A TEMPORAL DYNAMICS: A GENERALISED NEWTONIAN AND WAVE MECHANICS

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Abstract: In this work we discuss the possibility of reconciling quantum mechanics with classical mechanics by formulating a temporal dynamics, which is a dynamics caused by the rate of change of time with respect to distance. First, we show that a temporal dynamics can be derived from the time dilation formula in Einstein's theory of special relativity. Then we show that a short-lived time-dependent force derived from a dynamical equation that is obtained from the temporal dynamics in a 1-dimensional temporal manifold can be used to describe Bohr's postulates of quantum radiation and quantum transition between stable orbits in terms of classical dynamics and differential geometry. We extend our discussions on formulating a temporal dynamics to a 3-dimensional temporal manifold. With this generalisation we are able to demonstrate that a sub-quantum dynamics is a classical dynamics.

1. Introduction

In 1935, Einstein and his co-authors published a paper that raises the question of whether quantum mechanical description of physical reality can be considered complete [1]. Even though their non-locality arguments have been disproved by theory and experiments [2,3], fundamentally and epistemologically, their question remains unresolved. Despite the fact that the modern quantum theory is based on a more sophisticated and advanced mathematical formulation, many paradoxical and counter-intuitive aspects in physics still remain. This situation has an epistemological consequence that underpins the very foundation of quantum physics. If we do not simply embrace the current probabilistic interpretation of the quantum mechanical formalism then we need to find a way to interpret quantum physics. On the other hand, if general relativity could account for all forces and contain in it the quantum description of physical reality then it would be a complete theory. Because the dynamical foundation of quantum mechanics is based on classical mechanics, we may ask a question of whether the classical mechanics itself is incomplete. In this work we attempt to answer this question by formulating a new dynamics in addition to Newtonian dynamics to explain quantum phenomena. We will discuss the possibility of reconciling quantum mechanics with classical mechanics by formulating a temporal dynamics, which is a dynamics produced by the rate of change of time with respect to distance. In Section 2, we show that a temporal dynamics can be derived from the time dilation formula in Einstein's theory of special relativity [4,5]. In Section 3, we show that a short-lived time-dependent force derived from a dynamical equation that is obtained from the temporal dynamics in a temporal manifold can be used to describe Bohr's postulates of quantum radiation and quantum transition between stable orbits in terms of classical dynamics and differential geometry. Despite the fact that Bohr's model of a hydrogen-like atom is very basic compared to the modern theory of quantum mechanics, if we attempt to look for a connection between classical and quantum physics, we need to consider it first, because Bohr's model itself is fundamentally classical, except for Bohr's postulates [6]. These postulates assume the existence of discrete entities that could not be explained in terms of classical mechanics itself. In order to describe the dynamics of the electron in a deterministic manner we need to formulate a new dynamics that can be used to describe how the electron transits between stable orbits, because quantum mechanics based on Newtonian mechanics cannot explain this type of dynamical characteristics of the electron. Time has been considered to be part of the fundamental structure of space-time, either absolute, when time is assumed to flow with the same rate in all coordinate systems, or relative, when the rate of the flow of time is associated with relative motions. In this work, we will show that if time exhibits a dynamic character then a temporal dynamics will yield new physical insights that can be used to account for problems in quantum physics, such as the processes of quantum transition and quantum radiation in the Bohr's model of a hydrogen-like atom in terms of classical dynamics. In Section 4, we extend our discussions on formulating a temporal dynamics to a 3-dimensional temporal manifold. With this generalisation we are able to demonstrate that a sub-quantum dynamics is a classical dynamics.

For references, a summary of the postulates of Bohr's model of a hydrogen-like atom may be summarised as follows:

- The centripetal force required for the electron to orbit the nucleus in a stable circle is the Coulomb force $F = kq^2/r^2$. Using Newton's second law, $ma = F$, we obtain
  $$\frac{mv^2}{r} = \frac{kq^2}{r^2} \tag{1}$$
- The permissible orbits are those that satisfy the condition that the angular momentum of the electron equals $n\hbar$, that is
  $$mv_n r_n = n\hbar \tag{2}$$
  where the subscript *n* in v_n and r_n denotes the *n*th orbit.
- When the electron moves in one of the stable orbits it does not radiate. However, it will radiate when it makes a transition between them.

From Equations (1) and (2) we can derive the following

$$r_n = n^2 a_0 \tag{3}$$

$$v_n = \frac{\hbar}{mna_0} \tag{4}$$

where $a_0 = \hbar^2/mk\,q^2$ is the Bohr radius. As in classical mechanics, using the kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}kq^2/r$ and the potential energy $V = -kq^2/r$, the total energy $E = T + V$ is calculated as

$$E = -\frac{1}{2}\frac{kq^2}{r}. \tag{5}$$

Using Equation (3), the energy levels of the permissible orbits are

$$E_n = -\frac{1}{2}\frac{kq^2}{n^2 a_0}. \tag{6}$$

It is also noted that, as in circular motion, the relationship between velocity v, radius r and angular frequency ω is given by $\omega = v/r$, for stable orbits of the electron, from Equations (3) and (4) we obtain

$$\omega_n = \frac{v_n}{r_n} = \frac{\hbar}{ma_0^2 n^3} \tag{7}$$

It is seen from Equation (7) that when $n \to \infty$, $\omega_n \to 0$.

2. The concept of time in Special Relativity

In the next section we will formulate a temporal dynamics in terms of the rate of change of time of a physical object, similar to Newtonian dynamics. However, in this section we want to show that such temporal dynamics can be derived from Einstein's theory of special relativity. Consider two inertial reference systems S and S' with coordinates (x_1, x_2, x_3, t) and (x_1', x_2', x_3', t'). If the system S' moves relative to the system S along the x_1-axis with the velocity $\mathbf{v}$, then according to Newtonian physics the transformation of the coordinates of the two systems is the Galilean transformation [7]

$$x_1 = x_1' + vt, \qquad x_2 = x_2', \qquad x_3 = x_3', \qquad t = t' \tag{8}$$

The concept of absolute time in Newtonian physics was changed when Einstein proposed his theory of special relativity. Instead of the Galilean transformation, in special relativity the transformation of the coordinates adopts the Lorentz transformation

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad x_2 = x_2', \qquad x_3 = x_3', \qquad t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{9}$$

