A THEORY OF TEMPORAL RELATIVITY

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Abstract: In this work we develop a theory of temporal relativity, which includes a temporal special relativity and a temporal general relativity, on the basis of a generalised Newtonian temporal dynamics. We then show that a temporal relativity can be used to study the dynamics of quantum radiation of an elementary particle from a quantum system.

1. Introduction

In physics, space and time are combined to form a continuum in which space is regarded as consisting of three dimensions and time as one dimension. The requirement that time should have only one dimension can be seen as a result our perception of existence, even though the combination of space and time to form a unified space-time is totally asymmetric. Can we perceive time as a three dimensional continuum? Is there any form of physical processes that suggests that time should exist as a three dimensional continuum? If space and time are totally symmetric then from the symmetry of space and time we expect that physical laws which govern the spatial dynamics and the temporal dynamics of a particle should have identical forms, except for their roles to be reversed. It is known that in classical and wave mechanics, a dynamical equation that is used to describe a physical system can be derived from the law of conservation of energy, which in turns is derived from the concept of work done. In the following we show that there is a similarity between spatial and temporal physical entities that are used to formulate physics using the concepts of work done and energy. In classical mechanics, the work done is defined as [1]

$$W = \int_{r_1}^{r_2} \mathbf{F}.d\mathbf{r}. \tag{1}$$

Consider an inverse square field with a force given by the form

$$F(r) = \frac{A}{r^2} \tag{2}$$

where A is a constant. This force can be derived from the potential

$$V(r) = \frac{A}{r} \tag{3}$$

The potential given be Equation (3) in turns can be seen as a weak field of the Schwarzschild solutions of Einstein's field equations of general relativity. In spatial dynamics, with the form of force given in Equation (2), the work done defined by Equation (1) on a particle along its radial motion becomes

$$W = \int_{r_1}^{r_2} \frac{A}{r^2} dr. \tag{4}$$

It is interesting to note that de Broglie's relation [2] in quantum mechanics $p = h/\lambda$ can be written in the form given in Equation(4) as follows

$$p = \frac{h}{\lambda} = \int_{\lambda}^{\infty} \frac{h}{r^2} dr \tag{5}$$

In quantum physics, Planck's quantum of energy is stated through the relationship

$$\Delta E = \frac{h}{T} \tag{6}$$

where h is Planck's constant and T is the period of wave motion of a quantum particle [3]. It is seen that the Planck's quantum of energy given in Equation (6) can be put in the following form

$$W = \frac{h}{T} = \int_{T}^{\infty} \frac{h}{t^2} dt. \tag{7}$$

Equation (7) can be generalised to take a more general form as [4]

$$W = \int_{t_1}^{t_2} \frac{B}{t^2} dt, \tag{8}$$

where B is a constant. The form given in Equation (8) is similar to that of Equation (4) except the roles of space and time are reversed. From the similarity between space and time through Equations (4) and (8), we can define a temporal force as

$$F(t) = \frac{B}{t^2} \tag{9}$$

Similar to spatial dynamics of classical mechanics, if the force in Equation (9) can be derived from a temporal potential $V(t)$ as $F(t) = -dV/dt$, then with the force given by Equation (9), the temporal potential $V(t)$ can be found as

$$V(t) = \frac{B}{t} \tag{10}$$

In the following we will consider the dynamics of an elementary particle therefore the constant B will be identified with Planck's constant h, and we will show that the type of potential given in Equation (10) can be derived from a temporal relativity.

Consider an elementary particle of mass m under the influence of a temporal force given in Equation (9) by a quantum system. We need to find the corresponding force $\mathbf{F}$ as defined in classical mechanics for this temporal force. In classical mechanics, the work done W defined by Equation (1) for a force **F** that moves an object with velocity **v** from time t_1 to time t_2 is re-written as

$$W = \int_{t_1}^{t_2} \mathbf{F}.\mathbf{v}dt \tag{11}$$

It should be emphasised here that the force **F** in Equation (11) is a normal force defined in classical physics, unlike the temporal force given in Equation (9) which is defined in a temporal manifold. From Equations (8) and (11) we obtain the relation

$$\mathbf{F}.\mathbf{v} = \frac{h}{t^2}\,. \tag{12}$$

Assuming $\mathbf{F}.\mathbf{v} = Fv$ and using Newton's second law $F = m\,dv/dt$, the following equation is obtained

$$mv\frac{dv}{dt} = \frac{h}{t^2}\,. \tag{13}$$

If we consider the condition that at the initial time $t_1 = T$, the velocity of the particle is $v = v_0$, then solutions to Equation (13) are found as follows

$$\frac{mv^2}{2} = \frac{mv_0^2}{2} + h\left[\frac{1}{T} - \frac{1}{t}\right]. \tag{14}$$

