A DERIVATION OF THE RICCI FLOW

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Abstract: In this work we show that by restricting the coordinate transformations to the group of time-independent coordinate transformations it is possible to derive the Ricci flow from the contracted Bianchi identities.

In differential geometry, the Ricci flow is a geometric process that can be employed to smooth out irregularities of a Riemannian manifold. The Ricci flow is introduced by the equation [1]

$$\frac{\partial g_{\alpha\beta}}{\partial t}=-2R_{\alpha\beta} \tag{1}$$

In general, the partial time derivative of a tensor is not a tensor therefore the Ricci flow given in Equation (1) is not a tensorial equation. However, as will be shown later, the Ricci flow is a tensorial equation within the group of coordinate transformations that are time-independent. From the Ricci flow, various geometric quantities will evolve when the metric $g_{\alpha\beta}$ evolves with time. The following results have been obtained [1,2]

Inverse metric $g^{\alpha\beta}$:

$$\frac{\partial g^{\alpha\beta}}{\partial t}=2R^{\alpha\beta} \tag{2}$$

Affine connection $\Gamma^{\sigma}_{\alpha\beta}$:

$$\frac{\partial \Gamma^{\sigma}_{\alpha\beta}}{\partial t}=-g^{\sigma\gamma}\left(\nabla_\alpha R_{\beta\gamma}+\nabla_\beta R_{\alpha\gamma}-\nabla_\gamma R_{\alpha\beta}\right) \tag{3}$$

Riemann curvature tensor$R_{\alpha\beta\gamma\sigma}$:

$$\begin{aligned}\frac{\partial R_{\alpha\beta\gamma\sigma}}{\partial t}=\Delta R_{\alpha\beta\gamma\sigma}&+2\left(B_{\alpha\beta\gamma\sigma}-B_{\alpha\beta\sigma\gamma}-B_{\alpha\sigma\beta\gamma}+B_{\alpha\gamma\beta\sigma}\right)\\&-g^{\rho\tau}\left(R_{\rho\beta\gamma\sigma}R_{\tau\alpha}+R_{\alpha\rho\gamma\sigma}R_{\tau\beta}+R_{\alpha\beta\rho\sigma}R_{\tau\gamma}+R_{\alpha\beta\gamma\tau}R_{\tau\sigma}\right)\end{aligned} \tag{4}$$

where $B_{\alpha\beta\gamma\sigma}=g^{\rho\delta}g^{\tau\eta}R_{\rho\alpha\tau\beta}R_{\delta\gamma\eta\sigma}$.

Ricci curvature tensor $R_{\alpha\beta}$:

$$\frac{\partial R_{\alpha\beta}}{\partial t}=\Delta R_{\alpha\beta}+2g^{\rho\delta}g^{\tau\eta}R_{\rho\alpha\tau\beta}R_{\delta\eta}-2g^{\sigma\tau}R_{\sigma\alpha}R_{\tau\beta} \tag{5}$$

Ricci scalar curvature R:

$$\frac{\partial R}{\partial t} = \Delta R + 2 g^{\alpha\beta} g^{\gamma\sigma} R_{\alpha\sigma} R_{\beta\gamma} \tag{6}$$

In differential geometry, the contracted Bianchi identities can be written in the following from

$$\nabla_\beta R^{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} \nabla_\beta R \tag{7}$$