From Equations (9), for infinitesimal changes, the formulas for the length contraction and the time dilation are derived as

$$\frac{dx_1}{dx_1'} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{10}$$

$$\frac{dt}{dt'} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{11}$$

In the following we will consider the system S' as a moving particle in a reference system S. During its motion, if the particle interacts with other physical objects, as in the case of Compton's scattering between an electron and a photon, then the particle's direction and speed will change. From Equation (11), the proper time interval will change during the interaction if we assume the time of the reference frame still flows at a constant rate. In this case we have a rate of change of the proper time of the particle. However, if the speed of the particle remains small compared to the speed of light after this short period of interaction then the proper time flow will still be the same as that of the reference frame. Inversely, if we assume the proper time to flow at a constant rate during the interaction then the time interval of the time of the reference frame will change, and in this case we have the rate of change of the time of the reference frame. Furthermore, we will consider the case when the interaction happens only in a very short duration of time, therefore, we assume that the form in Equation (11) still remains valid even though the velocity of the particle changes continuously. This can be considered as an extension of the postulate of relativity. It should be mentioned here that this kind of extension of the postulate of relativity had led Einstein to develop his general theory of relativity. The extended principle of relativity is stated as: "*The law of physics must be of such a nature that they apply to systems of reference in any kind of motion*" [4]. With the assumption that the relation given by Equation (11) remains valid for a continuous change of velocity, we obtain the second rate of change of the time of the reference frame with respect to the proper time as

$$\frac{d^2t}{dt'^2} = \frac{\mathbf{v}.d\mathbf{v}/dt}{c^2\left(1 - \frac{v^2}{c^2}\right)^{3//2}} \frac{dt}{dt'}. \tag{12}$$

Using Equation (11) and multiplying both sides of Equation (12) by the relativistic mass of the particle, $m = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the particle's rest mass, we obtain

$$m_0 \frac{d^2t}{dt'^2} = \frac{\mathbf{v}.m_0 d\mathbf{v}/dt}{c^2\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \tag{13}$$

Equation (13) can be re-written as

$$m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^2 \frac{d^2t}{dt'^2} = \mathbf{v}.\mathbf{F} \tag{14}$$

where $\mathbf{F} = m_0\, d\mathbf{v}/dt$ is a Newtonian force that is responsible for the rate of change of the time of the particle. The dynamical equation given by Equation (14) needs to be reformulated in terms of variables in the reference system in order to be applied. For example, if the force $\mathbf{F}$ is the Coulomb force that acts along the radial direction given by the position vector $\mathbf{r}(t)$

and the particle moves in circular orbits, such as the motion of the electron in Bohr's model, then $\mathbf{v}.\mathbf{F} = 0$. In the case of Bohr's model, the speed of the electron in a stationary orbit is constant therefore according to Equation (11) the time rates of the reference frame time and the proper time are proportional to each other. For the *n*th orbit, we have $v_n = \hbar/mna_0$, therefore $dt = dt'/\sqrt{1-(\hbar/mna_0\, c)^2}$. We assume further that the proper time is the same for all stationary orbits. If the electron transits from one stationary orbit to another then it must adjust its time rate as measured by an observer in a reference frame. In order to do this it would need a second rate of change of its time with respect to distance along the path between the two orbits. From this result, it is seen that the second rate of change of the time of the particle occurs only when the particle departs from its circular motion. We will discuss this problem further in the following.

Assume the transition of the electron between two stationary orbits is the radial motion. In this case we have $\mathbf{v}.m_0 d\mathbf{v}/dt = \mathbf{v}.\mathbf{F} = \pm vF$. In the case of Bohr's model of a hydrogen-like atom, this happens when the electron transits from one stationary orbit to another. When the electron moves out of a stationary orbit it is still under the influence of Coulomb force, therefore we can assume the force **F** in Equation (14) is the Coulomb force. Since the motion of the electron now is along the positive direction of the radial position vector and the Coulomb force is in opposite direction, we can write $\mathbf{v}.m_0 d\mathbf{v}/dt = \mathbf{v}.\mathbf{F} = -vF$. Then Equation (14) can be re-written as

$$-m_0c^2v\left(1-\frac{v^2}{c^2}\right)^2\frac{d^2t}{v^2dt'^2} = F \tag{15}$$

Using Equation (11), we can write $v^2dt'^2 = (1-v^2/c^2)v^2dt^2 = (1-v^2/c^2)dr^2$. With this result, Equation (15) becomes

$$-m_0c^2v\left(1-\frac{v^2}{c^2}\right)\frac{d^2t}{dr^2} = F \tag{16}$$

If we define a new physical quantity

$$D = -m_0c^2v\left(1-\frac{v^2}{c^2}\right) \tag{17}$$

then Equation (16) takes the form

$$D\frac{d^2t}{dr^2} = F \tag{18}$$

The dynamical equation given in Equation (18) has the form of Newton's second law of motion. However, in this case the roles of space and time are reversed. Equation (18) plays the crucial role in the new temporal dynamics. The new physical quantity D plays the role of the inertial mass of a particle in Newtonian mechanics. As in the case of special relativistic dynamics, this new physical quantity also depends on the velocity of the particle.

3. A temporal dynamics in one dimension

In Newtonian physics, time is an independent 1-dimensional Euclidean continuum, which is an essential component of the fundamental structure of the nature. Time is considered to be absolute and its properties are independent of any system of reference. The time intervals of time between two events are identical for all reference systems. In classical physics, the dynamics of a particle is a study of its motion in space with respect to time under the action of forces, where time is considered to be universal and to flow at a constant rate. Because time is considered to be 1-dimensional, therefore in the following we will discuss the dynamics of a particle in 1-dimension and extend the discussion to a 3-dimensional temporal manifold in the next section. Consider the motion of a particle in a straight line under the action of a force $\mathbf{F}$. Its displacement from an origin is represented by the position vector $\mathbf{r}$. In order to study the dynamics of the particle, we divide the 1-dimensional Euclidean time into equal intervals Δt and measure the distance Δr_i that the particle has travelled in the ith time interval. In this case we define the rate of change of the displacement of the particle as $\Delta r_i/\Delta t$. If these rates are equal then we say the particle is moving with a constant velocity $\mathbf{v}$. Let $\Delta t \to 0$, we have $\mathbf{v} = d\mathbf{r}/dt$. If the rates are different then we say the particle is moving with an acceleration $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$. When physical entities related to the particle, such as mass m and charge q, are introduced then we can formulate a classical dynamics such as Newtonian dynamics

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}. \qquad (20)$$

In the following we will term Newtonian dynamics as the spatial dynamics, in contrast to the temporal dynamics that we formulate as follows. We also consider the motion of the particle along a straight line as described above. Instead of dividing the time line into equal intervals, we divide the spatial line into equal spatial intervals Δr. After the particle moves through the ith spatial interval we measure the corresponding time interval Δt_i that the particle has taken to move through that distance. In this case we define the rate of change of the temporal displacement of the particle with respect to distance as $\Delta t_i/\Delta r$. The temporal displacement from a temporal origin is represented by the temporal vector $\mathbf{t}$. If these rates are equal then we say the particle is moving with a constant temporal velocity $\mathbf{v}_T$. Let $\Delta r \to 0$, we have $\mathbf{v}_T = d\mathbf{t}/dr$. If the rates are different then we say the particle is moving with a temporal acceleration $\mathbf{a}_T = d\mathbf{v}_T/dr = d^2\mathbf{t}/dr^2$. If the temporal dynamics of the particle is also caused by a force $\mathbf{F}$ then we can formulate a temporal dynamics similar to Newtonian dynamics

$$D\frac{d^2\mathbf{t}}{dr^2} = \mathbf{F}. \qquad (21)$$

where the physical quantity D, which plays the role of the inertial mass m in Newtonian mechanics needs to be determined. The quantity D has the dimension of the quantity given by Equation (17), i.e., $[D] = [ML^3T^{-3}]$. The form given by Equation (21) is similar to Newton's second law of motion given by Equation (20), except for the roles of space and time are reversed.

