Using a quotient polynomial to probe the solvability of polynomial potentials in one-dimensional quantum mechanics

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Abstract

Making use of the Bethe ansatz, we introduce a quotient polynomial and we show that the presence of intermediate terms in the quotient polynomial, i.e. terms other than the constant and the leading one, constitutes a non-solvability condition for the respective potential. In this context, both the exact solvability of the quantum harmonic oscillator and the quasi-exact solvability of the sextic anharmonic oscillator stem naturally from the quotient polynomial, as in the first case, it is an energy-dependent constant, while in the second case, it is a second-degree binomial with no linear term. In all other cases, the quotient polynomial has at least one intermediate term, the presence of which makes the respective potentials non-solvable.

Keywords: quotient polynomial, leading coefficient, constant term, intermediate terms, exactly solvable potentials, quasi-exactly solvable potentials, non-solvable potentials, polynomial potentials, harmonic oscillator, sextic anharmonic oscillator

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Introduction – Dimensionless units

Assume a particle of mass m moving in a one-dimensional polynomial potential $V(x)$ that is attractive (at least) at long distances, i.e. a potential that is a polynomial of even degree with positive leading coefficient.

Since $V(x) \to \infty$ as $|x| \to \infty$, all states of the particle are bound and its energy spectrum is discrete and non-degenerate **[1, 2]**.

Since the potential does not have singular points – it is infinitely many times differentiable in $\mathbb{R}$ – the nth-excited-state wave function of the particle will have n zeros **[3]**.

That is, the ground-state wave function will have no zeros, the first-excited-state wave function will have one zero, and so on.

Then, if a square-integrable wave function $\psi(x)$ satisfies the energy eigenvalue equation of the particle, i.e. the equation

$$\psi''(x) + \frac{2m}{\hbar^2}\left(E - V(x)\right)\psi(x) = 0 \quad (1)$$

and it has r zeros, it will be the rth-excited state wave function of the particle, with energy E.

For simplicity and convenience, we'll make the quantities of interest, i.e. the position, the potential, and the energy, dimensionless.

To this end, we introduce the dimensionless variable

$$\tilde{x} \equiv \frac{x}{l} \quad (2)$$

where l is a positive real constant with dimensions of length, which, at a later time, can be related to the length scale of the examined particle (see the section "The quantum harmonic oscillator").

Then, the wave function $\psi(x)$ becomes a function of $\tilde{x}$, i.e. $\psi(x) \to \psi(\tilde{x})$, and the potential $V(x)$ also becomes a function of $\tilde{x}$, i.e. $V(x) \to V(\tilde{x})$.

Using the chain rule and (2), the first derivatives with respect to x and $\tilde{x}$ are related by the equation

$$\frac{d}{dx} = \frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}} = \frac{1}{l}\frac{d}{d\tilde{x}}$$

and, similarly,

$$\frac{d^2}{dx^2} = \frac{1}{l^2}\frac{d^2}{d\tilde{x}^2}$$

Then, the second derivatives of the wave function with respect to x and $\tilde{x}$ are related by the equation

$$\frac{d^2\psi(x)}{dx^2} = \frac{1}{l^2}\frac{d^2\psi(\tilde{x})}{d\tilde{x}^2}$$

Using the previous equation, the energy eigenvalue equation (1) is written in terms of $\tilde{x}$ as

$$\frac{1}{l^2}\frac{d^2\psi(\tilde{x})}{d\tilde{x}^2}+\frac{2m}{\hbar^2}\left(E-V(\tilde{x})\right)\psi(\tilde{x})=0\Rightarrow\psi''(\tilde{x})+\frac{2ml^2}{\hbar^2}\left(E-V(\tilde{x})\right)\psi(\tilde{x})=0\Rightarrow$$

$$\Rightarrow\psi''(\tilde{x})+\left(\frac{2ml^2E}{\hbar^2}-\frac{2ml^2V(\tilde{x})}{\hbar^2}\right)\psi(\tilde{x})=0 \quad (3)$$

where now the primes denote differentiation with respect to $\tilde{x}$.
We observe that

$$\left[\frac{2ml^2}{\hbar^2}\right]=\left[\frac{mx^2}{p^2x^2}\right]=\left[\frac{m}{p^2}\right]\overbrace{=}^{E=\frac{p^2}{2m}}\left[\frac{1}{E}\right]$$

Thus, the quantity $\frac{\hbar^2}{2ml^2}$ has dimensions of energy and it can be used as an energy scale to make the related quantities, i.e. the energy and the potential dimensionless.
To this end, we set

$$\tilde{E}\equiv\frac{2ml^2E}{\hbar^2} \quad (4)$$

$$\tilde{V}(\tilde{x})\equiv\frac{2ml^2V(\tilde{x})}{\hbar^2} \quad (5)$$

The quantities $\tilde{E}$ and $\tilde{V}(\tilde{x})$ are dimensionless, we may call them "dimensionless energy" and "dimensionless potential", respectively, but for convenience, we'll still call them energy and potential.
By means of (4) and (5), the energy eigenvalue equation (3) becomes

$$\psi''(\tilde{x})+\left(\tilde{E}-\tilde{V}(\tilde{x})\right)\psi(\tilde{x})=0 \quad (6)$$

