SPACETIME STRUCTURES OF QUANTUM PARTICLES

Vu B Ho

Advanced Study, 9 Adela Court, Mulgrave, Victoria 3170, Australia

Abstract: In this work we propose a covariant formulation for the gravitational field and derive equations that can be used to construct the spacetime structures for short-lived and stable quantum particles. We also show that Schrödinger wavefunctions can be used to construct spacetime structures for the quantum states of a quantum system, such as the hydrogen atom. Even though our discussions in this work are focused on the microscopic objects, the results obtained can be applied equally to the macroscopic phenomena.

1. A covariant formulation of classical physics

In this section we show that the three main formulations of physics, namely, Newton's second law of motion, the field equations of the electromagnetic field and the field equations of the gravitational field can be formulated in similar covariant forms so that the formulations differ only by the nature of the geometrical objects that represent the corresponding physical entities. We will show that Newton's law can be represented by a scalar, the electromagnetic field by a symmetric affine connection and the gravitational field by a symmetric metric tensor. In classical physics, for conservative forces, Newton's second law can be written in terms of a potential energy V as follows [1]

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} \tag{1}$$

$$\mathbf{F} = -\boldsymbol{\nabla} V \tag{2}$$

and the conventional Maxwell field equations of the electromagnetic field are written as [2,3]

$$\boldsymbol{\nabla}.\mathbf{j}_e + \frac{\partial \rho_e}{\partial t} = 0 \tag{3}$$

$$\boldsymbol{\nabla}.\mathbf{E} = \frac{\rho_e}{\epsilon} \tag{4}$$

$$\boldsymbol{\nabla}.\mathbf{B} = 0 \tag{5}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{6}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \epsilon\mu\frac{\partial \mathbf{E}}{\partial t} = \mu\mathbf{j}_e \tag{7}$$

and Einstein field equations of the gravitational field are written in the covariant form [4]

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta} \tag{8}$$

However, Maxwell field equations of the electromagnetic field can be formulated in the following covariant form

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}} = \mu j^{\beta} \tag{9}$$

where the electromagnetic tensor $F^{\alpha\beta}$ is expressed in terms of the four-vector potential $A^{\mu} \equiv (V, \mathbf{A})$ as $F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$. The four-current j^{μ} is defined as $j^{\mu} \equiv (\rho_e, \mathbf{j}_e)$. Since the electromagnetic tensor $F^{\alpha\beta}$ is anti-symmetric, it can be expressed in terms of a dual vector. In terms of the electromagnetic tensor $F^{\alpha\beta}$, the electromagnetic energy-momentum tensor $T^{\alpha\beta}$ for the free electromagnetic field with the defined Lagrangian of the form $L = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}$ can be established

$$T^{\alpha\beta} = \epsilon_0 c^2 \left(\eta_{\mu\nu} F^{\alpha\mu} F^{\nu\beta} + \frac{1}{4} \eta^{\alpha\beta} F_{\lambda\nu} F^{\lambda\nu} \right) \tag{10}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric tensor [2,3]. We now show how a covariant form as given in Equation (9) for the electromagnetic field can be formulated for Newton's law of mechanical dynamics and the field equations of the gravitational field.

In order to formulate Newton's law of dynamics covariantly, we write Newton's second law given in Equations (1) and (2) in terms of the potential energy V with the coordinate notation $x^{\mu} = (x, y, z)$ as follows

$$m \frac{d^2 x^{\mu}}{dt^2} + \frac{\partial V}{\partial x^{\mu}} = 0 \tag{11}$$

From the definition of work done in classical mechanics, which is defined as the line integral of a force $\mathbf{F}$ along a path C, $W = \int_C \mathbf{F}.d\mathbf{r}$, the conserved energy E for a particle of inertial mass m is established as

$$E = \frac{1}{2} m \sum_{\mu=1}^{3} \left(\frac{dx^{\mu}}{dt} \right)^2 + V \tag{12}$$

From Equation (12), we obtain the following relation by differentiation

$$\frac{\partial E}{\partial x^{\mu}} = m \frac{d^2 x^{\mu}}{dt^2} + \frac{\partial V}{\partial x^{\mu}} \tag{13}$$

If the dynamics of the particle satisfies Newton's second law $m\, d^2 x^{\mu}/dt^2 + \partial V / \partial x^{\mu} = 0$ then we obtain

$$\frac{\partial E}{\partial x^{\mu}} = 0 \tag{14}$$

It is seen that Equation (14) has the covariant form similar to Maxwell equations of the electromagnetic field given in Equation (9). However, the covariant equation for Newton's dynamics is related to a scalar rather than a tensor as in the field equations for electromagnetism. Furthermore, it is interesting to observe the following. While the total energy $E = T + V$ of a physical system is the sum of the kinetic energy and the potential energy, the Lagrangian $L = T - V$ of a physical system is essentially the difference between the kinetic energy and the potential energy. As in the case for the total energy E, we can write

$$\frac{\partial L}{\partial x^{\mu}} = m\frac{d^2 x^{\mu}}{dt^2} - \frac{\partial V}{\partial x^{\mu}} \tag{15}$$