It can be shown that the dynamics of a particle obtained from Equation (21) is equivalent to the dynamics obtained from Equation (20). For example, consider the simple harmonic motion with the force in Equation (20) is given by the Hooke's law

$$\mathbf{F} = -k\mathbf{r}. \tag{22}$$

Using Newton's second law of motion

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}, \tag{23}$$

we obtain the dynamic equation for the simple harmonic motion

$$m\frac{d^2\mathbf{r}}{dt^2} = -k\mathbf{r}. \tag{24}$$

Since the motion is 1-dimensional, Equation (24) can be re-written in the form using only the magnitude of the position vector as

$$m\frac{d^2r}{dt^2} = -kr. \tag{25}$$

A general solution of Equation (25) with the initial condition $t = 0, r = 0$ is given by

$$r = A\sin\left(\sqrt{\frac{k}{m}}t\right). \tag{26}$$

The inverse obtained from Equation (26) is

$$t = \sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{r}{A}\right). \tag{27}$$

From Equation (27) we can derive the second rate of change of the time *t* with respect to *r* as

$$\frac{d^2t}{dr^2} = \sqrt{\frac{m}{k}}\frac{r}{(A^2 - r^2)^{3/2}}. \tag{28}$$

If we multiply both sides of Equation (28) by a dimensional constant *D* so that the right hand side of Equation (28) has the dimension of a force F, then we have

$$D\frac{d^2t}{dr^2} = F \tag{29}$$

Equation (29) has the form of Newton's second law similar to Equation (18) with the corresponding force given by

$$F = \sqrt{\frac{m}{k}} \frac{Dr}{(A^2 - r^2)^{3/2}} . \tag{30}$$

Since the motion is 1-dimensional, Equations (29) and (30) can be re-written in vector form

$$D \frac{d^2\mathbf{t}}{dr^2} = \mathbf{F}. \tag{31}$$

with the force **F** becomes

$$\mathbf{F} = \sqrt{\frac{m}{k}} \frac{D\mathbf{r}}{(A^2 - r^2)^{3/2}} . \tag{32}$$

It is seen that, at least formally, the dynamics that arises from the system of Equations (31) and (32) is equivalent to the Newtonian dynamics that arises from the system of Equations (22) and (23), in the sense that either of them can be used to describe the dynamics of the simple harmonic motion. In this case, however, the quantity *D* is not required to be determined for a complete description of the simple harmonic motion.

Now we consider an example that can be used to explain Bohr's model. That is the motion along the radial direction of a particle of spatial mass *m* under an inverse square field with the force $F = -k/r^2$. Applying Newtonian dynamics with Equation (20) we obtain

$$m \frac{d^2 r}{dt^2} = -\frac{k}{r^2} \tag{33}$$

The solution of this equation with the initial condition $t = 0, r = r_0$ is given by[8]

$$t = \int_{r_0}^{r} \frac{1}{\sqrt{\frac{2}{m}\left(E + \frac{k}{r}\right)}} dr. \tag{34}$$

where *E* is the total energy of the particle. From Equation (34) we obtain the following rates of change of the time *t* with respect to the distance *r*

$$\frac{dt}{dr} = \frac{1}{\sqrt{\frac{2}{m}\left(E + \frac{k}{r}\right)}} \tag{35}$$

$$\frac{d^2 t}{dr^2} = \frac{k\sqrt{m}}{2\sqrt{2}r^2\left(E + \frac{k}{r}\right)^{3/2}} \tag{36}$$

If we multiply both sides by the temporal mass *D* then we have

$$D \frac{d^2 t}{dr^2} = \frac{Dk\sqrt{m}}{2\sqrt{2}r^2\left(E + \frac{k}{r}\right)^{3/2}} \tag{37}$$

Hence the temporal force is found as

$$F = \frac{Dk\sqrt{m}}{2\sqrt{2}r^2\left(E + \frac{k}{r}\right)^{3/2}} \tag{38}$$

Inversely, if the temporal dynamics from Equation (21) together with the force from Equation (38) are given then the dynamics of the particle under the inverse square law can be recovered.

Instead of Newtonian dynamics, we now consider the temporal dynamics of a particle by applying the dynamical law given by Equation (18). If the force is the Coulomb's inverse square law with magnitude $|\mathbf{F}| = kq^2/r^2$, then we obtain the equation

$$D\frac{d^2t}{dr^2} = \frac{kq^2}{r^2}. \tag{39}$$

With the condition $r \to \infty, D/v \to 0$, and the initial condition $t = 0, r = r_0$, a general solution to Equation (39) can be found as

$$r = r_0 e^{-\frac{D}{kq^2}t}. \tag{40}$$

The corresponding Newtonian force $F = m\,d^2r/dt^2$ derived from Equation (40) is

$$F = m\left(\frac{D}{kq^2}\right)^2 r_0 e^{-\frac{D}{kq^2}t} = m\left(\frac{D}{kq^2}\right)^2 r. \tag{41}$$

Equation (39) can be used to explain in terms of classical dynamics how the electron of Bohr's model of a hydrogen-like atom transits between stable orbits. It can be re-written as

$$D\frac{d}{dr}\left(\frac{1}{v}\right) = \frac{kq^2}{r^2} \tag{42}$$

If we also assume the condition $r \to \infty$, $D/v \to 0$ then we have the relation

$$\frac{D}{v} = -\frac{kq^2}{r} \tag{43}$$

It is noted that the energy due to the temporal dynamics is equal to the potential energy of the electron in the Coulomb field of the nucleus. Using Equation (3) and Equation (4), the value of *D* for the *n*th stable orbit of Bohr's model can be calculated as

$$D = -\frac{kq^2\hbar}{ma_0^2n^3} \tag{44}$$

Using this value of *D*, with the initial condition $t = 0, r = r_n$, the solution given in Equation (40) becomes

$$r = r_n e^{\frac{\hbar}{m a_0^2 n^3} t}. \tag{45}$$

If we consider the time t in Equation (45) is the duration for the electron to transit between stable orbits, then the transition time from r_n to r_m can be calculated as

$$t = \frac{m a_0^2 n^3}{\hbar} \ln\left(\frac{r_m}{r_n}\right) \tag{46}$$

For example, the time for the electron to transit from the first stable orbit to the second stable orbit of Bohr's model is $t \sim 3.0 \times 10^{-17}$s. The Newtonian force found in Equation (41) is re-written as

$$F = m\left(\frac{\hbar}{m a_0^2 n^3}\right)^2 \left(r_n e^{\frac{\hbar}{m a_0^2 n^3} t}\right) = m\left(\frac{\hbar}{m a_0^2 n^3}\right)^2 r \tag{47}$$

