pi) = c \int \left[\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right) e^M \wedge d\Sigma^{IJ} + \varepsilon_{IJKL}\Sigma^{IJ} \wedge \left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^{K_1} \wedge e^{K_2}\right],$$

or

$$S(e,\pi) = c \int \left[\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right) e^M \wedge d\Sigma^{IJ} + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) \Sigma^{IJ} \wedge \Sigma^{K_1K_2}\right]$$

By inserting our imposing:

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK},$$

we get

$$\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right) e^M = \varepsilon_{IJKL}\pi^{MKL} e_M = \varepsilon_{IJKL}\pi_N \varepsilon^{NMKL} e_M = -2\left(\pi_I e_J - \pi_J e_I\right).$$

We can write

$$\Sigma^{IJ} \wedge \Sigma^{K_1K_2} \to \varepsilon^{IJK_1K_2}\Sigma^{01} \wedge \Sigma^{23},$$

by it we have

$$\begin{aligned}
\varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\Sigma^{IJ} \wedge \Sigma^{K_1K_2} &= \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\varepsilon^{IJK_1K_2}\Sigma^{01} \wedge \Sigma^{23} \\
&= 2\left(\pi_L{}^K{}_M\right)\left(\pi_K{}^{ML}\right)\Sigma^{01} \wedge \Sigma^{23} = 2\left(\pi_{LKM}\right)\left(\pi^{KML}\right)\Sigma^{01} \wedge \Sigma^{23} \\
&= 2\left(\pi_{KML}\right)\left(\pi^{KML}\right)\Sigma^{01} \wedge \Sigma^{23} = 2\pi^I \varepsilon_{IKML}\pi_J \varepsilon^{JKML}\Sigma^{01} \wedge \Sigma^{23} \\
&= -12\pi^2\Sigma^{01} \wedge \Sigma^{23}.
\end{aligned}$$

Therefore, the Action becomes

$$S(e,\pi,\Sigma) = c \int \left[-2\left(\pi_I e_J - \pi_J e_I\right) \wedge d\Sigma^{IJ} - 12\pi_I \pi^I \Sigma^{01} \wedge \Sigma^{23}\right].$$

Because the area field Σ^{IJ} is anti-symmetry, we can write

$$S(e,\pi,\Sigma) = c\int \left[-4\pi_I e_J \wedge d\Sigma^{IJ} - 12\pi_I\pi^I\Sigma^{01}\wedge\Sigma^{23}\right],$$

and by using $\varepsilon_{0123} = -1$, we can rewrite it like

$$S(e,\pi,\Sigma) = c\int \left[-4\pi_I e_J \wedge d\Sigma^{IJ} + 12\pi_I\pi^I\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}/4!\right],$$

or

$$S(e,\pi,\Sigma) = c\int \left[-4\pi_I e_J \wedge d\Sigma^{IJ} + \frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right].$$

The path integral over momentum π^I vanishes unless (the equation of motion)

$$\frac{\delta}{\delta\pi_I}\int \left[-4\pi_I e_J \wedge d\Sigma^{IJ} + \frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right] = 0.$$

But it is not easy to separate Σ from e. It is like the gravity field, it is separable only in weak gravity(background spacetime). Therefore, we solve it in the background spacetime. On arbitrary spacetime, we get the integral:

$$\int \left(-4\pi_I e_J \wedge d\Sigma^{IJ} + \frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right)$$

$$\to \int \left(-4\pi_I e_{\mu J}\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} + \frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x.$$

The background spacetime approximation is

$$e^I_\mu(x) \to \delta^I_\mu + h^I_\mu(x)\ , \quad e \to 1 + \delta e,$$

by it the area field becomes

$$\Sigma^{IJ}_{\mu\nu} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right) \to \frac{1}{2}\left(\delta^I_\mu\delta^J_\nu - \delta^I_\nu\delta^J_\mu\right) + \frac{1}{2}\left(h^I_\mu\delta^J_\nu - h^I_\nu\delta^J_\mu\right) + \frac{1}{2}\left(\delta^I_\mu h^J_\nu - \delta^I_\nu h^J_\mu\right).$$