Taking the positive sign for v from Equation (14) we obtain a propulsive force

$$F = \frac{h}{t^2\sqrt{v_0^2 + \frac{2h}{m}\left[\frac{1}{T} - \frac{1}{t}\right]}}\,. \tag{15}$$

The negative sign of v from the solutions (15) may be considered when the particle is being absorbed by a quantum system. It is also noted that by using de Broglie's relation given in Equation (5), it can be shown that the momentum of the emitted particle satisfies the relation

$$mv = mv_0 + h\left[\frac{1}{\lambda} - \frac{1}{r}\right] \tag{16}$$

where λ is the wavelength of the particle's wave motion before it is emitted from a quantum system. It is seen from these results that there is a continuous transfer of energy and momentum between a quantum system and a microscopic object during a quantum radiation process. The total energy transferred to the particle is equal to the Planck's quantum of energy, which is a total work done on the particle, and the momentum in de Broglie's relation is a total transferred momentum.

From the similarity between space and time discussed above, the aim of this work is to formulate a theory of temporal relativity, which is a relativistic theory of a three dimensional temporal continuum. To achieve this aim first we need to formulate a temporal dynamics which is a dynamics with regards to the rate of change of time. Even though the formulation of a temporal dynamics has been discussed in details in our other works [4], for clarity, an outline of the main features of the temporal dynamics is given in Section 2 and 3. We will develop a special theory of temporal relativity in Section 4 and in Section 5 we will generalise and formulate a general theory of temporal relativity.

2. The concept of time in Einstein's theory of special relativity

In this section we want to show that a temporal dynamics, which is similar to Newtonian dynamics, can be derived from Einstein's theory of special relativity. Consider two inertial reference systems S and S' with coordinates (x_1, x_2, x_3, t) and (x_1', x_2', x_3', t'). If the system S' moves relative to the system S along the x_1-axis with the velocity $\mathbf{v}$, then according to Newtonian physics the transformation of the coordinates of the two systems is the Galilean transformation [5]

$$x_1 = x_1' + vt, \qquad x_2 = x_2', \qquad x_3 = x_3', \qquad t = t' \tag{17}$$

The concept of absolute time in Newtonian physics was changed when Einstein proposed his theory of special relativity. Instead of the Galilean transformation, in special relativity the transformation of the coordinates adopts the Lorentz transformation

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad x_2 = x_2', \qquad x_3 = x_3', \qquad t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{18}$$

From Equations (18), for infinitesimal changes, the formulas for the length contraction and the time dilation are derived as

$$\frac{dx_1}{dx_1'} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{19}$$

$$\frac{dt}{dt'} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{20}$$

In the following we will consider the system S' as a moving particle in a reference system S. During its motion, if the particle interacts with other physical objects, as in the case of Compton's scattering between an electron and a photon, then the particle's direction and speed will change. From Equation (20), the proper time interval will change during the interaction if we assume the time of the reference frame still flows at a constant rate. In this case we have a rate of change of the proper time of the particle. However, if the speed of the particle remains small compared to the speed of light after this short period of interaction then the proper time flow will still be the same as that of the reference frame. Inversely, if we assume the proper time to flow at a constant rate during the interaction then the time interval of the time of the reference frame will change, and in this case we have the rate of change of the time of the reference frame. Furthermore, we will consider the case when the interaction happens only in a very short duration of time, therefore, we assume that the form in Equation (20) still remains valid even though the velocity of the particle changes continuously. This can be considered as an extension of the postulate of relativity. It should be mentioned here that this kind of extension of the postulate of relativity had led Einstein to develop his general theory of relativity. The extended principle of relativity is stated as: "*The law of physics must be of such a nature that they apply to systems of reference in any kind of motion*" [6]. With the assumption that the relation given by Equation (20) remains valid for a continuous change of velocity, we obtain the second rate of change of the time of the reference frame with respect to the proper time as

$$\frac{d^2t}{dt'^2} = \frac{\mathbf{v}.d\mathbf{v}/dt}{c^2\left(1-\frac{v^2}{c^2}\right)^{3//2}}\frac{dt}{dt'}. \tag{21}$$

Using Equation (20) and multiplying both sides of Equation (21) by the relativistic mass of the particle, $m = m_0/\sqrt{1-v^2/c^2}$, where m_0 is the particle's rest mass, we obtain

$$m_0\frac{d^2t}{dt'^2} = \frac{\mathbf{v}.m_0 d\mathbf{v}/dt}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \tag{22}$$