This is a repulsive force. This force is opposite to the attractive Coulomb force. It should be mentioned here that from Equations (3) and (4), the speed of the electron can be calculated as $v_n = 2.1877 \times 10^6/n$ m/s, which is much less than the speed of light in vacuum. Therefore, according to the time dilation given in Equation (11), the proper time flow and the time flow of the reference frame can be considered as flowing at the same rate. The magnitude of the force given in Equation (47) at $t = 0$ is $F = ke^2/a_0^2 n^4$. This is equal to the magnitude of the Coulomb force $F = ke^2/r^2$ calculated at the distance $r_n = n^2 a_0$. In this case, because the net force on the electron equals zero, the electron does not radiate according to classical electrodynamics. From this analysis, we conclude that the effect of the force on the electron of a hydrogen-like atom of Bohr's model given in Equation (47) must be very short-lived. This is in fact consistent with our assumption that this force only appears when there is a second rate of change of the time. It is possible to use the concept of an exchange of virtual photons in electrostatic interaction between two charges to address this situation [9]. We assume that each time the electron absorbs a virtual photon, in a very short duration of time the electron changes its speed and this will give rise to the second rate of change of the time. We have a different situation when the electron absorbs a real photon. The duration of the time during the process of absorption, as can be calculated using Equation (46), is long enough and the force given in Equation (47) will cause the electron to transit to the higher stable orbit. The average force given to the electron by the photon can be estimated according to classical dynamics as follows. For the electron to make a transition from the n_1th stable orbit with $r_{n_1} = n_1^2 a_0$ to the n_2th stable orbit with $r_{n_2} = n_2^2 a_0$, the average force provided by the photon is

$$F = \frac{\Delta E}{\Delta d} = \frac{ke^2}{2a_0^2 (n_1 n_2)^2} \tag{48}$$

For example, if $n_1 = 1$ and $n_2 = 2$ then the average force is $F \sim 1.027325 \times 10^{-8}$N. This force has the same order of magnitude of the Coulomb force on the electron in its ground orbit, which can be calculated as $F = ke^2/a_0^2 n^4 \sim 8.218598 \times 10^{-8}$N.

4. Time as a 3-dimensional manifold

In this section, we will generalise to formulate a 3-dimensional temporal dynamics that involves the second rate of change of time with respect to distance. Mathematically, space-time can be assumed to be a six-dimensional metrical continuum, which is a union of a 3-dimensional spatial manifold and a 3-dimensional temporal manifold. The spatial manifold is a simply connected Euclidean space R^3 and the temporal manifold is also a simply connected Euclidean manifold R^3. The points of this space-time are expressed as $(x^1, x^2, x^3, x^4, x^5, x^6)$, where (x^4, x^5, x^6) representing (t^1, t^2, t^3), and the square of the infinitesimal space-time length is of a quadratic form $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. For the purpose of this work, however, as in Newtonian physics, we will consider space-time as two separate Euclidean manifolds which exist together. However, as shown below, these spatial and temporal manifolds are connected dynamically. In this case, the quadratic forms for the infinitesimal spatial arc length and the temporal arc length are reduced respectively to the forms $ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$ and $d\tau^2 = (dt_1)^2 + (dt_2)^2 + (dt_3)^2$. In Newtonian physics, the dynamics of a particle is a description of the rate of change of its position in space with respect to time according to Newton's laws of motion, where time is assumed to flow at a constant rate and is considered to be a 1-dimensional continuum. In the following, we will generalise this formulation by considering the dynamics of a particle as a description of the mutual rates of change of the position and the time of a particle with respect to one another, where not only space but time is also considered to be a 3-dimensional manifold. As shown below, this generalisation will yield new insights that can be used to explain physical phenomena. Especially, it is shown that matter, space and time of a particle are connected through the spatial mass m and the temporal mass D.

Consider a particle of inertial mass m that occupies a position in space. In a coordinate system S, the position of the particle at the time τ is determined by the position vector $\mathbf{r}(\tau) = x_1(\tau)\mathbf{i} + x_2(\tau)\mathbf{j} + x_3(\tau)\mathbf{k}$. We have assumed the Newtonian time is the temporal arc length τ. As in classical physics, the classical dynamics of the particle is governed by Newton's laws of motion. We will term Newton's laws as spatial laws. These laws are stated as follows:

- First spatial law: In an inertial reference frame, unless acted upon by a force, an object either remains at rest or continues to move at a constant velocity.
- Second spatial law:

  $$m\frac{d^2\mathbf{r}}{d\tau^2} = \mathbf{F}. \tag{49}$$

  This law is used to determine the spatial trajectory of the particle in space with respect to time.
- Third spatial law: for every action, there is an equal and opposite reaction.

These spatial laws determine the dynamics of a particle in space with the assumption that time is 1-dimensional, universal and flowing at a constant rate. For example, within this formulation, Equation (24) for the simple harmonic motion should have been written as $m\,d^2\mathbf{r}/d\tau^2 = -k\mathbf{r},$ where τ is the temporal arc length in the 3-dimensional temporal manifold and $d\tau^2 = (dt_1)^2 + (dt_2)^2 + (dt_3)^2$.

Similar to the case of 1-dimensional time, we can establish a dynamics for a 3-dimensional temporal manifold by considering space as an independent variable. However, due to the symmetry between space and time we may use the following argument to formulate. As in classical dynamics, in order for a particle to change its position it needs a flow of time. So, similarly, we assume that in order for the particle to change its time it would need an expansion of space. We consider the motion of a particle in space as its local spatial expansion. This assumption then allows us to define the rate of change of time with respect to space. From this mutual symmetry between space and time, a temporal dynamics, which is identical to Newtonian dynamics, can be assumed. Consider a particle of a temporal mass D that occupies a time in the 3-dimensional temporal manifold. In the coordinate system S, the time of the particle at the position specified by the spatial vector $\mathbf{r}$ is determined by the temporal vector $\mathbf{t}(s) = t_1(s)\mathbf{i} + t_2(s)\mathbf{j} + t_3(s)\mathbf{k}$, where s is the spatial arc length in the 3-dimensional spatial manifold and $ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$. We assume the temporal dynamics of the particle is governed by dynamical laws which are similar to Newton's laws of motion in space. In the following we will term these laws as temporal laws. These laws are stated as follows:

- First temporal law: In an inertial reference frame, unless acted upon by a force, the time of an object either does not flow or flows at a constant rate. This is a generalisation of Newtonian concept of time, which is considered to be universal and flowing at a constant rate independent of the state of motion of the particle.
- Second temporal law:

$$D\frac{d^2\mathbf{t}}{ds^2} = \mathbf{F}. \tag{50}$$

  The constant D is a dimensional constant which plays the role of the inertial mass m of the particle in space. We can choose a unit for D so that the force $\mathbf{F}$ in Equation (50) remains a force. This law is used to determine the temporal trajectory of the particle in the time manifold with respect to space.
- Third temporal law: for every action, there is an equal and opposite reaction.

