Regular and singular rational extensions of the harmonic oscillator with two known eigenstates

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Abstract

Exactly solvable rational extensions of the harmonic oscillator have been constructed as supersymmetric partner potentials of the harmonic oscillator **[1]** as well as using the so-called prepotential approach **[2]**. In this work, we use the factorization property of the energy eigenfunctions of the harmonic oscillator and a simple integrability condition to construct and examine series of regular and singular rational extensions of the harmonic oscillator with two known eigenstates, one of which is the ground state. Special emphasis is given to the interrelation between the special zeros of the wave function, the poles of the potential, and the excitation of the non-ground state. In the last section, we analyze specific examples.

Contents

Dimensionless units

In order to make the quantities that interest us dimensionless, we introduce the dimensionless variable $\tilde{x} \equiv \frac{x}{l}$, where x is the position and l is a positive real constant with dimensions of length, which later can be related to the length scale of the examined particle.

Then, the wave function $\psi(x)$ of the particle and the potential $V(x)$ become functions of $\tilde{x}$, i.e. $\psi(x) \to \psi(\tilde{x})$ and $V(x) \to V(\tilde{x})$.

Also, we have

$$\frac{d}{dx} = \frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}} = \frac{1}{l}\frac{d}{d\tilde{x}}$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \frac{1}{l}\frac{d}{d\tilde{x}}\frac{1}{l}\frac{d}{d\tilde{x}} = \frac{1}{l^2}\frac{d^2}{d\tilde{x}^2}$$

Thus

$$\frac{d^2}{dx^2} = \frac{1}{l^2}\frac{d^2}{d\tilde{x}^2}$$

Then, the second derivatives of the wave function with respect to x and $\tilde{x}$ are related by the equation

$$\frac{d^2\psi(x)}{dx^2} = \frac{1}{l^2}\frac{d^2\psi(\tilde{x})}{d\tilde{x}^2}$$

The energy eigenvalue equation of the particle in the potential $V(x)$, i.e. the equation

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\left(E - V(x)\right)\psi(x) = 0$$

is then written as

$$\frac{1}{l^2}\frac{d^2\psi(\tilde{x})}{d\tilde{x}^2} + \frac{2m}{\hbar^2}\left(E - V(\tilde{x})\right)\psi(\tilde{x}) = 0 \Rightarrow \psi''(\tilde{x}) + \frac{2ml^2}{\hbar^2}\left(E - V(\tilde{x})\right)\psi(\tilde{x}) = 0 \Rightarrow$$

$$\Rightarrow \psi''(\tilde{x}) + \left(\frac{2ml^2E}{\hbar^2} - \frac{2ml^2V(\tilde{x})}{\hbar^2}\right)\psi(\tilde{x}) = 0 \quad (1)$$

where the primes now denote differentiation with respect to $\tilde{x}$.

We observe that

$$\left[\frac{2ml^2}{\hbar^2}\right] = \left[\frac{mx^2}{p^2x^2}\right] = \left[\frac{m}{p^2}\right] \overset{E=\frac{p^2}{2m}}{=} \left[\frac{1}{E}\right]$$

The quantity $\frac{\hbar^2}{2ml^2}$ has then dimensions of energy and we use it to make the energy and the potential dimensionless.

To this end, we set

$$\tilde{E} \equiv \frac{2ml^2E}{\hbar^2} \quad (2)$$

$$\tilde{V}(\tilde{x}) \equiv \frac{2ml^2V(\tilde{x})}{\hbar^2} \quad (3)$$

The quantities $\tilde{E}$ and $\tilde{V}(\tilde{x})$ are then dimensionless.

For convenience, we'll call the dimensionless variables $\tilde{x}$, $\tilde{E}$, and $\tilde{V}(\tilde{x})$, dimensionless position, dimensionless energy, and dimensionless potential, respectively.

Substituting (2) and (3) into (1), we obtain

$$\psi''(\tilde{x}) + \left(\tilde{E} - \tilde{V}(\tilde{x})\right)\psi(\tilde{x}) = 0 \quad (4)$$

This is the energy eigenvalue equation in terms of the dimensionless position, energy, and potential.