Equation (22) can be re-written as

$$m_0c^2\left(1-\frac{v^2}{c^2}\right)^2\frac{d^2t}{dt'^2} = \mathbf{v}.\mathbf{F} \tag{23}$$

where $\mathbf{F} = m_0\,d\mathbf{v}/dt$ is a Newtonian force that is responsible for the rate of change of the time of the particle. The dynamical equation given by Equation (23) needs to be reformulated in terms of variables in the reference system in order to be applied. For example, consider the radial motion. In this case we have $\mathbf{v}.m_0 d\mathbf{v}/dt = \mathbf{v}.\mathbf{F} = \pm vF$. Then Equation (23) can be re-written as

$$\pm m_0c^2v\left(1-\frac{v^2}{c^2}\right)^2\frac{d^2t}{v^2dt'^2} = F \tag{24}$$

Using Equation (20), we can write $v^2dt'^2 = (1-v^2/c^2)v^2dt^2 = (1-v^2/c^2)dr^2$. With this result, Equation (24) becomes

$$\pm m_0c^2v\left(1-\frac{v^2}{c^2}\right)\frac{d^2t}{dr^2} = F \tag{25}$$

If we define a new physical quantity

$$D = \pm m_0 c^2 v \left(1 - \frac{v^2}{c^2}\right) \tag{26}$$

then Equation (25) takes the form

$$D \frac{d^2 t}{dr^2} = F \tag{27}$$

The dynamical equation given in Equation (27) has the form of Newton's second law of motion. However, in this case the roles of space and time are reversed. The new physical quantity D plays the role of the inertial mass of a particle in Newtonian mechanics. As in the case of special relativistic dynamics, this new physical quantity also depends on the velocity of the particle.

3. Time as a 3-dimensional manifold

In Newtonian physics, time is an independent 1-dimensional Euclidean continuum, which is an essential component of the fundamental structure of the nature. Time is considered to be absolute and its properties are independent of any system of reference. The time intervals of time between two events are identical for all reference systems. In classical physics, the dynamics of a particle is a study of its motion in space with respect to time under the action of forces, where time is considered to be universal and to flow at a constant rate. Because time is considered to be 1-dimensional, therefore we will discuss the dynamics of a particle in 1-dimension first and then extend the discussion to a 3-dimensional temporal manifold. Consider the motion of a particle in a straight line under the action of a force **F**. Its displacement from an origin is represented by the position vector **r**. In order to study the dynamics of the particle, we divide the 1-dimensional Euclidean time into equal intervals Δt and measure the distance Δr_i that the particle has travelled in the ith time interval. In this case we define the rate of change of the displacement of the particle as $\Delta r_i/\Delta t$. If these rates are equal then we say the particle is moving with a constant velocity **v**. Let $\Delta t \to 0$, we have $\mathbf{v} = d\mathbf{r}/dt$. If the rates are different then we say the particle is moving with an acceleration $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$. When physical entities related to the particle, such as mass m and charge q, are introduced then we can formulate a classical dynamics such as Newtonian dynamics

$$m \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{F}. \tag{28}$$

In the following we will term Newtonian dynamics as the spatial dynamics, in contrast to the temporal dynamics that we formulate as follows. We also consider the motion of the particle along a straight line as described above. Instead of dividing the time line into equal intervals, we divide the spatial line into equal spatial intervals Δr. After the particle moves through the ith spatial interval we measure the corresponding time interval Δt_i that the particle has taken to move through that distance. In this case we define the rate of change of the temporal displacement of the particle with respect to distance as $\Delta t_i/\Delta r$. The temporal displacement from a temporal origin is represented by the temporal vector **t**. If these rates are equal then we say the particle is moving with a constant temporal velocity $\mathbf{v}_T$. Let $\Delta r \rightarrow 0$, we have $\mathbf{v}_T = d\mathbf{t}/dr$. If the rates are different then we say the particle is moving with a temporal acceleration $\mathbf{a}_T = d\mathbf{v}_T/dr = d^2\mathbf{t}/dr^2$. If the temporal dynamics of the particle is also caused by a force **F** then we can formulate a temporal dynamics similar to Newtonian dynamics

$$D\frac{d^2\mathbf{t}}{dr^2} = \mathbf{F}. \tag{29}$$

where the physical quantity D, which plays the role of the inertial mass m in Newtonian mechanics needs to be determined. The quantity D has the dimension of the quantity given by Equation (26), i.e., $[D] = [ML^3T^{-3}]$. The form given by Equation (29) is similar to Newton's second law of motion given by Equation (28), except for the roles of space and time are reversed.