The quotient polynomial

Since the function $\psi(\tilde{x})$ is an energy eigenfunction, it is, by definition, linearly independent, and thus it cannot be identically zero.
Thus, dividing (6) by $\psi(\tilde{x})$ and solving for $\tilde{V}(\tilde{x})$, we obtain

$$\tilde{V}(\tilde{x})=\frac{\psi''(\tilde{x})}{\psi(\tilde{x})}+\tilde{E} \quad (7)$$

Using the Bethe ansatz **[4]**, we seek eigenfunctions $\psi(\tilde{x})$ having the form

$$\psi(\tilde{x};m,n)=A_np_n(\tilde{x})\exp\left(g_{2m}(\tilde{x})\right) \quad (8)$$

where $p_n(\tilde{x})$ is a dimensionless, real polynomial of degree $n\geq 0$ and $g_{2m}(\tilde{x})$ is an also dimensionless, real polynomial of degree $2m\geq 2$, i.e. $m\geq 1$, with negative leading coefficient, so that the wave function (8) is square integrable.
It is important to remember that the number of (real) zeros of $p_n(\tilde{x})$ is not, in general, equal to its degree n.

Depending on its coefficients, the polynomial $p_n(\tilde{x})$ can have from 0 up to n (real) zeros. Thus, if r is the number of zeros of $p_n(\tilde{x})$, then, in general, $r = 0,1,...,n$.

The constant A_n is the normalization constant, and since $p_n(\tilde{x})$ is dimensionless, A_n has the same dimensions as the wave function, i.e. it has dimensions of $(\text{length})^{-1/2}$.

We do not incorporate the normalization constant into $p_n(\tilde{x})$ because we want to keep $p_n(\tilde{x})$ dimensionless.

Since the leading coefficient of $p_n(\tilde{x})$ is, by definition, non-zero, we can write $p_n(\tilde{x})$ as

$$p_n(\tilde{x}) = p_n\left(\tilde{x}^n + \frac{p_{n-1}}{p_n}\tilde{x}^{n-1} + ... + \frac{p_0}{p_n}\right),$$

and the wave function (8) is then written as

$$\psi(\tilde{x};m,n) = A_n p_n\left(\tilde{x}^n + \frac{p_{n-1}}{p_n}\tilde{x}^{n-1} + ... + \frac{p_0}{p_n}\right)\exp\left(g_{2m}(\tilde{x})\right)$$

Defining a new normalization constant $A_n' \equiv A_n p_n$, with the same dimensions as A_n, since p_n is dimensionless, we incorporate the leading coefficient of $p_n(\tilde{x})$ into the normalization constant, and the wave function is written as

$$\psi(\tilde{x};m,n) = A_n' p_n(\tilde{x})\exp\left(g_{2m}(\tilde{x})\right),$$

with $p_n = 1$, i.e. the polynomial $p_n(\tilde{x})$ is monic.

In what follows, we'll drop the prime from A_n' and write the wave function in the form (8) with $p_n(\tilde{x})$ being monic, i.e. $p_n = 1$.

Using (8), the first derivative of $\psi(\tilde{x};m,n)$ with respect to $\tilde{x}$ is then

$$\psi'(\tilde{x};m,n) = A_n p_n'(\tilde{x})\exp\left(g_{2m}(\tilde{x})\right) + A_n g_{2m}'(\tilde{x})\,p_n(\tilde{x})\exp\left(g_{2m}(\tilde{x})\right) =$$
$$= A_n\left(p_n'(\tilde{x}) + g_{2m}'(\tilde{x})\,p_n(\tilde{x})\right)\exp\left(g_{2m}(\tilde{x})\right)$$

That is

$$\psi'(\tilde{x};m,n) = A_n\left(p_n'(\tilde{x}) + g_{2m}'(\tilde{x})\,p_n(\tilde{x})\right)\exp\left(g_{2m}(\tilde{x})\right)$$

Differentiating the previous equation once more with respect to $\tilde{x}$, we obtain

$$\psi''(\tilde{x};m,n) = \left(A_n\left(p_n'(\tilde{x}) + g_{2m}'(\tilde{x})\,p_n(\tilde{x})\right)\exp\left(g_{2m}(\tilde{x})\right)\right)' =$$

$$= A_n\left(p_n'(\tilde{x}) + g_{2m}'(\tilde{x})\,p_n(\tilde{x})\right)'\exp\left(g_{2m}(\tilde{x})\right) + A_n\left(p_n'(\tilde{x}) + g_{2m}'(\tilde{x})\,p_n(\tilde{x})\right)g_{2m}'(\tilde{x})\exp\left(g_{2m}(\tilde{x})\right) =$$
$$= A_n\left(p_n''(\tilde{x}) + g_{2m}''(\tilde{x})\,p_n(\tilde{x}) + g_{2m}'(\tilde{x})\,p_n'(\tilde{x})\right)\exp\left(g_{2m}(\tilde{x})\right) +$$

$$+A_n\left(g_{2m}'(\tilde{x})p_n'(\tilde{x})+g_{2m}'^2(\tilde{x})p_n(\tilde{x})\right)\exp\left(g_{2m}(\tilde{x})\right)=$$

$$=A_n\left(p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})+\left(g_{2m}'^2(\tilde{x})+g_{2m}''(\tilde{x})\right)p_n(\tilde{x})\right)\exp\left(g_{2m}(\tilde{x})\right)$$