In what follows, we'll work with the dimensionless position, energy, and potential, but for simplicity and convenience, we'll keep denoting them by x, E, and $V(x)$.

General analysis

It is known that **[3]**, leaving aside the normalization constants, the energy eigenfunctions of the harmonic oscillator are products of its ground-state wave function and the respective Hermite polynomials, i.e. $\psi_n(x) \sim H_n(x)\psi_0(x)$, where $\psi_0(x)$ and $\psi_n(x)$ are, respectively, the ground-state and the n-th excited-state wave functions, and $H_n(x)$ is the respective Hermite polynomial.

Using this property as our starting point, we'll search for real potentials $V(x)$ having a bound eigenstate of energy E_1, which is described by a wave function $\psi_1(x)$, and a bound eigenstate of energy $E_2 \neq E_1$, which is described by a wave function of the form

$$\psi_2(x) = A_2 p(x)\psi_1(x) \quad (5)$$

where A_2 is the normalization constant of the wave function $\psi_2(x)$.

For the moment, we'll assume that $p(x)$ is a general, appropriate function, and later we'll restrict our attention to polynomials of definite parity.

Since both wave functions describe bound energy eigenstates, they must both be square integrable.

The wave functions $\psi_1(x)$ and $\psi_2(x)$ satisfy, respectively, the energy eigenvalue equations

$$\psi_1''(x) + \left(E_1 - V(x)\right)\psi_1(x) = 0 \quad (6)$$

$$\psi_2''(x) + \left(E_2 - V(x)\right)\psi_2(x) = 0 \quad (7)$$

Using (5), the first derivative of $\psi_2(x)$ is

$$\psi_2'(x) = p'(x)\psi_1(x) + p(x)\psi_1'(x)$$

Then, the second derivative of $\psi_2(x)$ is

$$\psi_2''(x) = p''(x)\psi_1(x) + p'(x)\psi_1'(x) + p'(x)\psi_1'(x) + p(x)\psi_1''(x) =$$
$$= p''(x)\psi_1(x) + 2p'(x)\psi_1'(x) + p(x)\psi_1''(x)$$

That is

$$\psi_2''(x) = p''(x)\psi_1(x) + 2p'(x)\psi_1'(x) + p(x)\psi_1''(x) \quad (8)$$

Besides, from (6) we obtain

$$\psi_1''(x) = (V(x) - E_1)\psi_1(x)$$

Substituting the expression of $\psi_1''(x)$ into (8), we obtain

$$\psi_2''(x) = p''(x)\psi_1(x) + 2p'(x)\psi_1'(x) + p(x)(V(x) - E_1)\psi_1(x) =$$
$$= 2p'(x)\psi_1'(x) + ((V(x) - E_1)p(x) + p''(x))\psi_1(x)$$

That is

$$\psi_2''(x) = 2p'(x)\psi_1'(x) + ((V(x) - E_1)p(x) + p''(x))\psi_1(x) \quad (9)$$

The equations (5) and (9) give us $\psi_2(x)$ and its second derivative in terms of $\psi_1(x)$, $p(x)$, and their derivatives. Observe that the second derivative of $\psi_2(x)$ contain only the first derivative of $\psi_1(x)$, not the second.

Then, substituting (5) and (9) into the energy eigenvalue equation for $\psi_2(x)$ (eq. (7)), we obtain a first-order, linear – and homogeneous – differential equation for $\psi_1(x)$. We have