By inserting it in the action:

$$S\left(e,\Sigma\right) = c\int \left(-4\pi_I e_{\mu J}\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} + \frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x,$$

it becomes

$$S\left(e,\Sigma\right)\rightarrow S\left(h,\delta\Sigma\right)=c\int\left(-4\pi_I\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon_J{}^{\nu\rho\sigma}+\frac{1}{2}\pi^2\left(-24\right)+\ldots\right)d^4x.$$

Therefore the condition(equation of motion):

$$\frac{\delta}{\delta\pi_I}\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right]=0$$

approximates to

$$\frac{\delta}{\delta\pi_I}\int\left(-4\pi_I\partial_\nu\Sigma^I{}_{J\rho\sigma}\varepsilon^{J\nu\rho\sigma}+\frac{1}{2}\pi^2\left(-24\right)\right)d^4x=0.$$

Its solution is

$$\pi^I=-\frac{1}{6}\partial_\nu\Sigma^I_{J\rho\sigma}\varepsilon^{J\nu\rho\sigma}=-\frac{1}{6}\partial^\nu\Sigma^{IJ\rho\sigma}\varepsilon_{J\nu\rho\sigma}.$$

By that, the action in background spacetime is approximated to

$$S(\Sigma)\rightarrow c\int\left[\frac{2}{3}\partial^{\nu_1}\Sigma^{IJ_1\rho_1\sigma_1}\varepsilon_{J_1\nu_1\rho_1\sigma_1}\partial_\nu\Sigma_{IJ\rho\sigma}\varepsilon^{J\nu\rho\sigma}+...\right]d^4x.$$

By defining inner product $\Sigma^{IJ_1\rho_1\sigma_1}\Sigma_{IJ\rho\sigma}=\Sigma^2\delta^{J_1}_J\delta^{\rho_1}_\rho\delta^{\sigma_1}_\sigma$, we get

$$S(\Sigma)\rightarrow c\int\left(-4\partial_\mu\Sigma^{\nu\rho}_{IJ}\partial^\mu\Sigma^{IJ}_{\nu\rho}+...\right)d^4x\quad\text{with}\quad\partial_\mu\Sigma^{\mu\rho}_{IJ}=0.$$

This is the action of the area field in weak gravity field (background spacetime). It is like the electromagnetic field. The corresponding Lagrange is

$$L_0(\Sigma)\rightarrow-4c\left(\partial_\mu\Sigma^{\nu\rho}_{IJ}\right)\left(\partial^\mu\Sigma^{IJ}_{\nu\rho}\right)\ \text{with}\quad\partial_\mu\Sigma^{\mu\rho}_{IJ}=0.$$

We rewrite it like

$$L_0(\Sigma)d^4x=-4c\left(\partial_\mu\Sigma^{\nu\rho}_{IJ}\right)\left(\partial^\mu\Sigma^{IJ}_{\nu\rho}\right)ed^4x+...$$

We replace c with $-c$, as we did in deriving the Lagrange of the gravity field. This constant is determined in the static gravity field potential $c\succ0$. Therefore,

$$L_0(\Sigma)d^4x\rightarrow4c\left(\partial_\mu\Sigma^{\nu\rho}_{IJ}\right)\left(\partial^\mu\Sigma^{IJ}_{\nu\rho}\right)ed^4x+....\qquad(3.1)$$

We can get rid of opposite sign by the comparing with the free electromagnetic Lagrange in background spacetime $e \to 1 + \delta e$. We can replace ∂_μ with $i\partial_\mu$, it is equivalent to the replacing $k_\mu \to ik_\mu$ in the free solutions: $e^{ikx} \to e^{-kx}$ *or* e^{kx} in the background spacetime. By that the area field becomes a classical field, we can consider it as background field that the gravity field propagates over it.

By using the selfdual projection (2.1) and (2.2):

$$e = -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! \to -\frac{2i}{4!} \left(P^i\right)_{IJ} \left(P_i\right)_{KL} \Sigma^{IJ}_{\mu\nu} \Sigma^{KL}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma},$$

the Lagrange (3.1) becomes

$$L_0(\Sigma) e d^4x = -8ci \left(\partial_\mu \Sigma^{\nu\rho}_{IJ} \partial^\mu \Sigma^{IJ}_{\nu\rho}\right) \left(\Sigma^i_{\mu\nu} \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}/4!\right) d^4x + ...$$