Equation (15) can be used to describe, for example, the expansion of a physical system in which the term $\partial V/\partial x^{\mu}$ can be considered as a repulsive force. If the Lagrangian of a physical system is conserved

$$\frac{\partial L}{\partial x^{\mu}} = 0 \tag{16}$$

then we obtain

$$m\frac{d^2 x^{\mu}}{dt^2} - \frac{\partial V}{\partial x^{\mu}} = 0 \tag{17}$$

We now show that the field equations of the gravitational field can be proposed and formulated in a covariant form similar to the covariant form of the electromagnetic field given in Equation (9). It is shown in differential geometry that the Ricci tensor $R^{\alpha\beta}$ satisfies the Bianchi identities [5]

$$\nabla_{\beta} R^{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R \tag{18}$$

where $R = g^{\alpha\beta} R_{\alpha\beta}$ is the Ricci scalar curvature. Even though Equation (18) is purely geometrical, it has a covariant form similar to Equation (9) for the electromagnetic tensor $\partial_{\alpha} F^{\alpha\beta} = \mu j^{\beta}$. If the quantity $\frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R$ can be identified as a physical entity, such as a four-current of gravitational matter, then Equation (18) has the status of a dynamical law of a physical theory. In this case a four-current $j^{\alpha} = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as

$$j^{\alpha} = \frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R \tag{19}$$

In later sections we will show that the purely geometrical four-current j^{α} defined by Equation (19) can be established as physical entities, however, in the following we want to show that for the case of a purely gravitational field in which $\frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R = 0$, the proposed field equations given in Equation (18) also arrive at the same results as those from Einstein's formulation of the gravitational field. For a purely gravitational field, Equation (18) reduces to the equation

$$\nabla_\beta R^{\alpha\beta} = 0 \tag{20}$$

The field equations given in Equation (20) play the role of Maxwell field equations for the free electromagnetic field. Even though rigorous solutions to the dynamical field equations given in Equation (20) would require laborious mathematical investigations, we can obtain solutions that are found from the original Einstein field equations, such as Schwarzschild solution, by observing that, since $\nabla_\mu g^{\alpha\beta} \equiv 0$, Equation (20) implies

$$R^{\alpha\beta} = \Lambda g^{\alpha\beta} \tag{21}$$

where Λ is an undetermined constant. Equation (21) can also be written in a covariant form as

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{22}$$

Using the identities $g_{\alpha\beta}g^{\alpha\beta} = 4$ and $g_{\alpha\beta}R^{\alpha\beta} = R$, we obtain $\Lambda = R/4$. If we consider a centrally symmetric gravitational field with the metric

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{23}$$

then the Schwarzschild solution can be found as [6]

$$ds^2 = \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)c^2dt^2 - \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{24}$$

It is observed that, as in the case of the free electromagnetic field, with the Schwarzschild solution obtained, the energy-momentum tensor $T_{\alpha\beta}$ for the gravitational field can be established if we define it through the relation $T_{\alpha\beta} = \frac{1}{\kappa}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)$.

2. Probabilistic characteristics of geometrical objects

In this section, we discuss the possibility of identifying geometrical objects with physical entities and we show that the identifications provide a route to formulate dynamical equations that describe probabilistic processes in physical theories. In particular, by assuming the purely geometrical Bianchi identities as covariant field equations of the gravitational field we are able to derive a geometric diffusion equation and a Schrödinger-like wave equation that can be used to describe random movement of particles as spacetime structures. First we want to show that the probabilistic characteristics of geometrical objects also manifest even in semi-classical theory such as Bohr's theory of the hydrogen atom [7]. As shown in the appendix 1, the momentum p of the particle and the curvature κ of its trajectory in a plane are related through the relation $p = \hbar\kappa$. According to the canonical formulation of classical physics, the particle dynamics is governed by the action principle $\delta S = \delta \int p ds = 0$. Using the relationship $p = \hbar\kappa$ and the expression of the curvature of a trajectory $f(x)$ in a plane, $\kappa = f''/(1 + f'^2)^{3/2}$, the action integral S takes the form

$$S = \int \hbar\kappa ds = \int \frac{\hbar f''}{1 + f'^2} dx \tag{25}$$

It is shown in the calculus of variations that to extremise the integral $S = \int L(f, f', f'', x)dx$, the function $f(x)$ must satisfy the differential equation [8]

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial f''} = 0 \tag{26}$$

However, with the functional of the form given in Equation (25), $L = \hbar f''/(1 + f'^2)$, it is straightforward to verify that the differential equation (26) is satisfied by any function $f(x)$. This result may be considered as a foundation for the Feynman's path integral formulation of quantum mechanics, which uses all classical trajectories of a particle in order to calculate the transition amplitude of a quantum mechanical system [9]. Since any path can be taken by a particle moving in a plane, if the orbits of the particle are closed, it is possible to represent each class of paths of the fundamental homotopy group of the particle by a circular path, since topologically, any path in the same equivalence class can be deformed continuously into a circular path. This validates Bohr's assumption of circular motion for the electron in a hydrogen-like atom. This assumption then leads immediately to the Bohr quantum condition

$$\oint pds = \hbar \oint \kappa ds = \hbar \oint \frac{ds}{r} = \hbar \oint d\theta = nh \tag{27}$$

The Bohr quantum condition possesses a topological character in the sense that the principal quantum number n is identified with the winding number, which is used to represent the fundamental homotopy group of paths of the electron in the hydrogen atom.

Now we show that the geometrical objects that are identified as physical entities from the covariant form of the field equations of the gravitational field given in Equation (18) also manifest the probabilistic characteristics. From the four-current of matter given in Equation (19), by letting $\alpha = 0$, we obtain the matter density component of the four-current as

$$\rho = j^0 = \frac{1}{2} g^{0\beta} \nabla_\beta R = \frac{1}{2} g^{0\beta}\, \partial_\beta R \tag{28}$$