We now generalise to formulate a 3-dimensional temporal dynamics that involves the second rate of change of time with respect to distance. Mathematically, space-time can be assumed to be a six-dimensional metrical continuum, which is a union of a 3-dimensional spatial manifold and a 3-dimensional temporal manifold. The spatial manifold is a simply connected Euclidean space R^3 and the temporal manifold is also a simply connected Euclidean manifold R^3. The points of this space-time are expressed as $(x^1, x^2, x^3, x^4, x^5, x^6)$, where (x^4, x^5, x^6) representing (t^1, t^2, t^3), and the square of the infinitesimal space-time length is of a quadratic form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. For the purpose of this work, however, as in Newtonian physics, we will consider space-time as two separate Euclidean manifolds which exist together. However, as shown below, these spatial and temporal manifolds are connected dynamically. In this case, the quadratic forms for the infinitesimal spatial arc length and the temporal arc length are reduced respectively to the forms $ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$ and $d\tau^2 = (dt_1)^2 + (dt_2)^2 + (dt_3)^2$. In Newtonian physics, the dynamics of a particle is a description of the rate of change of its position in space with respect to time according to Newton's laws of motion, where time is assumed to flow at a constant rate and is considered to be a 1-dimensional continuum. In the following, we will generalise this formulation by considering the dynamics of a particle as a description of the mutual rates of change of the position and the time of a particle with respect to one another, where not only space but time is also considered to be a 3-dimensional manifold. As shown below, this generalisation will yield new insights that can be used to explain physical phenomena. Especially, it is shown that matter, space and time of a particle are connected through the spatial mass *m* and the temporal mass *D*.

Consider a particle of inertial mass m that occupies a position in space. In a coordinate system S, the position of the particle at the time τ is determined by the position vector $\mathbf{r}(\tau) = x_1(\tau)\mathbf{i} + x_2(\tau)\mathbf{j} + x_3(\tau)\mathbf{k}$. We have assumed the Newtonian time is the temporal arc length τ. As in classical physics, the classical dynamics of the particle is governed by Newton's laws of motion. We will term Newton's laws as spatial laws. These laws are stated as follows:

- First spatial law: In an inertial reference frame, unless acted upon by a force, an object either remains at rest or continues to move at a constant velocity.
- Second spatial law:

$$m\frac{d^2\mathbf{r}}{d\tau^2} = \mathbf{F}. \tag{30}$$

  This law is used to determine the spatial trajectory of the particle in space with respect to time.
- Third spatial law: for every action, there is an equal and opposite reaction.

These spatial laws determine the dynamics of a particle in space with the assumption that time is 1-dimensional, universal and flowing at a constant rate. For example, within this formulation, Equation (24) for the simple harmonic motion should have been written as $m\,d^2\mathbf{r}/d\tau^2 = -k\mathbf{r}$**,** where τ is the temporal arc length in the 3-dimensional temporal manifold and $d\tau^2 = (dt_1)^2 + (dt_2)^2 + (dt_3)^2$.

Similar to the case of 1-dimensional time, we can establish a dynamics for a 3-dimensional temporal manifold by considering space as an independent variable. However, due to the symmetry between space and time we may use the following argument to formulate. As in classical dynamics, in order for a particle to change its position it needs a flow of time. So, similarly, we assume that in order for the particle to change its time it would need an expansion of space. We consider the motion of a particle in space as its local spatial expansion. This assumption then allows us to define the rate of change of time with respect to space. From this mutual symmetry between space and time, a temporal dynamics, which is identical to Newtonian dynamics, can be assumed. Consider a particle of a temporal mass D that occupies a time in the 3-dimensional temporal manifold. In the coordinate system S, the time of the particle at the position specified by the spatial vector $\mathbf{r}$ is determined by the temporal vector $\mathbf{t}(s) = t_1(s)\mathbf{i} + t_2(s)\mathbf{j} + t_3(s)\mathbf{k}$, where s is the spatial arc length in the 3-dimensional spatial manifold and $ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$. We assume the temporal dynamics of the particle is governed by dynamical laws which are similar to Newton's laws of motion in space. In the following we will term these laws as temporal laws. These laws are stated as follows:

- First temporal law: In an inertial reference frame, unless acted upon by a force, the time of an object either does not flow or flows at a constant rate. This is a generalisation of Newtonian concept of time, which is considered to be universal and flowing at a constant rate independent of the state of motion of the particle.
- Second temporal law:

$$D\frac{d^2\mathbf{t}}{ds^2} = \mathbf{F}. \tag{31}$$

The constant D is a dimensional constant which plays the role of the inertial mass m of the particle in space. We can choose a unit for D so that the force $\mathbf{F}$ in Equation (31) remains a force. This law is used to determine the temporal trajectory of the particle in the time manifold with respect to space.