That is

$$\psi''(\tilde{x};m,n)=A_n\left(p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})+\left(g_{2m}'^2(\tilde{x})+g_{2m}''(\tilde{x})\right)p_n(\tilde{x})\right)\exp\left(g_{2m}(\tilde{x})\right) \quad (9)$$

Using (8) and (9), we have

$$\frac{\psi''(\tilde{x};m,n)}{\psi(\tilde{x};m,n)}=\frac{p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})+\left(g_{2m}'^2(\tilde{x})+g_{2m}''(\tilde{x})\right)p_n(\tilde{x})}{p_n(\tilde{x})}=$$

$$=\frac{p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})}{p_n(\tilde{x})}+g_{2m}'^2(\tilde{x})+g_{2m}''(\tilde{x})$$

That is

$$\frac{\psi''(\tilde{x};m,n)}{\psi(\tilde{x};m,n)}=\frac{p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})}{p_n(\tilde{x})}+g_{2m}'^2(\tilde{x})+g_{2m}''(\tilde{x}) \quad (10)$$

The primes denote differentiation with respect to $\tilde{x}$.
By means of (10), (7) is written as

$$\tilde{V}(\tilde{x};m,n)=\frac{p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})}{p_n(\tilde{x})}+g_{2m}'^2(\tilde{x})+g_{2m}''(\tilde{x})+\tilde{E} \quad (11)$$

The potential $\tilde{V}(\tilde{x};m,n)$ is a polynomial, and so is the function $g_{2m}'^2(\tilde{x})+g_{2m}''(\tilde{x})+\tilde{E}$.

Thus, the difference $\tilde{V}(\tilde{x};m,n)-\left(g_{2m}'^2(\tilde{x})+g_{2m}''(\tilde{x})+\tilde{E}\right)$ is also a polynomial, as the difference of two polynomials, and then, from (11), we derive that the function $\frac{p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})}{p_n(\tilde{x})}$ is also a polynomial, which means that the denominator of the fraction divides the numerator, i.e. the polynomial $p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})$ contains, as a factor, the polynomial $p_n(\tilde{x})$, i.e.

$$p_n''(\tilde{x})+2g_{2m}'(\tilde{x})p_n'(\tilde{x})=-q(\tilde{x};m,n)p_n(\tilde{x}) \quad (12)$$

where $q(\tilde{x};m,n)$ is the quotient polynomial or, simply, the polynomial q.
The minus sign in (12) is put in for convenience, it has not any physical – or other – significance.

The degree of q

We observe that

$\deg\left(g'p'\right)=\deg\left(g'\right)+\deg\left(p'\right)=\left(2m-1\right)+\left(n-1\right)=2m+n-2$

and

$\deg\left(p''\right)=n-2<2m+n-2$, since $m\geq 1$

Since $\deg\left(g'p'\right)>\deg\left(p''\right)$,

$\deg\left(p''+2g'p'\right)=\deg\left(2g'p'\right)=2m+n-2$,

and then, from (12), the degree of q is such that

$\deg\left(qp\right)=2m+n-2\Rightarrow\deg\left(q\right)+\deg\left(p\right)=2m+n-2$,

and since $\deg\left(p\right)=n$, we end up to

$\deg\left(q\right)=2\left(m-1\right)$ (13)

Since $m\geq 1$, $\deg\left(q\right)\geq 0$, which is necessary for (12) to hold for a polynomial q.
Since the degree of q is $2\left(m-1\right)$, we'll denote it by $q_{2(m-1)}\left(\tilde{x};n\right)$ instead of $q\left(\tilde{x};m,n\right)$.
We also note that $\deg\left(q\right)=\deg\left(g''\right)$ and that q is always of even degree.

From (13), we see that the degree of the polynomial q is independent of n, i.e. it is independent of the degree of the polynomial p, and thus it is the same for all polynomials p of fixed degree n.

The potential

Using (12), the potential (11) is written as

$\tilde{V}\left(\tilde{x};m,n\right)=g_{2m}'^{2}\left(\tilde{x}\right)+g_{2m}''\left(\tilde{x}\right)-q_{2(m-1)}\left(\tilde{x};n\right)+\tilde{E}$ (14)

As m,n change, the potential, in general, changes too, and thus (14) gives sequences of m,n-dependent potentials.
We observe that

$\deg\left(g'^{2}\right)=2\deg\left(g'\right)=2\left(2m-1\right)=4m-2$

$\deg\left(g''\right)=2m-2<4m-2$, as $m\geq 1$

Also

$\deg\left(q\right)=\deg\left(g''\right)$

Thus

$\deg\left(V\right)=\deg\left(g'^{2}\right)=4m-2$

That is

$\deg\left(V\right)=2\left(2m-1\right)$ (15)

The potential is also of even degree and since the leading coefficient of g'^2 is positive, from (14) we derive that the leading coefficient of the potential is also positive. Thus, the potential is an even-degree polynomial with positive leading coefficient, which means that it is attractive at (least at) long distances, and this justifies the form of the ansatz solution (8) for the energy eigenfunctions.