It is seen from Equation (28) that in order to be able to define matter in terms of geometrical objects, the Ricci scalar must depend explicitly on the coordinates x^μ. In particular, if the metric tensor $g^{\alpha\beta}$ is diagonal and the Ricci tensor depends explicitly on the temporal coordinate $x^0 = ct$ then we have the geometrical density

$$\rho = \frac{1}{2c} g^{00}\, \partial_t R \tag{29}$$

In order to give the entity ρ a physical content, we introduce a dimensional constant k_1 and Equation (29) is rewritten as

$$\rho = \frac{k_1}{2c} g^{00}\, \partial_t R \tag{30}$$

We will assume that the field equations of general relativity given in Equation (18) can be applied to the microscopic space-time structures of quantum particles and, furthermore, in order to specify a particular form for the quantity g^{00}, we will adopt Weyl's postulate, even though the postulate has mainly been used for considerations of macroscopic phenomena. Weyl's postulate requires that the geodesics of the perfect fluid particles are orthogonal to a family of spacelike hypersurfaces. As a consequence, a commoving frame can be introduced such that the line element can be written in the form [5]

$$ds^2 = g_{00}c^2dt^2 - a^2(t)g_{\alpha\beta}dx^\alpha dx^\beta \tag{31}$$

It is noted that Weyl's hypothesis allows us to think of the geometry in which spatial structures evolve over time. With this view, quantum particles that are formed from the microscopic space-time structures can be regarded as normal elementary particles in three-dimensional Euclidean space. As a consequence, we will assume that the matter density ρ in Equation (30) also satisfies the Poisson's equation for a potential V in classical physics

$$\nabla^2 V = 4\pi k_2 \rho \tag{32}$$

where k_2 is a dimensional constant. Normally, Poisson's equation is used to describe the potential field of a conservative force, which is time-independent. However, Poisson's equation can also be used for time-dependent potentials if the Coulomb gauge is applied. In fact, Poisson's equation can also be formulated for non-conservative forces in which the potentials are time-dependent [10]. And, even though it conserves the energy-momentum tensor, general relativity is non-conservative. Therefore, we can assume that the potential in Poisson's equation given in Equation (32) is time-dependent. As in the case of Einstein theory of general relativity in which the field equations are proposed by observing the similarity between the Bianchi identities $\nabla_\beta\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right) \equiv 0$ and the conservation of the energy-momentum tensor $\nabla_\beta T^{\alpha\beta} \equiv 0$, in the following we will assume that the scalar potential V and the Ricci scalar to be related by the relation

$$V = k_3 R \tag{33}$$

where k_3 is an undetermined dimensional constant. With the above assumptions, from Equations (30), (32) and (33) we obtain

$$\nabla^2(k_3 R) = \frac{4\pi k_2 k_1}{2c} g^{00}\,\partial_t R \tag{34}$$

Equation (34) is rewritten as

$$\partial_t R = \frac{ck_3}{2\pi k_1 k_2 g^{00}} \nabla^2 R \tag{35}$$

In order to investigate further we need to specify the time component g^{00} of the metric tensor $g^{\alpha\beta}$. It is seen from Equations (2) and (4) given in the appendix 2 that a real spacetime structure that is described by the Ricci tensor can admit a real metric tensor or an imaginary metric tensor. If the metric tensor $g^{\alpha\beta}$ is real then we obtain a diffusion equation

$$\partial_t R = k\nabla^2 R \tag{36}$$

where the constant $k = ck_3/(2\pi k_1 k_2 g^{00})$. While investigating the theory of the Brownian movement of particles suspended in a liquid, Einstein derived the following one-dimensional differential equation for diffusion [11]

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2} \tag{37}$$

where $f(x,t)$ can be identified with the concentration per unit length of the number of particles or of the substance under study, and D is the coefficient of diffusion. The solution to Equation (37) is

$$f(x,t) = \frac{M}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}} \tag{38}$$

where $M = \int_{-\infty}^{\infty} f(x,t)dx$ is the total number of particles or the total mass of the substance. For the case of a three-dimensional diffusion equation given in Equation (36), solutions can be found to take the form [12]

$$R(x,y,z,t) = \frac{M}{\left(\sqrt{4\pi kt}\right)^3}e^{-\frac{x^2+y^2+z^2}{4kt}} \tag{39}$$

Equation (39) determines the probabilistic distribution of an amount of geometrical substance M which is defined via the Ricci scalar R and manifests as observable matter. On the other hand, if the metric tensor $g^{\alpha\beta}$ is imaginary, then since the Ricci tensor $R_{\alpha\beta}$ is real and the Ricci scalar R is a contraction of the metric tensor and the Ricci tensor given by the relation $R = g^{\alpha\beta}R_{\alpha\beta}$, the Ricci scalar R is imaginary. If we let $R = i\Psi$ then Equation (36) can be written as

$$\partial_t \Psi = ik\nabla^2 \Psi \tag{40}$$

Equation (40) is similar to the free particle Schrödinger wave equation in quantum mechanics

$$\frac{\partial \psi(\mathbf{r},\mathbf{t})}{\partial t} = i\frac{\hbar}{2\mu}\nabla^2\psi(\mathbf{r},\mathbf{t}) \tag{41}$$

The similarity between Equations (40) and (41) suggests that the Schrödinger wavefunction $\psi(\mathbf{r},\mathbf{t})$ may intrinsically be related to the geometrical structure of spacetime that can be materialised to become observable as quantum particles. However, it is seen that unless $\partial_t \mathrm{R} = 0$, Equation (40) can only be realised within the existing framework of mathematics if there exist real functions whose rates of change are imaginary functions. In fact, such functions can be used to describe real physical processes without their property of producing imaginary rates being realised. For example, if the rates of change of a real function $f(x)$ are given as $df(x)/dx = ik_1 f(x)$ and $d(df(x)/dx)/dx = ik_2\, df(x)/dx$, where k_1 and k_2 are real, then $d^2f(x)/dx^2 + k_1k_2 f(x) = 0$ is a real equation, which can be used to describe a wave motion. If we generalise Equation (36) by assuming that the Ricci scalar R can take complex-valued values then a complex solution to Equation (36) can be obtained as

$$\mathrm{R}(x,y,z,t)=\frac{M}{\left(\sqrt{4\pi ikt}\right)^{3}}e^{i\left(\frac{x^2+y^2+z^2}{4kt}\right)} \tag{42}$$