To discover the area field behavior, we test one wave; $\cos\left(k_\mu x^\mu\right)$. We get

$$L_0(\Sigma) e d^4x \to -8ci \left(k^\mu k_\mu \Sigma^{\nu\rho}_{IJ} \Sigma^{IJ}_{\nu\rho}\right) \left(\Sigma^i_{\mu\nu} \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}/4!\right) d^4x.$$

The action of that is

$$e^{iLed^4x} \to \exp 8c \left(k_\mu k^\mu \Sigma^{\nu\rho}_{IJ} \Sigma^{IJ}_{\nu\rho}\right) \left(\Sigma^i_{\mu\nu} \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}/4!\right) d^4x,$$

it yields

$$e^{i\delta S} \to \exp 8c \left(-k_0^2 + \vec{k}^2\right) \left(\Sigma^{\nu\rho}_{IJ} \Sigma^{IJ}_{\nu\rho}\right) \left(\Sigma^i_{\mu\nu} \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}/4!\right) d^4x. \qquad (3.2)$$

It is opposite to the behavior of the gravity field (2.3). In the time-like region $(-k_0^2 + \vec{k}^2 \prec 0)$, the area field is annihilated $\Sigma^{JK}_{\nu\rho} \to e^I_\mu$, while in the space-like region $(-k_0^2 + \vec{k}^2 \succ 0)$, the area field is created $e^I_\mu \to \Sigma^{JK}_{\nu\rho}$. This is the duality $e^I_\mu \leftrightarrow \Sigma^{JK}_{\nu\rho}$. It preserves the reality condition. It is like the duality of left and right spinor field under Lorentz transformation and party.

The opposite behavior is with the anti-selfdual representation, the hermitian conjugate of the selfdual

$$2i \left(P^i\right)^{IJ} \left(P_i\right)^{KL} - 2i \left(\bar{p}^i\right)^{IJ} \left(\bar{P}_i\right)^{KL} \to \varepsilon^{IJKL},$$

which is projection from $I \neq J$ and $K \neq L$ in the left to $I \neq J \neq K \neq L$ in the right.

The first term is for the selfdual, while the second is for the anti-selfdual. The tensor product of them satisfies the reality:

$$e^{i\Delta L(selfdual)d^4x}e^{i\Delta L(anti-selfdual)d^4x} = real.$$

Instead of that, we can satisfy the reality by gravity-area duality:

$$e^{i\Delta L(e)d^4x}e^{i\Delta L(\Sigma)d^4x} = real : invariant\ for\ selfdual.$$

For one wave, it becomes

$$e^{\frac{2}{48c}\frac{1}{2}\left(k_0^2-\vec{k}^2\right)\left(e_I^\nu e_\nu^I\right)\left(\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)}e^{8c'\left(-k_0^2+\vec{k}^2\right)\left(\Sigma^{\nu\rho}_{IJ}\Sigma^{IJ}_{\nu\rho}\right)\left(\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)},$$

where we wrote c' to distinguish it from c. For

$$\frac{2}{48c}\frac{1}{2}\left(e_I^\nu e_\nu^I\right) = 8c'\left(\Sigma^{\nu\rho}_{IJ}\Sigma^{IJ}_{\nu\rho}\right),$$

the Tensor product of them equals one, this satisfies the reality. By that, we can determine c', like to choose $(48c)^{-1} = 16c'$, with

$$\left(e_I^\nu e_\nu^I\right) = \frac{1}{2}\left(\Sigma^{\nu\rho}_{IJ}\Sigma^{IJ}_{\nu\rho}\right) = \frac{1}{2}\left(\Sigma^{\nu\rho}_i\Sigma^i_{\nu\rho} + \bar{\Sigma}^{\nu\rho}_i\bar{\Sigma}^i_{\nu\rho}\right),$$

where the hermitian conjugate $\bar{\Sigma}^{\nu\rho}_i\bar{\Sigma}^i_{\nu\rho}$ is represented in anti-selfdual: $\bar{\Sigma}^i = \bar{P}^i_{IJ}\Sigma^{IJ}$.

As done for the left and right spinor fields; in the left spinor field representation, the right spinor field is zero. And in the right spinor field representation, the left spinor field is zero[3]. Therefore, in the selfdual representation, we assume that the anti-selfdual is zero. Like that in the anti-selfdual representation.