Also, since it is of even degree, the potential can be of even parity but cannot be of odd parity.

If we demand that the potential be of even parity, then **[1, 2]** the energy eigenfunctions (8) have definite parity, i.e. they are either even or odd functions.

This means that the polynomial g has definite parity – otherwise the energy eigenfunctions (8) cannot have definite parity – and since g is an even-degree polynomial, it can only be of even parity. Then, g' is of odd parity, and thus g'^2 is of even parity, as is g''.

Thus, the polynomial q is also of even parity, since from (14) it is written as

$$q_{2(m-1)}\left(\tilde{x};n\right)={g_{2m}}'^2\left(\tilde{x}\right)+{g_{2m}}''\left(\tilde{x}\right)+\tilde{E}-\tilde{V}\left(\tilde{x};m,n\right),$$

i.e. it is a sum of even-parity polynomials.

Thus, if the potential is of even parity, the polynomial q is also of even parity.

The ground-state energy

For $n=0$, $p_0\left(\tilde{x}\right)=1$, as it is monic, and thus $p_0'\left(\tilde{x}\right)=p_0''\left(\tilde{x}\right)=0$.

Then, (12) gives

$$q_{2(m-1)}\left(\tilde{x};0\right)=0 \quad (16)$$

for every $m=1,2,...$

Also, since $p_0\left(\tilde{x}\right)=1$, for $n=0$ (8) gives

$$\psi\left(\tilde{x};m,0\right)=A_0\exp\left(g_{2m}\left(\tilde{x}\right)\right)$$

The wave function $\psi\left(\tilde{x};m,0\right)$ has no zeros, and thus it is the ground-state wave function, which we'll denote by $\psi_0\left(\tilde{x};m\right)$, i.e.

$$\psi_0\left(\tilde{x};m\right)=A_0\exp\left(g_{2m}\left(\tilde{x}\right)\right) \quad (17)$$

The respective potential is given by (14) for $n=0$. Thus, using (16), we have

$$\tilde{V}\left(\tilde{x};m,0\right)={g_{2m}}'^2\left(\tilde{x}\right)+{g_{2m}}''\left(\tilde{x}\right)+\tilde{E}_{\min} \quad (18)$$

Thus, (17) is the ground-state wave function of the potential (18), with the energy $\tilde{E}_{\min}$ being the ground-state energy.

From (18), we see that, for any given polynomial g, to determine the potential uniquely, we need its value at a reference point, which also fixes the ground-state energy.

For polynomial potentials, the reference point can be the point $\tilde{x}=0$. If the value of the potential at $\tilde{x}=0$ is known, i.e. if $\tilde{V}(0;m,0)$ is known, then from (18) we calculate the ground-state energy $\tilde{E}_{\min}$, assuming that $g_{2m}(\tilde{x})$ is known, and the potential is determined uniquely.

Since a constant term in the potential does not have any physical significance, we can set $\tilde{V}(0;m,0)=0$, without losing generality, and then (18) gives

$$\tilde{E}_{\min}=-\left(g_{2m}'^{2}(0)+g_{2m}''(0)\right) \quad (19)$$

Since g is a real polynomial of degree $2m$ with negative leading coefficient, it will have the general form

$$g_{2m}(\tilde{x})=-g_{2m}{}^{2}\tilde{x}^{2m}+\sum_{k=0}^{2m-1}g_{k}\tilde{x}^{k} \quad (20)$$

with $m\geq 1$.

Thus, its first and second derivatives are, respectively,

$$g_{2m}'(\tilde{x})=-2mg_{2m}{}^{2}\tilde{x}^{2m-1}+\sum_{k=1}^{2m-1}kg_{k}\tilde{x}^{k-1}$$

$$g_{2m}''(\tilde{x})=-2m(2m-1)g_{2m}{}^{2}\tilde{x}^{2m-2}+\sum_{k=2}^{2m-1}k(k-1)g_{k}\tilde{x}^{k-2}$$

Then

$$g_{2m}'(0)=g_{1}$$

and

$g_{2m}''(0)=2g_{2}$ if $m\geq 2$, or $-2g_{2}{}^{2}$ if $m=1$.

Substituting into (19), we obtain

$$\tilde{E}_{\min}=-\left(g_{1}{}^{2}+2g_{2}\right) \quad (21)$$

if $m\geq 2$, or

$$\tilde{E}_{\min}=-\left(g_{1}{}^{2}-2g_{2}{}^{2}\right) \quad (22)$$

if $m=1$.

In any case, the ground-state energy depends only on the linear and quadratic coefficients of the exponential polynomial factor g.

The constant term g_0 of the polynomial g can also be incorporated into the normalization constant of the wave function (8), as it corresponds to the constant factor $\exp(g_0)$.