Equation (42) can also be rewritten in the form

$$\mathrm{R}(x,y,z,t)=\frac{-M}{\left(\sqrt{4\pi kt}\right)^{3}}\left(\frac{\sqrt{2}+i\sqrt{2}}{2}\right)\left(\cos\left(\frac{x^2+y^2+z^2}{4kt}\right)+i\sin\left(\frac{x^2+y^2+z^2}{4kt}\right)\right) \tag{43}$$

3. The spacetime structures of elementary particles

In this section we investigate spacetime structures of quantum particles by deriving equations that can be used to construct line elements for given Ricci scalar curvatures. The diffusion equation given in Equation (36) describes the density fluctuations of a geometrical substance that is undergoing diffusion. We assume that the geometrical substance materialises to appear as quantum particles from spacetime structures. For the Ricci scalar given in Equation (39), due to the spatial symmetry of the Ricci scalar, we seek a line element of the form

$$ds^2=D(cdt)^2-A(x,y,z,t)[(dx)^2+(dy)^2+(dx)^2] \tag{44}$$

where D is constant. As shown in the appendix 2, the quantity $A(x,y,z,t)$ satisfies the following differential equation

$$-\frac{3}{c^2DA}\frac{\partial^2A}{\partial t^2}+\frac{2}{A^2}\nabla^2A+\frac{3}{2A^3}(\boldsymbol{\nabla}A)^2=\frac{M}{\left(\sqrt{4\pi kt}\right)^{3}}e^{-\frac{x^2+y^2+z^2}{4kt}} \tag{45}$$

Asymptotically, Equation (45) describes a wave motion for $t\to\infty$ given the gauge condition that involves the first derivatives. Those quantum particles that can be described by Equation (45) are short-lived subatomic particles. They appear for a short time and then disappear into the purely gravitational field with a wave motion when $\mathrm{R}\to 0$. However, whether Equation (45) can be solved to obtain exact solutions requires further investigation. If the metric tensor of the line element given in Equation (44) is complex, then instead of Equation (45), we obtain the following equation for the complex function $A(x,y,z,t)$

$$-\frac{3}{c^2DA}\frac{\partial^2A}{\partial t^2}+\frac{2}{A^2}\nabla^2A+\frac{3}{2A^3}(\boldsymbol{\nabla}A)^2=\frac{M}{\left(\sqrt{4\pi ikt}\right)^{3}}e^{i\left(\frac{x^2+y^2+z^2}{4kt}\right)} \tag{46}$$

Now, it is noted that if the Ricci scalar is time-independent then Equation (36) reduces to Laplace equation

$$\nabla^2\mathrm{R}=0 \tag{47}$$

From the relation $j^{\alpha} = (\rho, \mathbf{j}_i) = \frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R = \frac{1}{2} g^{\alpha\beta}\, \partial_{\beta} R$, for the case of a symmetric metric tensor, we have $\rho = 0$, therefore the spatial structure of a quantum particle is visualised only in terms of the three-current $\mathbf{j}_i$. Apart from the interesting question that arises from this result about what form of matter the three-current $\mathbf{j}_i$ to represent, such physical structure is possible only when the metric tensor itself depends explicitly on time but the contraction $R = g^{\alpha\beta} R_{\alpha\beta}$ reduces the Ricci scalar to time-independent. For example, from the defined relation of the Ricci scalar $R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$, if $g^{00}R_{00}$ is time-independent and g^{11}, g^{11} and g^{11} are time-dependent, a time-independent Ricci scalar can be obtained if $g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \equiv 0$.

If the spatial structure of a quantum particle is considered to be spherically symmetric then its materialised spatial structure can be described using spherical coordinates (r, θ, ϕ). Equation (47) takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \mathrm{R}}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \mathrm{R}}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 \mathrm{R}}{\partial \phi^2}\right) = 0 \qquad (48)$$

The general solution for the Ricci scalar R can be found as [13]

$$\mathrm{R}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + B_{lm} r^{-(l+1)}\right) \mathrm{Y}_{lm}(\theta, \phi) \qquad (49)$$

where $\mathrm{Y}_{lm}(\theta, \phi)$ is the spherical harmonics and the coefficients A_{lm} and B_{lm} can be determined from the boundary conditions.

If the spatial structure of an elementary particle is considered to be cylindrically symmetric, such as a thin disc that forms the rotor of a gyroscope, then its materialised spatial structure can be described using cylindrical coordinates (ρ, ϕ, z). The Laplace equation given in Equation (47) now takes the form

$$\frac{\partial^2 \mathrm{R}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \mathrm{R}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \mathrm{R}}{\partial \phi^2} + \frac{\partial^2 \mathrm{R}}{\partial z^2} = 0 \qquad (50)$$

The general form of the solution for the boundary problem where the cylinder has a radius a and a height L is found as [13]

$$\mathrm{R}(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh\,(k_{mn} z)(A_{mn} \sin\,(m\phi) + B_{mn} \cos\,(m\phi)) \qquad (51)$$

where $J_m(x)$ is Bessel function, $k_{mn} = x_{mn}/a$ with x_{mn} are the roots of $J_m(x_{mn}) = 0$, and the coefficients A_{mn} and B_{mn} can be determined from the boundary conditions.