$$2p'(x)\psi_1'(x) + ((V(x) - E_1)p(x) + p''(x))\psi_1(x) + (E_2 - V(x))p(x)\psi_1(x) = 0 \Rightarrow$$
$$\Rightarrow 2p'(x)\psi_1'(x) + ((V(x) - E_1)p(x) + p''(x) + (E_2 - V(x))p(x))\psi_1(x) = 0 \Rightarrow$$
$$\Rightarrow 2p'(x)\psi_1'(x) + ((V(x) - E_1 + E_2 - V(x))p(x) + p''(x))\psi_1(x) = 0 \Rightarrow$$
$$\Rightarrow 2p'(x)\psi_1'(x) + ((E_2 - E_1)p(x) + p''(x))\psi_1(x) = 0 \Rightarrow$$
$$\Rightarrow \frac{\psi_1'(x)}{\psi_1(x)} + \frac{(E_2 - E_1)p(x) + p''(x)}{2p'(x)} = 0 \Rightarrow \frac{\psi_1'(x)}{\psi_1(x)} = -\frac{(E_2 - E_1)p(x) + p''(x)}{2p'(x)} =$$
$$= -\frac{E_2 - E_1}{2}\frac{p(x)}{p'(x)} - \frac{1}{2}\frac{p''(x)}{p'(x)}$$

That is

$$\frac{\psi_1'(x)}{\psi_1(x)} = -\frac{E_2 - E_1}{2}\frac{p(x)}{p'(x)} - \frac{1}{2}\frac{p''(x)}{p'(x)} \quad (10)$$

Since the two energy eigenstates are bound, each of the wave functions $\psi_1(x)$ and $\psi_2(x)$ is **[3]** the product of a real-valued function and a constant complex phase that can be incorporated into the respective normalization constant.
Thus, the function $p(x)$ can be assumed a real-valued function.
Also, omitting for the moment the normalization constants, we can assume that the two wave functions are also real-valued.
Then

$$\frac{p''(x)}{p'(x)} = \left(\ln\left|p'(x)\right| + C_1\right)' \quad (11)$$

and

$$\frac{\psi_1'(x)}{\psi_1(x)} = \left(\ln\left|\psi_1(x)\right| + C_2\right)' \quad (12)$$

with C_1, C_2 being real constants.
By means of (11) and (12), (10) is written as

$$\left(\ln\left|\psi_1(x)\right| + C_2\right)' = -\frac{E_2 - E_1}{2}\frac{p(x)}{p'(x)} - \frac{1}{2}\left(\ln\left|p'(x)\right| + C_1\right)' \Rightarrow$$

$$\Rightarrow \ln\left|\psi_1(x)\right| + C_2 = -\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)} - \frac{1}{2}\left(\ln\left|p'(x)\right| + C_1\right) \Rightarrow$$

$$\Rightarrow \ln\left|\psi_1(x)\right| = -\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)} - \frac{1}{2}\ln\left|p'(x)\right| - \frac{C_1}{2} - C_2 \Rightarrow$$

$$\Rightarrow \ln\left|\psi_1(x)\right| = -\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)} + \ln\left|p'(x)\right|^{-1/2} - \frac{C_1}{2} - C_2$$

Setting $C \equiv -\frac{C_1}{2} - C_2$ a new real constant, the previous equation is written as

$$\ln\left|\psi_1(x)\right| = -\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)} + \ln\left|p'(x)\right|^{-1/2} + C \quad (13)$$

We can also incorporate into C the constant of the integral $\int_x dy \frac{p(y)}{p'(y)}$, and thus we do the integration without adding a constant.
From (13) we obtain

$$\left|\psi_1(x)\right| = \exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)} + \ln\left|p'(x)\right|^{-1/2} + C\right) =$$
$$= \exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)\exp\left(\ln\left|p'(x)\right|^{-1/2}\right)\exp(C) =$$
$$= \exp(C)\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)\left|p'(x)\right|^{-1/2}$$

Setting $A_1 \equiv \exp(C)$, we obtain

$$\left|\psi_1(x)\right| = A_1\left|p'(x)\right|^{-1/2}\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)$$

or

$$\psi_1(x) = \pm A_1\left|p'(x)\right|^{-1/2}\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)$$