Therefore, the ground-state energy depends only on the coefficients of the two smallest powers in the polynomial g.

The case m=1 – The quantum harmonic oscillator

For $m=1$, $\deg(q)=0$, and thus the polynomial $q_0\left(\tilde{x};n\right)$ does not depend on $\tilde{x}$, it is an n-dependent constant, i.e.

$$q_0\left(\tilde{x};n\right)=q_0\left(n\right)$$

For $m=1$, the polynomial g is of second degree, i.e. it is a trinomial, which becomes a binomial if we incorporate its constant term into the normalization constant of the wave function (8).
Thus, if the potential is symmetric (i.e. of even parity), the polynomial g must also be of even parity – see the section “The potential” – and thus its linear term vanishes, and then we are left only with its quadratic term, i.e. $g_2\left(\tilde{x}\right)=-g_2{}^2\tilde{x}^2$, with $g_2\neq 0$.

For $g_2{}^2=-\frac{1}{2}$, $g_2\left(\tilde{x}\right)=-\frac{1}{2}\tilde{x}^2$ and $g_2{}'\left(\tilde{x}\right)=-\tilde{x}$.

Thus, for $m=1$ and $g_2{}^2=-\frac{1}{2}$, (12) takes the form

$$p_n''\left(\tilde{x}\right)-2\tilde{x}p_n'\left(\tilde{x}\right)=-q_0\left(n\right)p_n\left(\tilde{x}\right)$$

or

$$p_n''\left(\tilde{x}\right)-2\tilde{x}p_n'\left(\tilde{x}\right)+q_0\left(n\right)p_n\left(\tilde{x}\right)=0$$

In the next section, we’ll show that $q_0\left(n\right)=2n$ *, and thus the previous differential equation is written as

$$p_n''\left(\tilde{x}\right)-2\tilde{x}p_n'\left(\tilde{x}\right)+2np_n\left(\tilde{x}\right)=0\text{, with } n=0,1,\ldots$$

This is the Hermite differential equation and its polynomial solutions are the Hermite polynomials $H_n\left(x\right)$, i.e.

$$p_n\left(\tilde{x}\right)=H_n\left(\tilde{x}\right)$$

Besides, the second derivative of $g_2\left(\tilde{x}\right)=-\frac{1}{2}\tilde{x}^2$ is $g_2{}''\left(\tilde{x}\right)=-1$, and thus, substituting into the expression of the potential the first and second derivatives of $g_2\left(\tilde{x}\right)$ and that $q_0\left(\tilde{x};n\right)=q_0\left(n\right)=2n$, we obtain

$$\tilde{V}\left(\tilde{x};1,n\right)=\left(-\tilde{x}\right)^2-1-2n+\tilde{E}=\tilde{x}^2-\left(2n+1\right)+\tilde{E}$$

That is

$$\tilde{V}\left(\tilde{x};1,n\right)=\tilde{x}^2+\tilde{E}-\left(2n+1\right)$$

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* Observe that $q_0\left(0\right)=0$, as (16) dictates.

By means of (4) and (5), the previous equation is written as

$$\frac{2ml^2V\left(\tilde{x};1,n\right)}{\hbar^2}=\tilde{x}^2+\frac{2ml^2E}{\hbar^2}-\left(2n+1\right)\Rightarrow V\left(\tilde{x};1,n\right)=\frac{\hbar^2}{2ml^2}\tilde{x}^2+E-\left(n+\frac{1}{2}\right)\frac{\hbar^2}{ml^2}$$

Using (2), the last equation becomes

$$V\left(x;1,n\right)=\frac{\hbar^2}{2ml^4}x^2+E-\left(n+\frac{1}{2}\right)\frac{\hbar^2}{ml^2}\ (23)$$

For $\frac{\hbar^2}{2ml^4}=\frac{1}{2}m\omega^2$, the potential (23) is a harmonic oscillator potential plus the constant term $E-\left(n+\frac{1}{2}\right)\frac{\hbar^2}{ml^2}$, which is physically unimportant and setting it to zero yields

$$E_n=\left(n+\frac{1}{2}\right)\frac{\hbar^2}{ml^2}\ (24)$$

Besides, from the relation $\frac{\hbar^2}{2ml^4}=\frac{1}{2}m\omega^2$, we obtain

$$\frac{\hbar^2}{ml^4}=m\omega^2\Rightarrow\frac{1}{l^4}=\left(\frac{m\omega}{\hbar}\right)^2\Rightarrow l^4=\left(\frac{\hbar}{m\omega}\right)^2\Rightarrow l^2=\frac{\hbar}{m\omega}$$

Since $l>0$,

$$l=\sqrt{\frac{\hbar}{m\omega}}\ (25)$$

The quantity $\sqrt{\frac{\hbar}{m\omega}}$ is the length scale of the quantum harmonic oscillator (QHO). As noted in the beginning, the constant l is related – here, it is equal – to the length scale of the examined system, which, in this case, is the QHO.
By means of (25), (24) becomes

$$E_n=\left(n+\frac{1}{2}\right)\frac{\hbar^2}{m\frac{\hbar}{m\omega}}=\left(n+\frac{1}{2}\right)\hbar\omega$$