Although it is almost impossible to construct line elements for the whole value of the Ricci scalars given in Equations (49) and (51), it is possible to construct a line element for each quantum state for discrete values of l and m. For example, the quantum state with $l = 0$ and $m = 0$ for the spherically symmetric Ricci scalar given in Equation (49) is

$$\mathrm{R}(r) = \frac{1}{\sqrt{4\pi}}\left(\frac{B_{00}}{r} + A_{00}\right) \tag{52}$$

If we assume a spherically symmetric line element of the following form

$$ds^2 = D(cdt)^2 - A(r)(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\phi(d\phi)^2 \tag{53}$$

where D, A_{00} and B_{00} are undetermined constants, then using the result obtained in the appendix 2, we arrive at the differential equation

$$\frac{r}{2A^2}\frac{dA}{dr} + \frac{2}{r^2}\left(1 - \frac{1}{A}\right) = \frac{1}{\sqrt{4\pi}}\left(\frac{B_{00}}{r} + A_{00}\right) \tag{54}$$

Equation (54) is a first order non-linear differential equation and in general there exists a unique solution to the initial value problem that involves such equation.

4. A relationship between Schrödinger wavefunctions and spacetime structures

In this section, we show that there is a relationship between Schrödinger wavefunctions and the spacetime structures of a quantum system in the sense that Schrödinger wavefunctions are considered purely as mathematical objects that can be used for the construction of spacetime structures of the quantum states of a quantum system. In order to construct the spacetime structures for quantum particles, as discussed above, we observed the similarity between the equation $\rho = (k_1/2c)g^{00}\,\partial_t R$ and the equation $4\pi k_2\rho = \nabla^2 V$. We assumed that the scalar potential φ and the Ricci scalar to be related by the relation $V = k_3 R$, where k_1, k_2 and k_3 are undetermined dimensional constants. In the following we will discuss a procedure to construct spacetime structures for the quantum states of a quantum system in which the Schrödinger wavefunctions are employed as a pathway. We also assume that the relation $V = k_3 R$ is hold for any potential defined in classical physics. Since Schrödinger's original works were on the time-independent quantum states of the hydrogen atom, we first recapture the main ideas of Schrödinger's method to obtain the time-independent wave equation for the hydrogen atom. Schrödinger commenced with the Hamilton-Jacobi equation, written in terms of the Cartesian coordinates (x, y, z) as [14,15]

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2 - 2m\left(E + \frac{kq^2}{r}\right) = 0 \tag{55}$$

However, in order to obtain a partial differential equation that would give rise to the required results, Schrödinger introduced a new function ψ, which is real, single-valued and twice differentiable, through the relation

$$S = \hbar\ln\psi \tag{56}$$

where the action S is defined by

$$S = \int L dt \tag{57}$$

and L is the Lagrangian defined by

$$L = T - \varphi \tag{58}$$

with T is the kinetic energy and φ is the potential energy. In terms of the new function ψ, Equation (55) takes the form

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2 - \frac{2m}{\hbar^2}\left(E + \frac{kq^2}{r}\right)\psi^2 = 0 \tag{59}$$

Then by applying the principle of least action $\delta \int L dt = 0$, Schrödinger arrived at the required equation

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left(E + \frac{kq^2}{r}\right)\psi = 0 \tag{60}$$

Now we show that Schrödinger wavefunction ψ can be used to construct the spacetime structures of the quantum states of the hydrogen atom. By using the relations $L = dS/dt$, $dS/dt = \partial_t S + \sum_{\mu=1}^{3} \partial_\mu S(dx^\mu/dt)$, $T = m\sum_{\mu=1}^{3}(dx^\mu/dt)^2$ and $\varphi = T - L$, we obtain

$$\varphi = m\sum_{\mu=1}^{3}(dx^\mu/dt)^2 - \partial_t S + \sum_{\mu=1}^{3}\partial_\mu S(dx^\mu/dt) \tag{61}$$

In terms of the Schrödinger wavefunction ψ, Equation (61) can be rewritten as

$$\varphi = m\sum_{\mu=1}^{3}(dx^\mu/dt)^2 - \hbar\frac{\partial_t\psi + \sum_{\mu=1}^{3}\partial_\mu\psi(dx^\mu/dt)}{\psi} \tag{62}$$

From the assumed relations $V = k_3\mathrm{R}$ and $V = \varphi/m$, we arrive at the following relation between the Schrödinger wavefunction ψ and the Ricci scalar R

$$\mathrm{R} = \frac{1}{k_3}\left(\sum_{\mu=1}^{3}(dx^\mu/dt)^2 - \frac{\hbar}{m}\frac{\partial_t\psi + \sum_{\mu=1}^{3}\partial_\mu\psi(dx^\mu/dt)}{\psi}\right) \tag{63}$$

Since we will use spherical coordinates for the Schrödinger wave equation given in Equation (60), the Ricci scalar should also be written in terms of spherical coordinates (r, θ, ϕ). In terms of spherical coordinates, the Ricci scalar given in Equation (63) takes the form

$$\mathrm{R} = \frac{1}{k_3}\left(\left(\frac{dr}{dt}\right)^2 + r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 - \frac{\hbar}{m}\frac{\partial_t\psi + \partial_r\psi\frac{dr}{dt} + \partial_\theta\psi\frac{d\theta}{dt} + \partial_\phi\psi\frac{d\phi}{dt}}{\psi}\right) \tag{64}$$

In terms of the spherical coordinates (r, θ, ϕ), the eigenfunctions $\psi_{nlm}(r, \theta, \phi)$ for the hydrogen atom, which are solutions to the Schrödinger wave equation given in Equation (60), can be found [7]