Thus, in the selfdual representation, we have

$$\bar{\Sigma}^i = \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} - i\Sigma^{0i} = 0 \to \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} = i\Sigma^{0i}.$$

Therefore, the area field in the selfdual representation becomes

$$\Sigma^i = \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} + i\Sigma^{0i} = \varepsilon^{ijk}\Sigma_{jk},$$

which is real as required for satisfying the reality condition. It is equivalent to the replacing $x^0 \to -ix^0$. Same result we get in the anti-selfdual representation: $\Sigma^i = 0 \to \bar{\Sigma}^i = \varepsilon^{ijk}\Sigma_{jk}$. It is equivalent to the replacing $x^0 \to ix^0$, which allows the splitting: $SO(3,1) \to SU(2) \otimes SU(2)$.

In the two representations, the condition $\left(e_I^\nu e_\nu^I\right) = \frac{1}{2}\left(\Sigma_{IJ}^{\nu\rho}\Sigma_{\nu\rho}^{IJ}\right) = \frac{1}{2}\left(\Sigma_i^{\nu\rho}\Sigma_{\nu\rho}^i + \bar{\Sigma}_i^{\nu\rho}\bar{\Sigma}_{\nu\rho}^i\right)$ becomes $\left(e_I^\nu e_\nu^I\right) = \frac{1}{2}\varepsilon_{ij'k'}\Sigma_{\nu\rho}^{j'k'}\varepsilon^{ijk}\Sigma_{jk}^{\nu\rho} = \Sigma_{jk}^{\nu\rho}\Sigma_{\nu\rho}^{jk}$.

The difference between the selfdual and the anti-selfdual appeared in the opposite sign in the Lagrange:

$$L \to 8c'\left(-k_0^2 + \vec{k}^2\right)\left(\Sigma_i^{\nu\rho}\Sigma_{\nu\rho}^i\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right) \; for\ selfdual\ \bar{\Sigma}^i = 0,$$

and

$$L \to -8c'\left(-k_0^2 + \vec{k}^2\right)\left(\bar{\Sigma}_i^{\nu\rho}\bar{\Sigma}_{\nu\rho}^i\right)\left(\bar{\Sigma}_{\mu\nu}^i\bar{\Sigma}_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right) \; for\ anti\text{-}selfdual\ \Sigma^i = 0.$$

The opposite sign comes from the projection (2.2):

$$e \to -\frac{2i}{4!}\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4! + \frac{2i}{4!}\bar{\Sigma}_{\mu\nu}^i\bar{\Sigma}_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!.$$

We chose the selfdual representation because its tensor product with the gravity field satisfies the reality. It is like the duality of the left and right spinor fields under the Lorentz transformation and the party: $\psi_L \leftrightarrow \psi_R$.

4 The static potential of weak gravity

We derive the static potential of the scalar and spinor fields interactions with the weak gravity field in the static limit, the Newtonian gravitational potential. We find that this potential has the same structure for the both fields, it depends on the fields energy. By that, we determine the constant $c \succ 0$.

The action of scalar field in curved spacetime is[1]

$$S(e,\phi) = \int d^4x e\left(\eta^{IJ}e_I^\mu e_J^\nu D_\mu\phi^+ D_\nu\phi - V(\phi)\right).$$

In the weak gravity, the background spacetime approximation is given by

$$e_I^\mu(x) \to \delta_I^\mu + h_I^\mu(x) \quad , \quad e \to 1 + \delta e.$$

Thus, the action is approximated to

$$S(e,\phi) = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi + h^{\mu\nu}(x)\partial_\mu \phi^+ \partial_\nu \phi + h^{\nu\mu}(x)\partial_\mu \phi^+ \partial_\nu \phi - V(\phi) + ...\right).$$

The gravity field is symmetry, thus we get

$$S(e,\phi) = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu}(x)\partial_\mu \phi^+ \partial_\nu \phi - V(\phi) + ...\right).$$

The energy-momentum tensor of the scalar field is[3]

$$T_{\mu\nu} = \partial_\mu \phi^+ \partial_\nu \phi + g_{\mu\nu} L,$$

hence

$$\partial_\mu \phi^+ \partial_\nu \phi = T_{\mu\nu} - g_{\mu\nu} L.$$

By inserting it in the Lagrange, it becomes

$$L = \partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu}(x)\left(T_{\mu\nu} - g_{\mu\nu} L\right) - V(\phi) + ...$$