With the view that time is a 3-dimensional manifold, it follows that time flow is a complex description with regards to a physical process. Time is not simply specified as past, present and future, but also dependent on its direction of flow. Only when the direction of flow of time can be specified then the state and the dynamics of a particle can be determined completely. For example, if time is a 3-dimensional continuum whose topology is Euclidean R^3 then the time of a particle with a temporal distance of unit length from the origin of a reference system is a temporal sphere of unit radius. The 3-dimensional temporal manifold can be reduced to 1-dimensional continuum by considering the 3-dimensional temporal manifold as a compactified manifold of the form $\mathrm{R}\times S^2$, where S^2 is a 2-dimensional compact manifold whose size is much smaller than any length. However, in the following we will only consider forces that act along a radial spatial direction, such as the force of gravity and Coulomb force, therefore even though we can assume time as a 3-dimensional continuum whose topology is Euclidean R^3, we will also only consider the dynamics of a particle along its radial time. In this case time is effectively a 1-dimensional continuum. Therefore, in the following, otherwise stated, we will assume $ds = dr$ and $d\tau = dt$.

First, we want to investigate whether there are any forces that can produce the same dynamics for a physical system if we apply Equations (49) and (50) separately. Suppose the temporal dynamics of a particle and its spatial dynamics are influenced by the same force **F** that gives rise to the same physical process, then we have

$$m\frac{d^2\mathbf{r}}{dt^2} = D\frac{d^2\mathbf{t}}{dr^2} \tag{51}$$

Since *m* and *D* are constant, Equation (51) can be re-written in magnitude form as

$$m\frac{d^2 r}{dt^2} = D\frac{d^2 t}{dr^2} \tag{52}$$

The Equation (52) can be shown to take the form

$$\frac{d^2 r}{dt^2}\left(\frac{D}{m}+\left(\frac{dr}{dt}\right)^3\right) = 0. \tag{53}$$

From this equation we obtain the following equations

$$\frac{d^2 r}{dt^2} = 0 \tag{54}$$

and

$$\frac{dr}{dt} = \sqrt[3]{-D/m}\ . \tag{55}$$

Form these results, it is concluded that $\mathbf{F}\equiv 0$, and space and time are linearly related in this case. The speed given by Equation (55) may be the maximum speed of a particle of spatial mass *m* and temporal mass *D* in an empty space-time. If a particle has a mass *m* and its speed in empty space is v then from Equation (55) we obtain $D = -mv^3$. For example, if each particle is a galaxy of the observable universe then using Hubble's law $v = H_0 d$ the value of *D* can be determined [10]. In this case because the proper distance *d* can change over time, therefore the value of *D* can also change over time. Or, if photons are considered to have a mass of $m = 4.0\ \times 10^{-51} kg$ and their speed in empty space is $c = 3.0 \times 10^8 m/s$ then the value of *D* is $D = -2.9 \times 10^{-24}\mathrm{kgm}^3/\mathrm{s}^3$ [11]. These results also show that in order for the space to expand with the forward time, *D* must be negative.

Equation (54) when written in vector form has a more complex structure that may be associated with physical observables in quantum mechanics. The equation can be re-written in a vector form as follows

$$\frac{d^2\mathbf{r}}{dt^2} = 0 \tag{56}$$

If we apply the temporal rate with explicit partial derivatives to Equation (56) then besides the velocity, there exists a physical quantity that can be interpreted as spin in quantum mechanics. In Newtonian classical mechanics, the position vector which satisfies Equation (56) takes the form

$$\mathbf{r} = \mathbf{v}t + \mathbf{r}_0 \tag{57}$$

where $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is the constant velocity of the particle and $\mathbf{r}_0$ is an initial position. However, in terms of the 3-dimensional temporal manifold, Equation (56) becomes

$$\sum_{i,j=1}^{3} c_i c_j \frac{\partial^2\mathbf{r}}{\partial t_i \partial t_j} + \sum_{i=1}^{3} c_i \frac{\partial^2\mathbf{r}}{\partial t_i \partial t} + \frac{\partial^2\mathbf{r}}{\partial t^2} = 0 \tag{58}$$

where we have assumed $dt_i/dt = c_i$, with c_i's are constants. It is seen that the vector $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ plays the role of the velocity $\mathbf{v}$ in spatial dynamics. The position vector that satisfies Equation (58) is

$$\mathbf{r} = (a_{11}t_1 + a_{12}t_1 + a_{13}t_1 + v_1t)\mathbf{i} + (a_{21}t_1 + a_{22}t_1 + a_{23}t_1 + v_2t)\mathbf{j} \\ + (a_{31}t_1 + a_{32}t_1 + a_{33}t_1 + v_3t)\mathbf{k} \tag{59}$$

where a's and v's are arbitrary constants. Using the conditions $dt_i/dt = c_i$, this position vector can be re-written as

$$\mathbf{r} = (\mathbf{i} \quad \mathbf{j} \quad \mathbf{k})\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} t + \mathbf{v}t + \mathbf{r}_0 \tag{60}$$

It is seen from this form that if a 3-dimensional temporal manifold is introduced along side with the 3-dimensional spatial manifold then, in addition to the velocity $\mathbf{v}$, the motion of a free particle is also described by a matrix, which, even though the overall effect being a linear motion in space, can be represented as rotation in the temporal manifold if the matrix satisfies the orthogonality condition $\sum_{i=1}^{3} a_{ij}a_{ik} = \delta_{jk},\ j,k = 1,2,3$. It should be mentioned here that there is an isomorphism between the set of 3-dimensional orthogonal matrices and the set of pair of matrices (Q,-Q), where Q is a transformation matrix which represents the Cayley-Klein parameters $(\alpha, \beta, \gamma, \delta)$, which can be written in terms of the Euler angles (ϕ, θ, ψ) as follows [12],

$$\mathrm{Q} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} e^{\frac{i(\psi+\phi)}{2}}\cos\frac{\theta}{2} & ie^{\frac{i(\psi-\phi)}{2}}\sin\frac{\theta}{2} \\ ie^{\frac{i(\psi-\phi)}{2}}\sin\frac{\theta}{2} & e^{-\frac{i(\psi+\phi)}{2}}\cos\frac{\theta}{2} \end{pmatrix} \tag{61}$$

The half angles and the double-valued property of the isomorphism are related to the fact that the value of the spin of an elementary particle such as an electron is half integral.