- Third temporal law: for every action, there is an equal and opposite reaction.

With the view that time is a 3-dimensional manifold, it follows that time flow is a complex description with regards to a physical process. Time is not simply specified as past, present and future, but also dependent on its direction of flow. Only when the direction of flow of time can be specified then the state and the dynamics of a particle can be determined completely. For example, if time is a 3-dimensional continuum whose topology is Euclidean R^3 then the time of a particle with a temporal distance of unit length from the origin of a reference system is a temporal sphere of unit radius. The 3-dimensional temporal manifold can be reduced to 1-dimensional continuum by considering the 3-dimensional temporal manifold as a compactified manifold of the form $\mathrm{R} \times S^2$, where S^2 is a 2-dimensional compact manifold whose size is much smaller than any length. However, in the following we will only consider forces that act along a radial spatial direction, such as the force of gravity and Coulomb force, therefore even though we can assume time as a 3-dimensional continuum whose topology is Euclidean R^3, we will also only consider the dynamics of a particle along its radial time. In this case time is effectively a 1-dimensional continuum. Therefore, in the following, otherwise stated, we will assume $ds = dr$ and $d\tau = dt$.

4. A temporal special relativity

Consider two 1-dimensional inertial reference systems S and $S^{'}$ with coordinates (t, r) and (t', r'). We will extend the 1-dimensional temporal continuum to a 3-dimensional continuum with coordinates (t_1, t_2, t_3, r) and $\left(t_1^{'}, t_2^{'}, t_3^{'}, r^{'}\right)$, where $t_1 \equiv t$, $t_1' \equiv t'$ and other temporal coordinates can be set to be constant.

Now consider a particle which is located at the origin of the reference system S' in both space and time. In this work we assume that any flow of time must be associated with a spatial motion, therefore, if the particle remains at the origin then there will be no flow of time. The particle then starts to move and consequently the time of the particle starts to flow. The rate of flow of time will depend on the rate of displacement of the particle in space. This time is denoted by t'. The displacement of the particle at time t' is denoted by r'. If the system S' is not moving relative to the system S then we have the following coordinate transformation

$$t_1 = t_1', \qquad t_2 = t_2', \qquad t_3 = t_3', \qquad r = r' \tag{32}$$

If the system $S^{'}$ moves relative to the system S along the t_1-axis with a constant temporal velocity v_T, where, in general, a temporal velocity is defined as $\mathbf{v}_T = d\mathbf{t}/dr$, with $\mathbf{t}$ is the temporal vector, and if the system S' is not moving relative to the system S spatially, then the extra time that has passed is $v_T r'$, and the transformation of the coordinates of the particle in the two systems is the Galilean-like transformation

$$t_1 = t_1^{'} + v_T r', \qquad t_2 = t_2^{'}, \qquad t_3 = t_3^{'}, \qquad r = r^{'} \tag{33}$$

However, if we assume that time-flow must be associated with motion in space, then the transformation given in Equation (33) cannot be applied. In this case we need to use a more general transformation that specifies the motion of the system S' with respect to the system S. As in the case of Einstein's theory of special relativity, if we assume that there must be a maximal universal temporal speed, $v_{Tmax} = c_T$, then a transformation similar to Lorentz transformation can be formulated as follows

$$t_1 = \gamma(t_1' + v_T r'), \qquad t_2 = t_2', \qquad t_3 = t_3', \qquad r = \gamma(r' + v_T t_1'/c_T^2) \tag{34}$$

An expression for the quantity γ can be found if we assume the invariance of space-time intervals

$$(c_T r')^2 - (t_1')^2 = (c_T r)^2 - (t_1)^2 \tag{35}$$

Using Equations (34) and (35) we obtain

$$(c_T r')^2 - (t_1')^2 = \gamma^2[(1 - {v_T}^2/{c_T}^2)(c_T r')^2 - (1 - {v_T}^2/{c_T}^2)(t_1')^2] \tag{36}$$