By that, we have

$$L = \partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu} T_{\mu\nu} - V(\phi) - 2h^{\mu\nu} g_{\mu\nu} L + ...$$

Therefore, in the interaction term, we set the replacement:

$$\partial_\mu \phi^+ \partial_\nu \phi \to T_{\mu\nu} \quad \text{and} \quad L \to L - 2h^{\mu\nu} g_{\mu\nu} L.$$

Because the gravity field is weak (background spacetime), so $2h^{\mu\nu} g_{\mu\nu} L$ is neglected by comparing it with L.

We find the potential $V(r)$ of the exchanged virtual gravitons by two particles k_1 and k_2, using $M\left(k_1 + k_2 \to k_1' + k_2'\right)$ matrix element (like Born approximation to the scattering amplitude in non-relativistic quantum mechanics [7]).

For one diagram of Feynman diagrams, we have

$$iM\left(k_1+k_2\rightarrow k_1'+k_2'\right)=i\left(-ik_2'\right)_\mu\left(ik_2\right)_\nu\frac{\bar{\Delta}^{\mu\nu\rho\sigma}\left(q\right)}{i}i\left(-ik_1'\right)_\rho\left(ik_1\right)_\sigma,$$

with

$$q=k_1'-k_1=k_2-k_2'.$$

The propagator $\Delta^{\mu\nu\rho\sigma}\left(x_2-x_1\right)$ is the gravitons propagator (1.8), we get it from the Lagrange of the free gravity field (background spacetime) we had before:

$$L_0=\frac{1}{48c}\frac{1}{2}\eta_{IJ}e^I_\mu\left(g^{\mu\nu}\partial^2-\partial^\mu\partial^\nu\right)e^J_\nu\rightarrow\frac{1}{48c}\frac{1}{2}\eta_{IJ}h^I_\mu\left(g^{\mu\nu}\partial^2-\partial^\mu\partial^\nu\right)h^J_\nu.$$

With the gauge $\partial^\mu e^I_\mu=0$, we get

$$\Delta^{IJ}_{\mu\nu}\left(y-x\right)=\int\frac{d^4q}{\left(2\pi\right)^4}\bar{\Delta}^{IJ}_{\mu\nu}\left(q^2\right)e^{iq(y-x)}\ :\ \bar{\Delta}^{IJ}_{\mu\nu}\left(q^2\right)=48c\frac{g_{\mu\nu}\eta^{IJ}}{q^2-i\varepsilon}.$$

Therefore, the M matrix element becomes

$$iM\left(k_1+k_2\rightarrow k_1'+k_2'\right)=i48c\left(-ik_2'\right)_\mu\left(ik_2\right)_\rho\frac{g^{\mu\nu}g^{\rho\sigma}}{q^2}\left(-ik_1'\right)_\sigma\left(ik_1\right)_\nu,$$

where $g=\eta$ and $q=k_1'-k_1=k_2-k_2'$

By comparing it with[7]

$$iM\left(k_1+k_2\rightarrow k_1'+k_2'\right)=-i\bar{V}\left(q\right)\delta^4\left(k_{out}-k_{in}\right),$$

we get

$$\bar{V}\left(q^2\right)=-48c\left(-ik_2'\right)_\mu\left(ik_2\right)_\rho\frac{g^{\mu\nu}g^{\rho\sigma}}{q^2}\left(-ik_1'\right)_\sigma\left(ik_1\right)_\nu.$$

And by comparing this relation with the replacement:

$$\partial_\mu\phi^+\partial_\nu\phi\rightarrow T_{\mu\nu}\ \text{and}\quad L\rightarrow L-2h^{\mu\nu}g_{\mu\nu}L,$$

and by evaluating the inverse Fourier transform, we get

$$V\left(y-x\right)=-48cT_{\mu\rho}\left(y\right)g^{\mu\nu}g^{\rho\sigma}T_{\nu\sigma}\left(x\right)\frac{1}{4\pi\left|y-x\right|}=-48c\frac{T_{\mu\nu}\left(y\right)T^{\mu\nu}\left(x\right)}{4\pi\left|y-x\right|},$$

where $T^{\mu\nu}$ is transferred energy-momentum. In the static limit, for one particle, we approximate T^{00} to m: m is the mass of interacted particles.