We now extend our investigation of the similarity between the 3-dimensional temporal dynamics and the 3-dimensional spatial dynamics of a particle and show that the sub-quantum dynamics of an elementary particle is a classical dynamics. From the symmetry of space and time given by the dynamical equations (49) and (50), we anticipate that physical laws which govern the spatial dynamics and the temporal dynamics of a particle should have identical forms, except for their roles to be reversed. We know that in classical and wave mechanics, a dynamical equation that describes a physical system can be derived from the law of conservation of energy, which in turns is derived from the concept of work done. Therefore, we will apply this procedure in our investigation. In classical mechanics, the work done is defined as

$$W = \int_{r_1}^{r_2} \mathbf{F}.d\mathbf{r}. \tag{62}$$

In the following we will focus only on the work done of an inverse square field with the force $F = A/r^2$, where A is a constant. In spatial dynamics, with this form of force, the work done defined by Equation (62) on a particle along its radial motion becomes

$$W = \int_{r_1}^{r_2} \frac{A}{r^2} dr. \tag{63}$$

From the similarity between the spatial dynamics and the temporal dynamics of the particle, we may assume the work done in the temporal dynamics along the radial time to be written as

$$W = \int_{t_1}^{t_2} \frac{B}{t^2} dt, \tag{64}$$

where B is a constant. In fact, the form of work done given by Equation (64) can be realised, for example, by Planck's quantum of energy $\Delta E = h/T$ in quantum physics [13]. The Planck's quantum of energy can be put in this form as follows

$$W = \frac{h}{T} = \int_{T}^{\infty} \frac{h}{t^2} dt. \tag{65}$$

Besides Planck's quantum of energy, another fundamental relation in quantum mechanics is de Broglie's relation between the momentum and the wavelength of a particle, $p = h/\lambda$ [14]. This relation can be re-written in the form given by Equation (63) as

$$p = \frac{h}{\lambda} = \int_{\lambda}^{\infty} \frac{h}{r^2} dr \tag{66}$$

We will discuss later that these forms of work done and momentum can be used to show that at the sub-quantum level the dynamics of radiation is a classical dynamics. Now we show that using Equations (49), (50), (63) and (64) we can obtain different forms for the conservation law of energy and the corresponding wave equation for each form.

Form 1: Using Equations (49) and (63)

$$\int_{r_1}^{r_2} m\frac{d^2r}{dt^2}dr = \int_{r_1}^{r_2}\frac{A}{r^2}dr \tag{67}$$

we obtain the following form of the conservation law, where E is the total energy,

$$\frac{1}{2}mv^2 + \frac{A}{r} = E \tag{68}$$

A classical solution to Equation (68) for a particle moving along a radial direction with the initial condition $t = 0, r = r_0$ can be written as

$$t = \int_{r_0}^{r}\frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{A}{r}\right)}}dr. \tag{69}$$

Using the standard procedure in quantum mechanics to replace $E = i\hbar\,\partial/\partial t$ and $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$, the corresponding wave equation to Equation (68) is found as [15]

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{A}{r}\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{70}$$

The well-known Schrödinger wave equation for a hydrogen-like atom is then obtained by replacing $A = -kq^2$ in Equation (70). It should be emphasised here that we have used the motion along a radial direction in order to obtain a form for the law of conservation of energy. However, when we convert the form to a wave equation, the wave equation can be used to describe the wave dynamics of the particle in 3 spatial dimensions with respect to the 1-dimensional temporal arc length.

Form 2: Using Equations (50) and (63)

$$\int_{r_1}^{r_2} D\frac{d^2t}{dr^2}dr = \int_{r_1}^{r_2}\frac{A}{r^2}dr \tag{71}$$

we obtain the following form for the energy conservation law

$$\frac{D}{v} + \frac{A}{r} = E \tag{72}$$

A classical solution to Equation (72) can be found as

$$t = \frac{E}{D} r - \frac{A}{D} \ln(r) + c \tag{73}$$

This solution can be reduced to the form of the solution given in Equation (40). In order to convert Equation (72) into a wave equation, as in the case of the Schrödinger wave equation, we re-write Equation (72) in the following form

$$mD + \frac{A}{r} (p^2)^{\frac{1}{2}} = E (p^2)^{\frac{1}{2}} \tag{74}$$

Using the identity between the momentum operator $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$ and the Fractional Laplacian $(-\Delta)^{\alpha}$, $(p^2)^{\frac{1}{2}} = (-\hbar^2\nabla^2)^{\frac{1}{2}} = \hbar(-\Delta)^{\frac{1}{2}}$, the time-dependent wave equation corresponding to Equation (74) takes the form

$$\left(-i\hbar \frac{\partial}{\partial t}\right) (-\Delta)^{\frac{1}{2}} \psi - \frac{A}{r} (-\Delta)^{\frac{1}{2}} \psi = \frac{mD}{\hbar} \psi \tag{75}$$

Whether Equation (75) can be solved using the conventional mathematical analysis requires further investigation. However, according to our previous analysis, the resulting wave-function would be a description of the transition of an elementary particle in a quantum system, such as the transition of the electron from one stationary orbit to another of a hydrogen-like atom in Bohr's model when the constant A is replaced by $A = -kq^2$.

Form 3: Using Equations (49) and (64)

$$\int_{r_1}^{r_2} m \frac{d^2 r}{dt^2} dr = \int_{t_1}^{t_2} \frac{B}{t^2} dt \tag{76}$$

we obtain the following form for the energy conservation law

$$\frac{1}{2} m v^2 + \frac{B}{t} = E \tag{77}$$

A classical solution to Equation (77) is given by

$$r = \sqrt{\frac{2}{m}} t \sqrt{E - \frac{B}{t}} - \frac{B}{\sqrt{2mE}} \ln\left(2\sqrt{E} t \sqrt{E - \frac{B}{t}} + 2Et - B\right) + c \tag{78}$$

where c is a constant. The condition for the solution (78) to exist as real solution is $t \geq B/E$. If B is identified as the Planck constant, $B = h$, and E as the Planck quantum energy, $E = h/T$, then we have $t \geq T$, where T is the period of wave motion of the particle involved. In the next section, we will derive a dynamics that can be used to describe a physical process for $t < T$ only. Now we show that the relation (76) can be used to study the dynamics of the process of radiation of an elementary particle from a quantum system. In quantum physics, Planck's quantum of energy is considered as a complete unit of energy which is indivisible, even when it is being produced or absorbed by a quantum system [13,16]. This quantum hypothesis has led to the development of quantum mechanics and quantum field theories which are fundamental and essential to predictions and explanations of observations of sub-atomic physical processes [6,15]. Even though quantum theories have been proved to be highly successful in applications to physical problems, the quantum principles still remain paradoxical, especially when we try to reconcile the quantum dynamics, which seems to be probabilistic, to classical dynamics, which is deterministic. We suspect that this problem may be the result of an unknown dynamical process when a microscopic object is being emitted or absorbed by a quantum system. With this in mind, we will show that there is a propulsive force in action that transfers energy to the emitted particle and the total transferred energy is the Planck's quantum of energy. The work done W given in Equation (62) by a force **F** that moves an object with velocity **v** from time t_1 to time t_2 is re-written as