The quantity γ is deduced from Equation (36) as

$$\gamma = \frac{1}{\sqrt{1 - {v_T}^2/{c_T}^2}} \tag{37}$$

From Equations (34), for infinitesimal changes, the formulas for the length contraction and the time dilation are derived as

$$\frac{dt_1}{dt_1'} = \frac{1}{\sqrt{1 - \frac{{v_T}^2}{{c_T}^2}}}, \tag{38}$$

$$\frac{dr}{dr'} = \frac{1}{\sqrt{1 - \frac{{v_T}^2}{{c_T}^2}}}. \tag{39}$$

For two events that are infinite close to each other, the invariant infinitesimal interval ds can be defined as

$$ds^2 = {c_T}^2 dr^2 - {dt_1}^2 - {dt_2}^2 - {dt_3}^2 \tag{40}$$

5. A temporal general relativity

A general temporal relativity can be developed from the special temporal relativity formulated in Section 4. Consider two inertial reference systems S and S' with coordinates with coordinates (t_1, t_2, t_3, r) and (t_1', t_2', t_3', r'). We assume that the field equations of the temporal general relativity take the same form as that of Einstein's field equations, except for the roles of space and time are reversed, as follows [6]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{41}$$

where $T_{\mu\nu}$ is the temporal energy-momentum tensor, $g_{\mu\nu}$ is the temporal metric tensor, $R_{\mu\nu}$ is the temporal Ricci curvature tensor, R is the temporal scalar curvature, Λ is the temporal cosmological constant. For our discussions in this work, we assume a centrally symmetric field, and in this case the space-time metric can be written as

$$ds^2 = e^{\psi} c_T{}^2 dr^2 - e^{\chi} dt^2 - t^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{42}$$

Denoting the coordinates $(c_T r, t, \theta, \phi)$ by $\{x^\mu\}$ $(\mu = 0,1,2,3)$, the metric tensor $g_{\mu\nu}$ of this line element is

$$g_{\mu\nu} = \begin{pmatrix} e^{\psi} & 0 & 0 & 0 \\ 0 & -e^{\chi} & 0 & 0 \\ 0 & 0 & -t^2 & 0 \\ 0 & 0 & 0 & -t^2 \sin^2\theta \end{pmatrix} \tag{43}$$

In terms of the metric $g_{\mu\nu}$ given in Equation (43), the non-zero components of the connection

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\mu g_{\mu\lambda} - \partial_\mu g_{\mu\nu} \right) \tag{44}$$

are found as

$$\Gamma^0_{00} = \frac{1}{2} \frac{\partial \psi}{\partial r}, \quad \Gamma^1_{00} = \frac{e^{\psi - \chi}}{2} \frac{\partial \psi}{\partial t}, \quad \Gamma^0_{01} = \frac{1}{2} \frac{\partial \psi}{\partial t}, \quad \Gamma^1_{01} = \frac{1}{2} \frac{\partial \chi}{\partial r}$$

$$\Gamma^0_{11} = \frac{e^{\psi - \chi}}{2} \frac{\partial \chi}{\partial r}, \quad \Gamma^1_{11} = \frac{1}{2} \frac{\partial \chi}{\partial t}, \quad \Gamma^2_{12} = \frac{1}{t}, \quad \Gamma^3_{13} = \frac{1}{t}, \quad \Gamma^3_{23} = \cot\theta$$

$$\Gamma^1_{22} = -t e^{-\chi}, \quad \Gamma^1_{33} = -t \sin^2\theta e^{-\chi}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta \tag{45}$$

With the line element in Equation (42), and $\Lambda = 0$, the vacuum solutions satisfy the following system of equations

$$\frac{\partial \psi}{\partial t} + \frac{1}{t} - \frac{e^{\chi}}{t} = 0$$

$$\frac{\partial \chi}{\partial t} - \frac{1}{t} + \frac{e^{\chi}}{t} = 0$$

$$\frac{\partial \chi}{\partial r} = 0$$

$$2 \frac{\partial^2 \psi}{\partial t^2} + \left(\frac{\partial \psi}{\partial t} \right)^2 + \frac{2}{t} \left(\frac{\partial \psi}{\partial t} - \frac{\partial \chi}{\partial t} \right) - \frac{\partial \psi}{\partial t} \frac{\partial \chi}{\partial t} - e^{\psi - \chi} \left(2 \frac{\partial^2 \chi}{\partial r^2} + \left(\frac{\partial \chi}{\partial r} \right)^2 - \frac{\partial \psi}{\partial r} \frac{\partial \chi}{\partial r} \right) = 0 \tag{46}$$

These equations are not independent, since it can be verified that the last equation follows from the first three equations. Furthermore, the first two equations result in $\partial\psi/\partial t +$