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi) \tag{65}$$

where the spherical harmonics $Y_{lm}(\theta, \phi)$ and the radial functions $R_{nl}(r)$ are given as

$$Y_{lm}(\theta, \phi) = (-1)^m\left(\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right)^{\frac{1}{2}} P_l^m(\cos\theta)e^{im\phi} \tag{66}$$

$$R_{nl}(r) = -\left(\left(\frac{2}{na_0}\right)^3 \frac{(n-l-1)!}{2n\left((n+l)!\right)^3}\right)^{\frac{1}{2}} e^{-\frac{\rho}{2}}\rho^l L_{n+l}^{2l+1}(\rho) \tag{67}$$

where $\rho = 2r/na_0$ and $a_0 = 4\pi\varepsilon_0\hbar^2/mq^2$. The first few normalised wave functions for the hydrogen atom and their corresponding Ricci scalars are given below

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-\frac{r}{a_0}} \tag{68}$$

$$\mathrm{R} = \frac{1}{k_3}\left(\left(\frac{dr}{dt}\right)^2 + r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 + \frac{\hbar}{ma_0}\frac{dr}{dt}\right) \tag{69}$$

$$\psi_{200}(r, \theta, \phi) = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\left(2 - \frac{r}{a_0}\right) e^{-\frac{r}{2a_0}} \tag{70}$$

$$\mathrm{R} = \frac{1}{k_3}\left(\left(\frac{dr}{dt}\right)^2 + r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 + \frac{3\hbar}{ma_0}\frac{dr}{dt}\right) \tag{71}$$

$$\psi_{300}(r, \theta, \phi) = \frac{1}{81\sqrt{3\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\left(27 - \frac{18r}{a_0} + \frac{2r^2}{a_0^2}\right) e^{-\frac{r}{3a_0}} \tag{72}$$

$$\mathrm{R} = \frac{1}{k_3}\left(\left(\frac{dr}{dt}\right)^2 + r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 + \frac{\hbar}{m}\left(\frac{54 - 4r}{a_0}\right)\frac{dr}{dt}\right) \tag{73}$$

It is seen from the above results that unless particular conditions are imposed, such as the constancy of the kinetic energy of the electron on each quantum state, the mathematical construction of possible spacetime structures of the quantum states would require laborious mathematical investigations. However, we would like to give the following interesting discussion about the geometric structures of spacetime that are closely related to the undeterministic character of the quantum states of a quantum system. We would like to show that the undeterministic characteristics of a quantum system in quantum physics are the result of its geometrical structures. If we consider quantum spacetime structures as embedded surfaces in the Euclidean space $\mathbf{R}^3$ then the Ricci scalar curvature R is related to the Gaussian curvature $K = 1/k_1 k_2$ as

$$\mathrm{R} = \frac{2}{k_1 k_2} \tag{74}$$

where k_1 and k_2 are the principal radii of the surface. Consider a surface defined by the relation $x^3 = f(x^1, x^2)$ in Cartesian coordinates (x^1, x^2, x^3). The Ricci scalar curvature given in Equation (74) can be found as

$$\mathrm{R} = \frac{2(f_{11}f_{22} - (f_{12})^2}{(1 + f_1^2 + f_2^2)^2} \tag{75}$$

where $f_\mu = \partial f/\partial x^\mu$ and $f_{\mu\nu} = \partial^2 f/\partial x^\mu \partial x^\nu$ [16]. Let P be a 3-dimensional physical quantity which plays the role of the momentum p in the 2-dimensional space action integral. The quantity P can be identified with the surface density of a physical quantity, such as charge. Since the momentum p is proportional to the curvature κ, which determines the planar path of a particle, in the 3-dimensional space the quantity P should be proportional to the Ricci scalar curvature R, which is used to characterise a surface. If we consider a surface action integral of the form $S = \int P dA = \int (q/\pi)\, \mathrm{R} dA$, where q is a universal constant, which plays the role of Planck's constant, then we have

$$S = \frac{q}{\pi} \int \frac{f_{11}f_{22} - (f_{12})^2}{(1 + f_1^2 + f_2^2)^{3/2}} dx^1 dx^2 \tag{76}$$

According to the calculus of variations, in order to extremise the action integral of the form $S = \int L(f, f_\mu, f_{\mu\nu}, x^\mu) dx^1 dx^2$, the functional $L(f, f_\mu, f_{\mu\nu}, x^\mu)$ must satisfy the differential equations

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial f_\mu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu} \frac{\partial L}{\partial f_{\mu\nu}} = 0 \tag{77}$$

However, it is straightforward to verify that with the functional of the form obtained from Equation (76), $L = (q/2\pi)\,(f_{11}f_{22} - (f_{12})^2)/(1 + f_1^2 + f_2^2)^{3/2}$, the differential equation given by Equation (77) is satisfied by any surface. Hence, we can generalise Feynman's postulate to formulate a quantum theory in which the transition amplitude between states of a quantum mechanical system is a sum over random surfaces, provided the functional P in the action integral $S = \int P dA$ is taken to be proportional to the Ricci scalar curvature R of a surface. Consider a closed surface and assume that we have many such different surfaces which are described by the higher dimensional homotopy groups. As for the case of positive curvatures, we choose from among the homotopy class a representative spherical surface, in which case we can write

$$\oint PdA = \frac{q}{4\pi}\oint d\Omega, \tag{78}$$

where $d\Omega$ is an element of solid angle. Since $\oint d\Omega$ depends on the homotopy class of the sphere that it represents, we have $\oint d\Omega = 4\pi n$, where n is the topological winding number of the higher dimensional homotopy group. From this result we obtain a generalised Bohr quantum condition

$$\oint PdA = nq \tag{79}$$

From the result obtained in Equation (79), as in the case of Bohr's theory of quantum mechanics, we may consider a quantum process in which a physical entity transits from one surface to another with some radiation-like quantum created in the process. Since this kind of physical process can be considered as a transition from one homotopy class to another, the radiation-like quantum may be the result of a change of the topological structure of the physical system, and so it can be regarded as a topological effect. Furthermore, it is interesting to note that the action integral $(q/2\pi)\oint \mathrm{R}dA$ is identical to Gauss's law in electrodynamics and the constant q can be identified with the charge of a particle. In this case the charge q represents the topological structure of a physical system, and must exist in multiples of q. Hence, the charge of a physical system, such as an elementary particle, may depend on the topological structure of the system and is classified by the homotopy group of closed surfaces. This result may shed some light on why charge is quantised even in classical physics.