$$W = \int_{t_1}^{t_2} \mathbf{F}.\mathbf{v}dt \tag{79}$$

In the case of a microscopic particle emitted from a quantum system, the work done W is equal to Planck's quantum of energy that can be expressed in the form given by Equation (65). It is seen from Equation (65) that possible effects of a propulsive force are applied to the emitted particle only after the time T, which is the period of wave motion of the quantum particle when the particle is bounded inside a quantum system. The bounded state of the electron of a hydrogen-like atom is a wave-like state will be discussed in Section 6. In the following we consider these effects to be classical, which obey the generalised Newtonian dynamics. From Equations (65) and (79) we obtain the relation

$$\mathbf{F}.\mathbf{v} = \frac{h}{t^2}\,. \tag{80}$$

Assuming $\mathbf{F}.\mathbf{v} = Fv$ and using Newton's second law $F = m\,dv/dt$, the following equations are obtained

$$F = \frac{h}{vt^2}\,, \tag{81}$$

$$v\frac{dv}{dt} = \frac{h}{mt^2}\,. \tag{82}$$

If we consider the condition that at the initial time $t_1 = T$, the velocity of the particle is $v = v_0$, then solutions to Equation (82) are found as follows

$$\frac{mv^2}{2} = \frac{mv_0^2}{2} + h\left[\frac{1}{T} - \frac{1}{t}\right]. \tag{83}$$

Taking the positive sign for *v* from Equation (83) we obtain the required propulsive force

$$F = \frac{h}{t^2\sqrt{v_0^2 + \frac{2h}{m}\left[\frac{1}{T} - \frac{1}{t}\right]}}\,. \tag{84}$$

The negative sign of *v* from the solutions (83) may be considered when an elementary particle is being absorbed by a quantum system. Furthermore, using de Broglie's relation given in Equation (66), it can be shown that the momentum of the emitted particle satisfies the relation

$$mv = mv_0 + h\left[\frac{1}{\lambda} - \frac{1}{r}\right] \tag{85}$$

where λ is the wavelength of the particle's wave motion before it is emitted from a quantum system. It is seen from these results that there is a continuous transfer of energy and momentum between a quantum system and a microscopic object during a quantum radiation process. The total energy transferred to the particle is equal to the Planck's quantum of energy, which is a total work done on the particle, and the momentum in de Broglie's relation is a total transferred momentum.

If the constant B in Equation (77) is identified with the Planck's constant, $B = h$, the corresponding time-dependent wave equation to Equation (77) is obtained as

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{h}{t}\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{85}$$

First it is noted that when $t \to \infty$, Equation (85) reduces to the Schrödinger wave equation for a free particle in space. In reality, however, due to the smallness of the Planck constant, the duration is very short to an observer in a laboratory. During this transition time, Equation (85) can be solved as follows. Let $\psi(\mathbf{r}, t) = \phi(\mathbf{r})\chi(t)$, Equation (85) is reduced to the following two equations

$$-\frac{\hbar^2}{2m}\nabla^2\phi = \mathrm{E}\phi \tag{86}$$

$$\frac{d\chi}{dt} = -\frac{iE}{\hbar}\chi + \frac{h}{t} \tag{87}$$

Equation (86) is the Schrödinger wave equation for a free particle with a continuous spectrum of energy. Solutions to Equation (86) take the form of a plane wave

$$\phi = C\exp(i\mathbf{k}.\mathbf{r}) \tag{88}$$

On the other hand, solutions to Equation (87) are given by

$$\chi(t) = \left(c + h\left(-i\int_{\frac{Et}{\hbar}}^{\infty}\frac{\sin(\tau)}{\tau}d\tau + \int_{\frac{Et}{\hbar}}^{\infty}\frac{\cos(\tau)}{\tau}d\tau\right)\right)\exp\left(\frac{iEt}{\hbar}\right) \tag{89}$$

where c is an undetermined constant. As we discussed above, we assume that the particle emitted from the quantum system at the time $t = T$, which is the period of wave motion of the particle in a quantum system. At $t = T$ we have $ET/\hbar = 2\pi$, therefore the initial value of the solution is

$$\chi(T) = c + h\left(-i\int_{2\pi}^{\infty}\frac{\sin(\tau)}{\tau}d\tau + \int_{2\pi}^{\infty}\frac{\cos(\tau)}{\tau}d\tau\right) \tag{90}$$

Form 4: Using Equations (50) and (64)

$$\int_{r_1}^{r_2} D\frac{d^2t}{dr^2}dr = \int_{t_1}^{t_2}\frac{B}{t^2}dt \tag{91}$$

we obtain the following energy conservation law

$$\frac{D}{v} + \frac{B}{t} = E \tag{92}$$

A classical solution to Equation (92) is found as

$$r = \frac{D}{E^2}[B\ln(B - Et) + Et] + c \tag{93}$$

where c is an undetermined constant. The condition for the solution (93) to exist as a real solution is $t < B/E$. If we also identify B with the Planck constant and E with the Planck quantum energy, then we have $t < T$, where T is the period of wave motion of the particle. The initial condition of Equation (93) is given by $t = 0$, $r = \frac{D}{E^2}[h\ln(h)] + c$. If $c = 0$, then the initial distance r_0 is equal to $r_0 = \frac{D}{E^2}[h\ln(h)]$.

The relation given in Equation (91) can also be used to study the dynamics of the process of radiation of an elementary particle from a quantum system. Using the relation (50) and (83) we obtain the following system of equations

$$F = \frac{h}{vt^2}\,, \tag{94}$$

$$D\frac{d}{dt}\left(\frac{1}{v}\right) = \frac{h}{t^2}\,. \tag{95}$$

With the final conditions $t_2 = T, v = v_0$ the solution to Equation (95) is found as

$$\frac{D}{v_0} - \frac{D}{v} = h\left[\frac{1}{t} - \frac{1}{T}\right] \tag{96}$$

Equation (96) shows that there is a continuous transfer of energy from a quantum system to an emitted particle to increase its speed from $v = 0$ at the initial time $t = 0$ to the final value $v = v_0$ at $t = T$. At the time $t = T$, the amount of energy transferred from the system is also equal to Planck's quantum of energy h/T. The corresponding force given by Equation (94) is

$$F = \frac{h}{t^2}\left[\frac{1}{v_0} + \frac{h}{D}\left[\frac{1}{T} - \frac{1}{t}\right]\right] \tag{97}$$

In order to convert the relation given by Equation (92) into a wave equation, we re-write it as

$$mD + \frac{h}{t}(p^2)^{\frac{1}{2}} = E(p^2)^{\frac{1}{2}} \tag{98}$$

The corresponding wave equation to Equation (98) is

$$-\frac{2\pi\hbar^2}{t}(-\Delta)^{\frac{1}{2}}\psi + \hbar E(-\Delta)^{\frac{1}{2}}\psi = mD\psi \tag{99}$$

The time-dependent wave equation of Equation (99) is

$$-\frac{2\pi\hbar^2}{t}(-\Delta)^{\frac{1}{2}}\psi + \hbar\left(-i\hbar\frac{\partial}{\partial t}\right)(-\Delta)^{\frac{1}{2}}\psi = mD\psi \tag{100}$$

Because of the Fractional Laplacian, whether Equation (99) and Equation (100) can be solved using the conventional mathematical analysis also requires further investigation.