$\partial\chi/\partial t = 0$, which leads to $\psi + \chi = 0$, due to the possibility of an arbitrary transformation of the spatial coordinate. The system of equations (46) admits a temporal line element, which has the form of the Schwarzschild line element

$$ds^2 = \left(1 + \frac{B}{t}\right) {c_T}^2 dr^2 - \left(1 + \frac{B}{t}\right)^{-1} dt^2 - t^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{47}$$

The temporal field given by Equation (47) has a singularity at the time $t = 0$. However, whether it has a singular spherical surface at $t = B$ depends on the sign of B. For the case of a quantum particle that is emitted from a quantum system, as discussed in the introduction, we take $B > 0$. This line element yields the temporal potential

$$V(t) = \frac{B}{t} \tag{48}$$

If we define a temporal force $F(t)$ by the relation $F(t) = -\, dV/dt$, then we have

$$F(t) = \frac{B}{t^2} \tag{49}$$

Since $B > 0$, the force given in Equation (49) is a repulsive force. However, if the constant of integration B take negative values, $B < 0$, then besides the singularity at $t = 0$, we also have a singular spherical surface at $t = B$. This can be considered as a temporal black hole.

In the following we will extend our discussion for a non-zero energy-momentum tensor of the form

$$T_\mu^\nu = \begin{pmatrix} -\frac{\alpha\beta}{\kappa}\frac{e^{-\beta t}}{t^2} & 0 & 0 & 0 \\ 0 & -\frac{\alpha\beta}{\kappa}\frac{e^{-\beta t}}{t^2} & 0 & 0 \\ 0 & 0 & \frac{\alpha\beta^2}{2\kappa}\frac{e^{-\beta t}}{t} & 0 \\ 0 & 0 & 0 & \frac{\alpha\beta^2}{2\kappa}\frac{e^{-\beta t}}{t} \end{pmatrix} \tag{50}$$

where the constants α and β will be needed to be specified. It can be verified that the energy-momentum tensor given in Equation (50) satisfies the conservation law

$$\nabla_\nu T_\mu^\nu = \frac{1}{\sqrt{-g}} \frac{\partial T_\mu^\nu \sqrt{-g}}{\partial x^\nu} - \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu} T^{\lambda\sigma} = 0 \tag{51}$$

With the energy-momentum tensor given in Equation (50) together with the metric tensor (43), the field equations of the temporal general relativity are reduced to the following system of equations

$$e^{-\chi}\left(\frac{\partial\chi}{\partial t} - \frac{1}{t}\right) + \frac{1}{t} = -\alpha\beta\frac{e^{-\beta t}}{t}$$

$$e^{-\chi}\left(\frac{\partial\psi}{\partial t}-\frac{1}{t}\right)+\frac{1}{t}=-\alpha\beta\frac{e^{-\beta t}}{t}$$

$$\frac{\partial\chi}{\partial r}=0$$

$$-e^{\chi}\left(2\frac{\partial^2\psi}{\partial t^2}+\left(\frac{\partial\psi}{\partial t}\right)^2+\frac{2}{t}\left(\frac{\partial\psi}{\partial t}-\frac{\partial\chi}{\partial t}\right)-\frac{\partial\psi}{\partial t}\frac{\partial\chi}{\partial t}\right)+e^{\psi}\left(2\frac{\partial^2\chi}{\partial r^2}+\left(\frac{\partial\chi}{\partial r}\right)^2-\frac{\partial\psi}{\partial r}\frac{\partial\chi}{\partial r}\right)$$
$$=\alpha\beta\frac{e^{-\beta t}}{t} \quad (52)$$

The system of equations given in Equation (52) when integrated gives the following line element

$$ds^2=\left(1-\alpha\frac{e^{-\beta t}}{t}+\frac{Q}{t}\right)c_T{}^2dr^2-\left(1-\alpha\frac{e^{-\beta t}}{t}+\frac{Q}{t}\right)^{-1}dt^2-t^2(d\theta^2+\sin^2\theta d\phi^2) \quad (53)$$

The line element given in Equation (53) yields a mixed potential of the form

$$V(t)=-\alpha\frac{e^{-\beta t}}{t}+\frac{Q}{t} \quad (54)$$

The constants α, β and the constant of integration Q need to be specified with known physical quantities. From Equation (54) we obtain the following equations by differentiation

$$\frac{dV}{dt}=\alpha e^{-\beta t}\left[\frac{\beta}{t}+\frac{1}{t^2}\right]-\frac{Q}{t^2} \quad (55)$$

$$\frac{d^2V}{dt^2}=-\alpha e^{-\beta t}\left[\frac{\beta^2}{t}+\frac{2\beta}{t^2}+\frac{2}{t^3}\right]+\frac{2Q}{t^3} \quad (56)$$