As a concluding remark, we would like to mention here that even though our discussions have been focused on the quantum objects, the results are equally applied to macroscopic phenomena.

Appendix 1

In differential geometry, the position vector $\mathbf{r}(s)$, the unit tangent vector $\mathbf{t}(s)$, the unit principal normal vector $\mathbf{p}(s)$ and the unit binormal vector $\mathbf{b}(s)$, defined by the relation $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{p}(s)$, satisfy the Frenet equations [16]

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{p}, \qquad \frac{d\mathbf{p}}{ds} = -\kappa\mathbf{t} + \varrho\mathbf{b}, \qquad \frac{d\mathbf{b}}{ds} = -\varrho\mathbf{p} \tag{1}$$

where $\kappa(s)$ and $\varrho(s)$ are the curvature and the torsion respectively, and $ds = \sqrt{d\mathbf{r}.d\mathbf{r}}$ is the linear element. If we consider the motion of a particle in a plane, as in the case of Bohr's model of a hydrogen-like atom, the Frenet equations reduce to

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{p}, \qquad \frac{d\mathbf{p}}{ds} = -\kappa\mathbf{t} \tag{2}$$

By differentiation, we obtain the following system of differential equations

$$\frac{d^2\mathbf{t}}{ds^2} - \frac{d(\ln\kappa)}{ds}\frac{d\mathbf{t}}{ds} + \kappa^2\mathbf{t} = 0 \tag{3}$$

$$\frac{d^2\mathbf{p}}{ds^2} - \frac{d(\ln\kappa)}{ds}\frac{d\mathbf{p}}{ds} + \kappa^2\mathbf{p} = 0 \tag{4}$$

If the curvature $\kappa(s)$ is assumed to vary slowly along the curve $\mathbf{r}(s)$, so that the condition $d(\ln\kappa)/ds = 0$ can be imposed, then $\mathbf{t}(s)$ and $\mathbf{p}(s)$ may be regarded as being oscillating with a spatial period, or wavelength, λ, whose relationship to the curvature κ is found as

$$\kappa = \frac{2\pi}{\lambda} \tag{5}$$

In the case of the Bohr's planar model of a hydrogen-like atom with circular orbits, the condition $d(\ln\kappa)/ds = 0$ is always satisfied, since the curvature remains constant for each of the orbits. In order to incorporate this elementary differential geometry into quantum mechanics, we identify the wavelength defined in Equation (5) with the de Broglie's wavelength of a particle. This seems to be a natural identification since the spatial period λ is the wavelength of the unit tangent vector $\mathbf{t}(s)$. With this assumption, the momentum p of the particle and the curvature κ are related through the relation

$$p = \hbar\kappa \tag{6}$$

Appendix 2

In this appendix, we show in detail the derivation of the equations to determine the metric tensor of the line element given in Equations (45) and (54). In differential geometry, the Riemann curvature tensor $R^\alpha{}_{\mu\beta\nu}$ is defined in terms of the affine connection $\Gamma^\gamma_{\alpha\beta}$ as

$$R^\alpha{}_{\mu\beta\nu} = \frac{\partial\Gamma^\alpha_{\mu\beta}}{\partial x^\nu} - \frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\beta} + \Gamma^\lambda_{\mu\beta}\Gamma^\alpha_{\lambda\nu} - \Gamma^\lambda_{\mu\nu}\Gamma^\alpha_{\lambda\beta} \tag{1}$$

The contraction of the Riemann curvature tensor given in Equation (1) with respect to the indices α and β gives the Ricci tensor

$$R_{\mu\nu} = \frac{\partial\Gamma^\sigma_{\mu\nu}}{\partial x^\sigma} - \frac{\partial\Gamma^\sigma_{\mu\sigma}}{\partial x^\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} \tag{2}$$

On the other hand, the contraction of the Riemann curvature tensor with respect to the indices α and μ gives the segmental curvature tensor [17]

$$Q_{\alpha\beta} = \frac{\partial\Gamma^{\lambda}_{\lambda\beta}}{\partial x^{\alpha}} - \frac{\partial\Gamma^{\lambda}_{\lambda\alpha}}{\partial x^{\beta}} \tag{3}$$

It is seen from Equation (3) that if the affine connection $\Gamma^{\gamma}_{\alpha\beta}$ is symmetric with respect to the indices α and β then the segmental curvature tensor $Q_{\alpha\beta}$ is anti-symmetric and in this case it might be used to formulate the electromagnetic field as given in Equations (9). In order to formulate the field equations for the gravitational field it is necessary to introduce a symmetric metric tensor $g_{\alpha\beta}$ in terms of which the affine connection $\Gamma^{\gamma}_{\alpha\beta}$ is defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \tag{4}$$

However, with the introduction of the symmetric metric tensor, it can be shown that the segmental curvature tensor vanishes. This result shows that the electromagnetic field and the gravitational field may not be formulated on the same structure of a Riemannian manifold.