Form 5: Using Equations (49) and (50)

$$\int_{r_1}^{r_2} m\frac{d^2r}{dt^2}dr = \int_{r_1}^{r_2} D\frac{d^2t}{dr^2}dr \tag{101}$$

we obtain the following energy conservation law

$$\frac{1}{2}mv^2 - \frac{D}{v} = E \tag{102}$$

General classical solutions to Equation (102) can be found to have a linear relationship between space and time in the form $r(t) = at + b$, which includes one real solution and two complex solutions. The real solution is expected because as we discussed before the relation (102) is that of a free particle. To convert the relation (102) into a wave equation, we re-write it in the form

$$\frac{1}{2m}(p^2)^{\frac{3}{2}} - E(p^2)^{\frac{1}{2}} = mD \tag{103}$$

The corresponding wave equation to Equation (103) is

$$\frac{\hbar^3}{2m}(-\Delta)^{\frac{3}{2}}\psi - E\hbar(-\Delta)^{\frac{1}{2}}\psi = mD\psi \tag{104}$$

The time-dependent wave equation is

$$\frac{\hbar^3}{2m}(-\Delta)^{\frac{3}{2}}\psi - \hbar\left(-i\hbar\frac{\partial}{\partial t}\right)(-\Delta)^{\frac{1}{2}}\psi = mD\psi \tag{104}$$

From our discussions before, we can say that this is the wave equation of a free particle in the unified space-time manifold.

Form 6: Using Equations (63) and (64)

$$\int_{r_1}^{r_2}\frac{A}{r^2}dr = \int_{t_1}^{t_2}\frac{B}{t^2}dt \tag{105}$$

we obtain the following energy conservation law

$$\frac{A}{r} - \frac{B}{t} = E \tag{106}$$

In this case we don't have a dynamics related to the motion of a particle but only a dynamics of the particle's space-time. This spatial-temporal dynamics is governed by the physical properties of the physical constants A and B associated with the particle. The relation in Equation (106) can be re-written as

$$r = \frac{A}{E}\left(1 - \frac{B/E}{t + B/E}\right) \quad (107)$$

It is interesting to note that at the initial time $t = 0, r = 0$ but when $t \to \infty, r \to A/E$. This result shows that the final size of the unified space-time of a particle is finite and inversely proportional its total energy. For a particle which has a negligible amount of energy, the size of its corresponding space-time is infinite. Inversely, the size of a particle will approach zero if it has an infinite amount of energy. The latter case is similar to that of a black hole in general relativity, while the former one can be considered to be that of a photon in terms of quantum physics.

Since the problem we deal with in this work is related to the wave-particle duality in quantum physics, it is therefore appropriate to discuss further this dual nature of an elementary particle. In classical mechanics a Lagrangian L is used to determine the trajectory of a particle by applying the principle of least action $\delta \int Ldt = 0$. Similarly, a Lagrangian density can be used to describe the wave dynamics of a classical field whose physical entities can be interpreted and identified. However, in Schrödinger's wave mechanics a Lagrangian is only related to the description of the dynamics of the phase of matter wave, but not a particular physical entity. This can be seen for the case of the motion of a free particle as follows [15,17]. In order to derive his wave equation, Schrödinger applied the Hamilton-Jacobi equation with a particular form of wavefunction $\psi = \psi_0 e^{\frac{2\pi}{h}\int Ldt}$. On the other hand, the solution obtained from Schrödinger's wave equation for a free particle is given by $\psi = \psi_0 e^{\frac{2\pi i}{h}(pr - Et)}$. Comparing these two wavefunctions we obtain $L = i(mv^2 - E)$. If we apply the Lagrangian equation in classical mechanics to this Lagrangian we arrive at the same equation of motion for the phase as that for a free particle in classical mechanics, that is $v = \text{constant}$. In our present problem, if the wave motion of an elementary particle at the time $t = T$ can be approximated to be that of a plane wave then this constant velocity can be identified as the initial velocity v_0 of the particle when it is emitted as given in Equation (83). This result may provide for an explanation of the interference pattern of a beam of elementary particles by considering their absorption and emission by a quantum system.

References

[1] A. Einstein, B. Podolsky and N. Rosen, *Physical Review*, **47**, 777 (1935).

[2] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press 1987).

[3] Alain Aspect, Jean Dalibard, Gérard Roger (1982), "Experimental Test of Bell's Inequalities Using Time-Varying Analyzers", *Phys. Rev. Lett.* **49** (25): 1804–7.

[4] A. Einstein, *The Principle of Relativity* (Dover Publications, New York, 1952).

[5] A. P. French, *Special Relativity* (Van Nostrand Reinhold (UK) Co. Ltd., England, 1983).

[6] N.Bohr, *Phil. Mag*. **26**, 1 (1913).

[7] L.D. Landau and E.M.Lifshitz, The Classical Theory of Fields (Pergamon Press, Sydney, 1987).

[8] H. Goldstein, Classical Mechanics (Addison-Wesley Inc., Sydney, 1980).

[9] D. H. Perkins, Introduction to High Energy Physics (Addison-Wesley, Sydney, 1987).

[10] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman and Company, New York, 1971).

[11] A. S. Goldhaber and M. M. Nieto, Phy. Rev. Lett. 21, 567 (1968).

[12] H. Goldstein, *Classical Mechanics*, (Addison-Wesley, London, 1980).

[13] M. Planck, *On the Theory of the Energy Distribution Law of the Normal Spectrum*, edited by D. ter Haar, The Old Quantum Theory (Pergamon Press, Oxford, 1967).

[14] L. de Broglie, *On the Theory of Quanta*, Ann. de. Phys. 10^{e}, t. III, *translated by* A.F. Kracklauer (Janvier-Fevrier, 1925).

[15] E. Schrödinger, *An Undulatory Theory of the Mechanics of Atoms and Molecules*, The Phys. Rev., **28**, 1926.

[16] A. Einstein, *Einstein's Proposal of the Photon Concept-a Translation of the Annalen der Physik Paper of 1905* by A. B. Arons and M. B. Peppard, 1965 Am. J. Phys**. 33** 367-374.

[17] W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory* (Dover Publications, New York, 1979).