The corresponding temporal force defined by the relation $F(t)=-dV/dt$ is

$$F(t)=-\alpha e^{-\beta t}\left[\frac{\beta}{t}+\frac{1}{t^2}\right]+\frac{Q}{t^2} \quad (57)$$

The force of interaction given by Equation (57) can be attractive or repulsive depending on the time of interaction. In particular, we can assume $F(t)=0$ at $t=1/\beta=T$ and with this assumption, from Equation (57), we obtain the relation

$$\beta^2\left(Q-\frac{2\alpha}{e}\right)=0 \quad (58)$$

Assume $\beta\neq 0$ then we have $Q=2\alpha/e$. The mixed potential in Equation (54) now takes the form

$$V(t)=-\frac{eQ}{2}\frac{e^{-\frac{t}{T}}}{t}+\frac{Q}{t} \quad (59)$$

and the force of interaction becomes

$$F(t) = -\frac{eQ}{2} e^{-\frac{t}{T}} \left[\frac{1}{Tt} + \frac{1}{t^2}\right] + \frac{Q}{t^2} \tag{60}$$

In order to investigate further we need to know the nature of the stationary point at $t = T$. The second derivative of $V(t)$ at $t = T$ is given by

$$\frac{d^2V}{dt^2} = -\frac{Q}{2T^3} \tag{61}$$

Depending on the signs of Q we have two different cases. In the case when $Q < 0$ we have $d^2V/dt^2 > 0$, therefore *V(t)* has a minimum at $t = T$. Since $F(t) = -dV/dt$, the force is repulsive for $t < T$ and attractive for $t > T$. In the following, we will focus on the case when $Q > 0$. In this case $d^2V/dt^2 < 0$, therefore *V(t)* has a maximum at $t = T$. Since $F(t) = -dV/dt$, the force is attractive for $t < T$ and repulsive for $t > T$. At $t \gg T$, we expect $V(t) = h/t$, where h is the Planck's constant, the mixed potential becomes

$$V(t) = -\frac{eh}{2}\frac{e^{-\frac{t}{T}}}{t} + \frac{h}{t} \tag{62}$$

The corresponding temporal force is found as

$$F(t) = -\frac{eh}{2} e^{-\frac{t}{T}} \left[\frac{1}{Tt} + \frac{1}{t^2}\right] + \frac{h}{t^2} \tag{63}$$

With the temporal force given by Equation (63), instead of Equation (13), we have the following equation

$$\mathbf{F}.\mathbf{v} = -\frac{eh}{2} e^{-\frac{t}{T}} \left[\frac{1}{Tt} + \frac{1}{t^2}\right] + \frac{h}{t^2} \tag{64}$$

Also assuming $\mathbf{F}.\mathbf{v} = Fv$ and using Newton's second law $F = m\,dv/dt$, we obtain

$$mv\frac{dv}{dt} = h\left[-\frac{e}{2} e^{-\frac{t}{T}} \left[\frac{1}{Tt} + \frac{1}{t^2}\right] + \frac{1}{t^2}\right] \tag{65}$$

If we also consider the condition that at the initial time $t_1 = T$, the velocity of the particle is $v = v_0$, then solutions to Equation (65) are found as

$$\frac{mv^2}{2} = \frac{mv_0^2}{2} + h\left[\frac{1}{T} - \frac{1}{t}\right] + \frac{h}{2}\left[\frac{e^{1-\frac{t}{T}}}{t} - \frac{1}{T}\right] \tag{66}$$

From Equation (66) it is seen that when $t \to \infty$, we have $mv^2/2 - mv_0^2/2 = \frac{1}{2}h/T$. This amount of energy that is transferred to an elementary particle from a quantum system is equal to the energy of the ground level of the harmonic oscillator.

It is also interesting to note that, as in the case of Einstein's general theory of relativity, the temporal general relativity may be formulated as a quantum theory by applying various quantisation methods used in quantum gravity [8-9]. In particular, if the field equations of the temporal general relativity $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$ are derived through the principle of least action $\delta S = 0$, where the action S is defined as $S = \int \left(\frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M \right) \sqrt{-g}\, d^4x$ [11], and if the energy-momentum tensor is defined by the relation $T_{\mu\nu} = (\Lambda/\kappa) g_{\mu\nu}$, then for the case when $n = 2$, because we have the identity $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$, the resulting field equations of the temporal general relativity are satisfied by any metric tensor $g_{\mu\nu}$. This result can be seen as a verification of Feynman's postulate in the path integral formulation of quantum mechanics [7,12].

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