It is interesting to note that Equation (7) has a covariant form of the field equations of the electromagnetic field written in a covariant form as

$$\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = \mu j^\beta \tag{8}$$

where the electromagnetic tensor $F^{\alpha\beta}$ is expressed in terms of the four-vector potential $A^\mu \equiv (V, \mathbf{A})$ as $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. The four-current j^μ is defined as $j^\mu \equiv (\rho_e, \mathbf{j}_e)$. From this similarity between Equation (7) and Equation (8) it is possible that a four-current of some form of matter $j^\alpha = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as [3]

$$j^\alpha = \frac{1}{2} g^{\alpha\beta} \nabla_\beta R \tag{9}$$

In this work, however, we will consider only for the case in which $\frac{1}{2} g^{\alpha\beta} \nabla_\beta R = 0$. In this case, Equation (7) reduces to the equation

$$\nabla_\beta R^{\alpha\beta} = 0 \tag{10}$$

We have two different situations that we will consider in detail in the following. First, since $\nabla_\mu g^{\alpha\beta} \equiv 0$ for a given metric tensor $g^{\alpha\beta}$, Equation (9) implies

$$R^{\alpha\beta} = \Lambda g^{\alpha\beta} \tag{11}$$

where Λ is an undetermined constant. Equation (10) can also be written in a covariant form as

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{12}$$

Using the identities $g_{\alpha\beta} g^{\alpha\beta} = 4$ and $g_{\alpha\beta} R^{\alpha\beta} = R$, we obtain $\Lambda = R/4$. As an example, if we consider a centrally symmetric gravitational field with the metric

$$ds^2 = e^\psi c^2 dt^2 - e^\chi dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{13}$$

then the Schwarzschild solution can be found as [4]

$$ds^2 = \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{14}$$

It is seen from the resulted line element given in Equation (14) that the contracted Bianchi identities given in Equation (7) could be identified as field equations for the gravitational field.

Now we consider the second situation and show how the Ricci flow can be derived from the field equation given in Equation (10). In differential geometry, the covariant derivative of a contravariant tensor of second rank $A^{\alpha\beta}$ is given by

$$\nabla_\gamma A^{\alpha\beta} = \partial_\gamma A^{\alpha\beta} + \Gamma^\alpha_{\sigma\gamma} A^{\sigma\beta} + \Gamma^\alpha_{\sigma\gamma} A^{\alpha\sigma} \tag{15}$$

The partial time derivative of Equation (15) is

$$\partial_t\left(\nabla_\gamma A^{\alpha\beta}\right) = \partial_t\left(\partial_\gamma A^{\alpha\beta}\right) + \left(\partial_t \Gamma^\alpha_{\sigma\gamma}\right) A^{\sigma\beta} + \Gamma^\alpha_{\sigma\gamma}\left(\partial_t A^{\sigma\beta}\right) + \left(\partial_t \Gamma^\alpha_{\sigma\gamma}\right) A^{\alpha\sigma} + \Gamma^\alpha_{\sigma\gamma}\left(\partial_t A^{\alpha\sigma}\right) \tag{16}$$

Under the coordinate transformation $x'^\alpha = f^\alpha(x^\beta)$, the tensor $A^{\alpha\beta}$ is transformed by the law

$$A'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\rho}\frac{\partial x'^\beta}{\partial x^\sigma} A^{\rho\sigma} \tag{17}$$

If the coordinate transformation is time-independent then the partial time derivative of the tensor $A^{\alpha\beta}$ is also a tensor which is transformed according to the rule

$$\frac{\partial A'^{\alpha\beta}}{\partial t} = \frac{\partial x'^\alpha}{\partial x^\rho}\frac{\partial x'^\beta}{\partial x^\sigma}\frac{\partial A^{\rho\sigma}}{\partial t} \tag{18}$$

In this case, we have

$$\nabla_\gamma\left(\partial_t A^{\alpha\beta}\right) = \partial_\gamma\left(\partial_t A^{\alpha\beta}\right) + \Gamma^\alpha_{\sigma\gamma}\left(\partial_t A^{\sigma\beta}\right) + \Gamma^\alpha_{\sigma\gamma}\left(\partial_t A^{\alpha\sigma}\right) \tag{19}$$