That is

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega\ (26)$$

with $n=0,1,...$
This is the spectrum of the QHO.
The potential (23) is then a harmonic oscillator potential, and it is independent of n, i.e. $V\left(x;1,n\right)=V\left(x\right)$.
The energy eigenfunctions (8), which are now the energy eigenfunctions of the QHO, take the form

$$\psi_n(\tilde{x}) = A_n H_n(\tilde{x}) \exp\left(-\frac{1}{2}\tilde{x}^2\right),$$

or, since $\tilde{x} = \frac{x}{\sqrt{\frac{\hbar}{m\omega}}} = \sqrt{\frac{m\omega}{\hbar}}x$,

$$\psi_n(x) = A_n H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \quad (27)$$

with $n = 0, 1, \ldots$ and the constants A_n are the normalization constants.

The leading coefficient of q

Returning to the equation (12), i.e.

$$p_n''(\tilde{x}) + 2g_{2m}'(\tilde{x})p_n'(\tilde{x}) = -q_{2(m-1)}(\tilde{x};n)p_n(\tilde{x}),$$

the polynomial $g_{2m}(\tilde{x})$ is of degree $2m$, with leading coefficient $-g_{2m}{}^2$.

Thus, the polynomial $g_{2m}'(\tilde{x})$ is of degree $2m-1$, with leading coefficient $-2mg_{2m}{}^2$.

Besides, the polynomial $p_n(\tilde{x})$ is of degree n and it is monic, i.e. its leading coefficient is 1.

Thus, the polynomial $p_n'(\tilde{x})$ is of degree $n-1$, with leading coefficient n.

Then, the degree of $2g_{2m}'(\tilde{x})p_n'(\tilde{x})$ is the sum of the degrees of $g_{2m}'(\tilde{x})$ and $p_n'(\tilde{x})$, i.e. it is $2m+n-2$, and its leading coefficient is $2\left(-2mg_{2m}{}^2\right)n = -4mng_{2m}{}^2$.

Also, the polynomial $p_n''(\tilde{x})$ is of degree $n-2 < 2m+n-2 \quad (m \geq 1)$.

Therefore, the polynomial $p_n''(\tilde{x}) + 2g_{2m}'(\tilde{x})p_n'(\tilde{x})$ is of degree $2m+n-2$, with leading coefficient $-4mng_{2m}{}^2$.

On the other hand, the polynomial $q_{2(m-1)}(\tilde{x};n)$ is of degree $2(m-1)$ with leading coefficient $q_{2(m-1)}(n)$.

Therefore, the polynomial $-q_{2(m-1)}(\tilde{x};n)p_n(\tilde{x})$ is of degree $2(m-1)+n = 2m+n-2$, with leading coefficient $-q_{2(m-1)}(n)$, as $p_n(\tilde{x})$ is monic.

Thus, equating the coefficients of the terms of degree $2m+n-2$ in both sides of (12), we obtain

$$-4mng_{2m}{}^2 = -q_{2(m-1)}(n)$$

or

$$q_{2(m-1)}(n) = 4mng_{2m}{}^2 \quad (28)$$

with $m = 1, 2, \ldots$ and $n = 0, 1, \ldots$

We see that the leading coefficient of q is equal to two times the degree of g (2m) times the degree of p (n) times the opposite of the leading coefficient of g.

If $m=1$ and $-g_{2m}{}^{2}=-\frac{1}{2}$, i.e. $g_{2m}{}^{2}=\frac{1}{2}$, (28) gives $q_0\left(n\right)=2n$, with $n=0,1,\ldots$, which is the case of the QHO we examined in the previous section.

If $n=0$, then, from (28), we obtain that the leading coefficient of the polynomial q is zero, which means that the polynomial q is zero, i.e. $q_{2(m-1)}\left(\tilde{x};0\right)=0$, as (16) dictates.

For a given polynomial $g_{2m}\left(\tilde{x}\right)$, m and $g_{2m}{}^{2}$ are fixed, and then from (28) we see that the leading coefficient of $q_{2(m-1)}\left(\tilde{x};n\right)$ depends only on the degree of $p_n\left(\tilde{x}\right)$, not on its coefficients, and thus

all different – i.e. all linearly independent – polynomials p of the same degree n, which satisfy (12), correspond to polynomials q with the same leading coefficient.

This is a consequence of the polynomial p being monic.

The constant term of q

We showed that the ansatz wave function (8) describes a bound eigenstate of energy $\tilde{E}$ of the polynomial potential (14), if the differential equation (12) is satisfied.