Therefore, we get the Newtonian gravitational potential:

$$V\left(y-x\right) = -48c\frac{m^2}{4\pi\left|y-x\right|} = -G\frac{m^2}{\left|y-x\right|} \to 48c = 4\pi G.$$

Therefore, the weak gravity Lagrange becomes

$$L_0 = \frac{1}{4\pi G}\frac{1}{2}\eta_{IJ}e^I_\mu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)e^J_\nu.$$

We do the same thing for the spinor fields interaction with the gravity. The action is[1]

$$S(e,\psi) = \int d^4xe\left(ie^\mu_I\bar{\psi}\gamma^I D_\mu\psi - m\bar{\psi}\psi\right),$$

where the covariant derivative D_μ is

$$D_\mu = \partial_\mu + \left(\omega_\mu\right)^I_J L^J_I + A^a_\mu \mathrm{T}^a.$$

In the background spacetime, it becomes

$$S(e,\psi) = \int d^4x\left(i\bar{\psi}\gamma^\mu D_\mu\psi + ih^\mu_I\bar{\psi}\gamma^I D_\mu\psi - m\bar{\psi}\psi + ...\right).$$

Let us consider only the terms:

$$\int d^4x\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi + ih^\mu_\nu\bar{\psi}\gamma^\nu\partial_\mu\psi - m\bar{\psi}\psi\right): \quad g=\eta.$$

The energy-momentum tensor of the spinor field is[3]

$$T^{\mu\nu} = -i\bar{\psi}\gamma^\mu\partial^\nu\psi + g^{\mu\nu}L.$$

As for the scalar field, in the interaction term, we have the replacement

$$i\bar{\psi}\gamma^\mu\partial^\nu\psi \to -T_{\mu\nu} \text{ and } L \to L + h^{\mu\nu}g_{\mu\nu}L.$$

The term $h^{\mu\nu}g_{\mu\nu}L$ is neglected by comparing it with the Lagrange L. We find M matrix element of the exchanged virtual gravitons $p_1+p_2 \to p'_1+p'_2$, for one diagram of Feynman diagrams[7]:

$$iM\left(p_1+p_2 \to p'_1+p'_2\right) = i48c\bar{u}\left(p'_1\right)\gamma^\mu\left(-ip_1\right)_\nu u\left(p_1\right)\frac{g_{\mu\sigma}g^{\nu\rho}}{q^2}\bar{u}\left(p'_2\right)\gamma^\sigma\left(-ip_2\right)_\rho u\left(p_2\right),$$

with

$$q = p_1' - p_1 = p_2 - p_2' \text{ and } \quad g = \eta,$$

we have

$$\bar{V}\left(q^2\right) = -48c\bar{u}\left(p_1'\right)\gamma^{\mu}\left(-ip_1\right)_{\nu} u\left(p_1\right)\frac{g_{\mu\sigma}g^{\nu\rho}}{q^2}\bar{u}\left(p_2'\right)\gamma^{\sigma}\left(-ip_2\right)_{\rho} u\left(p_2\right).$$

By comparing this relation with the replacement:

$$i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi \rightarrow -T_{\mu\nu} \text{ and } \quad L \rightarrow L + h^{\mu\nu}g_{\mu\nu}L,$$

and by evaluating the inverse Fourier transform, we get

$$V\left(y-x\right) = -48c\left(-T_{\mu\rho}\left(y\right)\right)g^{\mu\nu}g^{\rho\sigma}\left(-T_{\nu\sigma}\left(x\right)\right)\frac{1}{4\pi\left|y-x\right|} = -48c\frac{T_{\mu\nu}\left(y\right)T^{\mu\nu}\left(x\right)}{4\pi\left|y-x\right|},$$

where $T^{\mu\nu}$ is transferred energy-momentum. In the static limit, for one particle, we approximate T^{00} to m: m is the mass of interacted particles.

Therefore, we get the Newtonian gravitational potential:

$$V\left(y-x\right) = -48c\frac{m^2}{4\pi\left|y-x\right|} = -G\frac{m^2}{\left|y-x\right|} \rightarrow 48c = 4\pi G.$$

It is the same potential we found in the scalar field interaction with the gravity field.

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