It is observed from Equations (16) and (19) that if we impose the following condition on Equation (16)

$$\left(\partial_t \Gamma^\alpha_{\sigma\gamma}\right) A^{\sigma\beta} + \left(\partial_t \Gamma^\alpha_{\sigma\gamma}\right) A^{\alpha\sigma} = 0 \tag{20}$$

then we obtain the identity

$$\nabla_\gamma\left(\partial_t A^{\alpha\beta}\right) = \partial_t\left(\nabla_\gamma A^{\alpha\beta}\right) \tag{21}$$

In the case of a metric tensor $g^{\alpha\beta}$ then we have $\nabla_\gamma\left(\partial_t g^{\alpha\beta}\right) = \partial_t\left(\nabla_\gamma g^{\alpha\beta}\right) \equiv 0$, and in this case from the field equations $\nabla_\beta R^{\alpha\beta} = 0$ we arrive at

$$\partial_t g^{\alpha\beta} = -kR^{\alpha\beta} \tag{22}$$

where $-k$ is a scaling factor. Equation (22) can also be written in a covariant form of the Ricci flow as

$$\frac{\partial g_{\alpha\beta}}{\partial t} = kR_{\alpha\beta} \tag{23}$$

As an example, consider a simple line element of the form [3]

$$ds^2 = D(cdt)^2 - A(x,y,z,t)[(dx)^2 + (dy)^2 + (dx)^2] \tag{24}$$

where D is constant. If the coordinate transformations are time-independent then for a covariant metric tensor $g_{\alpha\beta}$ we have the following results

$$\nabla_\gamma g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} - \Gamma^\sigma_{\alpha\gamma} g_{\sigma\beta} - \Gamma^\sigma_{\beta\gamma} g_{\alpha\sigma} \tag{25}$$

$$\partial_t(\nabla_\gamma g_{\alpha\beta}) = \partial_t(\partial_\gamma g_{\alpha\beta}) - (\partial_t \Gamma^\sigma_{\alpha\gamma}) g_{\sigma\beta} - \Gamma^\sigma_{\alpha\gamma}(\partial_t g_{\sigma\beta}) - (\partial_t \Gamma^\sigma_{\beta\gamma}) g_{\alpha\sigma} - \Gamma^\sigma_{\beta\gamma}(\partial_t g_{\alpha\sigma}) \tag{26}$$

$$\nabla_\gamma(\partial_t g_{\alpha\beta}) = \partial_\gamma(\partial_t g_{\alpha\beta}) - \Gamma^\sigma_{\alpha\gamma}(\partial_t g_{\sigma\beta}) - \Gamma^\sigma_{\beta\gamma}(\partial_t g_{\alpha\sigma}) \tag{27}$$

Therefore, the relation $\nabla_\gamma(\partial_t g_{\alpha\beta}) = \partial_t(\nabla_\gamma g_{\alpha\beta})$ is obtained if we impose the conditions

$$(\partial_t \Gamma^\sigma_{\alpha\gamma}) g_{\sigma\beta} + (\partial_t \Gamma^\sigma_{\beta\gamma}) g_{\alpha\sigma} = 0 \tag{28}$$

With the line element given in Equation (24), using the non-zero components of the affine connection given in the appendix, the imposed conditions given in Equation (28) lead to the following conditions for the quantity A

$$\frac{\partial^2 A}{\partial t^2} = 0 \tag{29}$$

$$\frac{\partial^2 A}{\partial t^2} - \frac{1}{A}\left(\frac{\partial A}{\partial t}\right)^2 = 0 \tag{30}$$

$$\frac{\partial^2 A}{\partial t \partial x} - \frac{1}{A}\left(\frac{\partial A}{\partial t}\right)\left(\frac{\partial A}{\partial x}\right) = 0 \tag{31}$$

$$\frac{\partial^2 A}{\partial t \partial y} - \frac{1}{A}\left(\frac{\partial A}{\partial t}\right)\left(\frac{\partial A}{\partial y}\right) = 0 \tag{32}$$

$$\frac{\partial^2 A}{\partial t \partial z} - \frac{1}{A}\left(\frac{\partial A}{\partial t}\right)\left(\frac{\partial A}{\partial z}\right) = 0 \tag{33}$$