As we did for the ground-state energy – see the section "The ground-state energy" – we set the constant term of the polynomial potential (14) to zero. In other words, we demand that the potential vanish at $\tilde{x}=0$, i.e.

$$\tilde{V}\left(0;m,n\right)=0 \quad (29)$$

Using (14), the value of the potential at $\tilde{x}=0$ is

$$\tilde{V}\left(0;m,n\right)=g_{2m}'^{2}\left(0\right)+g_{2m}''\left(0\right)-q_{2(m-1)}\left(0;n\right)+\tilde{E}$$

But $q_{2(m-1)}\left(0;n\right)$ is equal to the constant term $q_0\left(n\right)$ of the polynomial q, and thus

$$\tilde{V}\left(0;m,n\right)=g_{2m}'^{2}\left(0\right)+g_{2m}''\left(0\right)-q_0\left(n\right)+\tilde{E}$$

Then, the condition (29) is written as

$$0=g_{2m}'^{2}\left(0\right)+g_{2m}''\left(0\right)-q_0\left(n\right)+\tilde{E}$$

or

$$q_0\left(n\right)=\tilde{E}+\left(g_{2m}'^{2}\left(0\right)+g_{2m}''\left(0\right)\right) \quad (30)$$

with $m=1,2,\ldots$ and $n=0,1,\ldots$

For a given polynomial $g_{2m}\left(\tilde{x}\right)$, the term $g_{2m}'^{2}\left(0\right)+g_{2m}''\left(0\right)$ is a constant, and thus

the constant term of the polynomial q is energy-dependent, and particularly, it is equal to the energy of the eigenstate plus a constant.

For polynomial potentials, which are smooth functions, the number of (real) zeros of an energy eigenfunction determines **[3]** the excitation of the respective eigenstate.
In our case, where the zeros of the wave function (8) are the zeros of the polynomial $p_n(\tilde{x})$, if the number of zeros of $p_n(\tilde{x})$ is r, then the eigenstate described by the wave function (8) is the rth-excited state of the respective potential, and its energy is $\tilde{E} \equiv \tilde{E}_r$.
Besides, since the polynomial q is of degree $2(m-1)$, it will have the form

$$q_{2(m-1)}(\tilde{x};n) = \sum_{k=0}^{2m-2} q_k(n)\tilde{x}^k \quad (31)$$

with $m = 1,2,...$ and $n = 0,1,...$
Using (31), the potential (14) takes the form

$$\tilde{V}(\tilde{x};m,n) = g_{2m}'^2(\tilde{x}) + g_{2m}''(\tilde{x}) - \left(q_{2m-2}(n)\tilde{x}^{2m-2} + q_{2m-3}(n)\tilde{x}^{2m-3} + ... + q_1(n)\tilde{x} + q_0(n)\right) + \tilde{E} =$$
$$= g_{2m}'^2(\tilde{x}) + g_{2m}''(\tilde{x}) - q_{2m-2}(n)\tilde{x}^{2m-2} - \left(q_{2m-3}(n)\tilde{x}^{2m-3} + ... + q_1(n)\tilde{x}\right) + \tilde{E} - q_0(n)$$

Substituting (28) and (30) into the last equality, we obtain – if $m \neq 1$ –

$$\tilde{V}(\tilde{x};m,n) = g_{2m}'^2(\tilde{x}) + g_{2m}''(\tilde{x}) - 4mng_{2m}{}^2 - \left(q_{2m-3}(n)\tilde{x}^{2m-3} + ... + q_1(n)\tilde{x}\right) - \left(g_{2m}'^2(0) + g_{2m}''(0)\right)$$

or

$$\tilde{V}(\tilde{x};m,n) = g_{2m}'^2(\tilde{x}) - g_{2m}'^2(0) + g_{2m}''(\tilde{x}) - g_{2m}''(0) - 4mng_{2m}{}^2 - \left(q_{2m-3}(n)\tilde{x}^{2m-3} + ... + q_1(n)\tilde{x}\right) \quad (32)$$

We see that the potential (32) does not depend on the constant term of the polynomial q. The condition (29), i.e. the condition that the physically unimportant constant term of the potential vanishes, removes the dependence of the potential on the constant term of q.

If $m = 1$, then $\deg(q) = 0$, which means that $q_0(\tilde{x};n) = q_0(n)$.
In this case, the potential takes the form

$$\tilde{V}(\tilde{x};1,n) = g_{2m}'^2(\tilde{x}) - g_{2m}'^2(0) + g_{2m}''(\tilde{x}) - g_{2m}''(0),$$

and it is independent of n, i.e. $V(x;1,n) = V(x)$.
As we saw in the section “The case m=1”, this is the case of the QHO.

The presence of intermediate terms in q as a non-solvability condition

We see that the potential (32) depends linearly on the intermediate coefficients of q, i.e. on the coefficients of degree from 1 up to $2m-3$ in $\tilde{x}$.

The intermediate coefficients of q depend* on the energy $\tilde{E}$ and they change when the energy changes.

Since the energy spectrum is non-degenerate, each energy corresponds to only one eigenstate.

Thus, the intermediate coefficients of q change when the eigenstate changes, and thus, from (32), we see that the potential also changes when the eigenstate changes.

Therefore, if the polynomial q has intermediate terms, we can find only one eigenstate for each potential (32).

If finding only one eigenstate is assumed a trivial case of quasi-exact solvability, we conclude that

the presence of intermediate coefficients in the polynomial q results in the respective potential (32) being neither exactly nor quasi-exactly solvable.