With the line element given in Equation (44), we obtain the following non-zero components of the affine connection [18]

$$\Gamma^{1}_{01} = \Gamma^{1}_{10} = \frac{1}{2cA}\frac{\partial A}{\partial t}, \qquad \Gamma^{2}_{02} = \Gamma^{2}_{20} = \frac{1}{2cA}\frac{\partial A}{\partial t}, \qquad \Gamma^{3}_{03} = \Gamma^{3}_{30} = \frac{1}{2cA}\frac{\partial A}{\partial t}$$

$$\Gamma^{0}_{11} = \frac{1}{2cD}\frac{\partial A}{\partial t}, \qquad \Gamma^{1}_{11} = \frac{1}{2A}\frac{\partial A}{\partial x}, \qquad \Gamma^{2}_{11} = -\frac{1}{2A}\frac{\partial A}{\partial y}, \qquad \Gamma^{3}_{11} = -\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^{1}_{12} = \Gamma^{1}_{21} = \frac{1}{2A}\frac{\partial A}{\partial y}, \quad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2A}\frac{\partial A}{\partial x}, \quad \Gamma^{1}_{13} = \Gamma^{1}_{31} = \frac{1}{2A}\frac{\partial A}{\partial z}, \quad \Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{2A}\frac{\partial A}{\partial x}$$

$$\Gamma^{0}_{22} = \frac{1}{2cD}\frac{\partial A}{\partial t}, \qquad \Gamma^{1}_{22} = \frac{1}{2A}\frac{\partial A}{\partial x}, \qquad \Gamma^{2}_{22} = \frac{1}{2A}\frac{\partial A}{\partial y}, \qquad \Gamma^{3}_{22} = -\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^{0}_{33} = \frac{1}{2cD}\frac{\partial A}{\partial t}, \qquad \Gamma^{1}_{33} = -\frac{1}{2A}\frac{\partial A}{\partial x}, \qquad \Gamma^{2}_{33} = -\frac{1}{2A}\frac{\partial A}{\partial y}, \qquad \Gamma^{3}_{33} = \frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^{2}_{23} = \Gamma^{2}_{32} = \frac{1}{2A}\frac{\partial A}{\partial z}, \qquad \Gamma^{3}_{23} = \Gamma^{3}_{32} = \frac{1}{2A}\frac{\partial A}{\partial y} \tag{5}$$

From the components of the affine connection given in Equation (5), we obtain

$$R_{11} = \frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{22} = \frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{33} = \frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{00} = -\frac{3}{2c^2A}\frac{\partial^2 A}{\partial t^2} + \frac{3}{4c^2A^2}\left(\frac{\partial A}{\partial t}\right)^2 \tag{6}$$

Using the relation $R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$ we obtain

$$R = -\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2} + \frac{2}{A^2}\nabla^2 A + \frac{3}{2A^3}(\boldsymbol{\nabla} A)^2 \tag{7}$$

With the line element given in Equation (53), we obtain the following non-zero components of the affine connection are [19]

$$\Gamma^1_{11} = \frac{1}{2A}\frac{dA}{dr}, \quad \Gamma^1_{22} = -\frac{r}{A}, \quad \Gamma^2_{12} = \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^1_{33} = -\frac{r}{A}\sin^2\theta, \quad \Gamma^2_{33} = -\cos\theta\sin\theta, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \tag{8}$$

From the components of the affine connection given in Equation (24), we obtain

$$R_{11} = -\frac{1}{rA}\frac{dA}{dr}$$

$$R_{22} = \frac{1}{A} - \frac{r}{2A^2}\frac{dA}{dr} - 1$$

$$R_{33} = \left(\frac{1}{A} - \frac{r}{2A^2}\frac{dA}{dr} - 1\right)\sin^2\theta$$

$$R_{00} = 0 \tag{9}$$

Using the relation $R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$ we obtain

$$\mathrm{R} = \frac{r}{2A^2}\frac{dA}{dr} + \frac{2}{r^2}\left(1 - \frac{1}{A}\right) \tag{10}$$

References

[1] H. Goldstein, *Classical Mechanics* (Addison-Wesley Inc., Sydney, 1980).

[2] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Sydney, 1987).

[3] A. O. Barut, *Electrodynamics and Classical theory of Fields and Particles* (Dover Publications, Inc., New York, 1980).

[4] A. Einstein, *The Principle of Relativity* (Dover Publications, New York, 1952).

[5] Ray D'Inverno, *Introducing Einstein's Relativity* (Clarendon Press, Oxford, 1992).

[6] D. Kramer, H. Stephanie, E. Hert and M. MacCallum, *Exact solutions of Einstein's field equations* (Cambridge University Press, 1980).

[7] N. Bohr, *Phil. Mag*. 26, 1 (1913).

[8] C. Lanczos, *The Variational Principles of Mechanics* (Dover Publications, New York, 1970).

[9] R. P. Feynman, *Rev. Mod. Phys*. **20**, 367 (1948).

[10] M.A.L. Marques et al., *Time-Dependent Density Functional Theory*, Lecture Notes in Physics 706 (Springer, Berlin Heidelberg, 2006).

[11] A. Einstein, *Investigations on the theory of the Brownian movement*, edited with notes by R. Furth and translated by A. D. Cowper (Dover Publications, Inc., New York, 1956).

[12] Walter A. Strauss, *Partial Differential Equation* (John Wiley & Sons, Inc., New York, 1992).

[13] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1975).

[14] Erwin Schrödinger, Collected Papers on Wave Mechanics (AMS Chelsea Publishing, New York, 1982).

[15] W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory* (Dover Publications, New York, 1979).

[16] E. Kreyszig, *Introduction to Differential Geometry and Riemannian Geometry* (University of Toronto Press, 1975).

[17] Nikodem J. Poplawski, *Spacetime and fields*, arXiv:0911.0334v1[gr-qc] 2 Nov 2009.

[18] Richard C. Tolman, *Relativity Thermodynamics and Cosmology* (Dover Publications, New York, 1987).

[19] Gary Oas, EPGY Summer Institute SRGR.