It is seen from the above conditions that the quantity A must be time-independent and the Ricci flow given in Equation (23) leads to the purely geometrical field $R_{\alpha\beta} = 0$ which admit only time-independent solutions. Since the equation $R_{\alpha\beta} = 0$ implies the equation $R = 0$, from Equation (4) in the appendix we obtain the following equation for the quantity A

$$\nabla^2 A + \frac{3}{4A}(\boldsymbol{\nabla} A)^2 = 0 \tag{34}$$

Appendix

In differential geometry, the affine connection $\Gamma^\gamma_{\alpha\beta}$ is defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right) \tag{1}$$

With the line element given in Equation (24), we obtain the following non-zero components of the affine connection [5]

$$\Gamma^{1}_{01} = \Gamma^{1}_{10} = \frac{1}{2cA}\frac{\partial A}{\partial t}, \qquad \Gamma^{2}_{02} = \Gamma^{2}_{20} = \frac{1}{2cA}\frac{\partial A}{\partial t}, \qquad \Gamma^{3}_{03} = \Gamma^{3}_{30} = \frac{1}{2cA}\frac{\partial A}{\partial t}$$

$$\Gamma^{0}_{11} = \frac{1}{2cD}\frac{\partial A}{\partial t}, \qquad \Gamma^{1}_{11} = \frac{1}{2A}\frac{\partial A}{\partial x}, \qquad \Gamma^{2}_{11} = -\frac{1}{2A}\frac{\partial A}{\partial y}, \qquad \Gamma^{3}_{11} = -\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^{1}_{12} = \Gamma^{1}_{21} = \frac{1}{2A}\frac{\partial A}{\partial y}, \quad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2A}\frac{\partial A}{\partial x}, \quad \Gamma^{1}_{13} = \Gamma^{1}_{31} = \frac{1}{2A}\frac{\partial A}{\partial z}, \quad \Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{2A}\frac{\partial A}{\partial x}$$

$$\Gamma^{0}_{22} = \frac{1}{2cD}\frac{\partial A}{\partial t}, \qquad \Gamma^{1}_{22} = \frac{1}{2A}\frac{\partial A}{\partial x}, \qquad \Gamma^{2}_{22} = \frac{1}{2A}\frac{\partial A}{\partial y}, \qquad \Gamma^{3}_{22} = -\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^{0}_{33} = \frac{1}{2cD}\frac{\partial A}{\partial t}, \qquad \Gamma^{1}_{33} = -\frac{1}{2A}\frac{\partial A}{\partial x}, \qquad \Gamma^{2}_{33} = -\frac{1}{2A}\frac{\partial A}{\partial y}, \qquad \Gamma^{3}_{33} = \frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^{2}_{23} = \Gamma^{2}_{32} = \frac{1}{2A}\frac{\partial A}{\partial z}, \qquad \Gamma^{3}_{23} = \Gamma^{3}_{32} = \frac{1}{2A}\frac{\partial A}{\partial y} \tag{2}$$

From the components of the affine connection given in Equation (2), we obtain

$$R_{11} = \frac{1}{2c^2 D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2 AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{22} = \frac{1}{2c^2 D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2 AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{33} = \frac{1}{2c^2 D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2 AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{00} = -\frac{3}{2c^2 A}\frac{\partial^2 A}{\partial t^2} + \frac{3}{4c^2 A^2}\left(\frac{\partial A}{\partial t}\right)^2 \tag{3}$$

Using the relation $R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$ we obtain

$$R = -\frac{3}{c^2 DA}\frac{\partial^2 A}{\partial t^2} + \frac{2}{A^2}\nabla^2 A + \frac{3}{2A^3}(\boldsymbol{\nabla} A)^2 \tag{4}$$

References

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