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* Consider a polynomial $p_n\left(\tilde{x}\right)$ of degree n, which satisfies the differential equation (12), i.e.

$$p_n''\left(\tilde{x}\right)+2g_{2m}'\left(\tilde{x}\right)p_n'\left(\tilde{x}\right)=-q_{2(m-1)}\left(\tilde{x};n\right)p_n\left(\tilde{x}\right).$$

Since $p_n\left(\tilde{x}\right)$ is of degree n, it can have up to n (real) zeros. Assuming that it has r zeros, then, in general, $0\le r\le n$. For more “transparency”, let us denote the polynomial $p_n\left(\tilde{x}\right)$ by $p_n\left(\tilde{x};r\right)$. Then, the wave function (8) is written as

$$\psi_r\left(\tilde{x};m,n\right)=A_r p_n\left(\tilde{x};r\right)\exp\left(g_{2m}\left(\tilde{x}\right)\right),$$

and it describes the rth-excited state of the respective potential $\tilde{V}\left(\tilde{x};m,n\right)$, which we’ll denote by $\left|r\right\rangle$.

The number of zeros of a fixed-degree polynomial depends solely on its coefficients, and thus the value of r is determined by the coefficients of $p_n\left(\tilde{x};r\right)$.

At the same time, the number of zeros of $p_n\left(\tilde{x};r\right)$, i.e. the value of r, determines the excitation of the eigenstate $\left|r\right\rangle$, and thus its energy.

Therefore, the coefficients of $p_n\left(\tilde{x};r\right)$ determine the energy of the eigenstate $\left|r\right\rangle$, or, in other words, the coefficients of $p_n\left(\tilde{x};r\right)$ – except the leading one, which is always 1 – are energy-dependent.

By equating the coefficients of the same-degree terms in both sides of the differential equation (12), we obtain equations relating the coefficients of $p_n\left(\tilde{x};r\right)$ to the intermediate coefficients of $q_{2(m-1)}\left(\tilde{x};n\right)$, and since the coefficients of $p_n\left(\tilde{x};r\right)$, except the leading one, are energy-dependent, so are the intermediate coefficients of $q_{2(m-1)}\left(\tilde{x};n\right)$.

Thus, unlike the leading coefficient of $q_{2(m-1)}\left(\tilde{x};n\right)$, which depends only on the degree of $p_n\left(\tilde{x};r\right)$, the intermediate coefficients of $q_{2(m-1)}\left(\tilde{x};n\right)$ depend on the coefficients of $p_n\left(\tilde{x};r\right)$, and thus they depend on the energy too, as does the constant term of $q_{2(m-1)}\left(\tilde{x};n\right)$ – which, as seen from (30), is equal to the energy plus a constant – but the constant term of $q_{2(m-1)}(\tilde{x};n)$ does not appear in the potential.

Solvability of polynomial potentials

If $m=1$, then $\deg(q)=0$, which means that the polynomial q is an n-dependent constant, i.e. $q_0(\tilde{x};n)=q_0(n)$, which is also energy-dependent, as seen from (30).

As we saw, in this case, the potential does not depend on n. Thus, not only the polynomials p of the same degree, but also the polynomials p of different degrees, correspond to the same potential, provided that they satisfy the differential equation (12).

This means that the respective potential – it is one because it does not depend on n – is exactly solvable.

As we saw, this is the case of the QHO, which is indeed an exactly solvable system.

If $m=2$, then $\deg(q)=2$, and the polynomial q has the form

$$q_2(\tilde{x};n)=q_2(n)\tilde{x}^2+q_1(n)\tilde{x}+q_0(n),$$

where the leading coefficient $q_2(n)$ is given by (28) for $m=2$, i.e.

$$q_2(n)=8ng_4{}^2.$$

Thus

$$q_2(\tilde{x};n)=8ng_4{}^2\tilde{x}^2+q_1(n)\tilde{x}+q_0(n)$$

In this case, we have only one intermediate term, the linear term $q_1(n)\tilde{x}$, which is of odd parity and thus it vanishes if the polynomial $q_2(\tilde{x};n)$ is of even parity.

As showed in the section "The potential", if the potential is symmetric (i.e. of even parity), the polynomial q is also of even parity, and thus all coefficients of its odd powers in $\tilde{x}$ must vanish.

For the case $m=2$, this means that $q_1(n)=0$, and thus the polynomial q has no intermediate terms, it is a binomial with a leading and a constant term.

Then, the respective potential (32) is of degree 6 – as seen from (15) – and it depends on n, i.e. (32) gives a series of n-dependent potentials.

In this case, only the polynomials p of the same degree n correspond to the same potential, meaning that the respective n-dependent potentials, given by (32), are quasi-exactly solvable.

This is the case of the sextic anharmonic oscillator, which is indeed **[5, 6]** a quasi-exactly solvable system.

In a following paper, we'll quasi-exactly solve the case $m=2$ using the polynomial q.

If $m\geq 3$, then $\deg(q)=2(m-1)\geq 4$, and thus the polynomial q contains at least one intermediate term of even parity – the term $q_2(n)\tilde{x}^2$ – which remains even if the potential is symmetric, and that makes the respective potentials non-solvable **[6]**.

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