Now, incorporating into $\psi_1(x)$ the normalization constant, which includes the complex phase, the constant A_1 becomes a complex constant.
Since the plus and minus wave functions are linearly dependent, we choose one of them, and thus, choosing the one with the plus sign, we end up to

$$\psi_1(x) = A_1\left|p'(x)\right|^{-1/2}\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right) \quad (14)$$

where A_1 is a complex constant.
If the function $p(x)$ has definite parity, i.e. if it is an even or an odd function, then $p'(x)$ has also definite parity, but different from $p(x)$, and thus $\frac{p(y)}{p'(y)}$ is always of odd parity. Then the function $\int_x dy\frac{p(y)}{p'(y)}$ is of even parity, as the indefinite integral of an odd-parity function. Thus, the exponential $\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)$ is also an even-parity function. Besides, since $p'(x)$ has definite parity, its absolute value $\left|p'(x)\right|$ is of even parity, and thus the function $\left|p'(x)\right|^{-1/2}$ is also of even parity. Thus, the product of $\left|p'(x)\right|^{-1/2}$ and $\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)$ is an even-parity function, i.e. the wave function $\psi_1(x)$ is of even-parity.
Then, since $\psi_1(x)$ is of even-parity, from (5) we see that $\psi_2(x)$ is of even/odd parity if and only if $p(x)$ is of even/odd parity, i.e. $\psi_2(x)$ has the same parity as $p(x)$.

Therefore, if $p(x)$ has definite parity, $\psi_1(x)$ is of even-parity and $\psi_2(x)$ has the same parity as $p(x)$.

Symmetric potentials

We'll consider symmetric potentials, i.e. potentials of even parity. Then, the energy eigenfunctions have definite parity **[3]**, and thus $\psi_1(x)$ and $\psi_2(x)$ have definite parity.
Since $\psi_1(x)$ and $\psi_2(x)$ have definite parity, and (5) is written as

$$p(x)=\frac{\psi_2(x)}{A_2\psi_1(x)},$$

we conclude that $p(x)$ has definite parity, and then, as we showed, $\psi_1(x)$ is of even-parity and $\psi_2(x)$ has the same parity as $p(x)$.
Therefore, $\psi_1(x)$ has the same parity as the potential, it has the symmetry of the potential.
Also, from (5), we see that if $\psi_1(x)$ has zeros, these are also zeros of $\psi_2(x)$, i.e. the zeros of $\psi_1(x)$ are common zeros of the two wave functions, and in this sense, they are special zeros, which are expected to result from the singularities of the potential.
Thus, the wave function $\psi_1(x)$ has the symmetry of the potential, i.e. it is of even parity, and it can have only special – or common – zeros resulting from any singularitites of the potential. This means that $\psi_1(x)$ is the ground-state wave function.
Then, $\psi_2(x)$ is an excited-state wave function, and thus $E_2>E_1$.
Since $\psi_2(x)$ is an excited-state wave function, it must have at least one simple zero **[4]**, which, as seen from (5), is also a zero of $p(x)$.
If $p(x)$ has r simple zeros, with $r=1,2,...$, then $\psi_2(x)$ is the r-th excited-state wave function **[4]**.
We remind that the two wave functions must be square integrable, and also, since the probability density must be finite everywhere, the two wave functions must be finite for every x.

The expression of the potential

Since the potential we consider is symmetric, i.e. $V(-x)=V(x)$, it is enough to calculate it in the domain $x\geq 0$ only.
In the region(s) of the domain $x\geq 0$ where the derivative $p'(x)$ is negative, i.e. $p'(x)<0$, (14) is written as

$$\psi_1(x)=A_1\left(-p'(x)\right)^{-1/2}\exp\left(-\frac{E_2-E_1}{2}\int_x dy\,\frac{p(y)}{p'(y)}\right) \quad (15)$$

Using (15), the first derivative of $\psi_1(x)$ is

$$\psi_1'(x) = -\frac{1}{2}A_1\left(-p'(x)\right)^{-3/2}\left(-p''(x)\right)\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)+$$

$$+A_1\left(-p'(x)\right)^{-1/2}\left(-\frac{E_2-E_1}{2}\frac{p(x)}{p'(x)}\right)\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)=$$

$$=\frac{1}{2}A_1\left(-p'(x)\right)^{-3/2}p''(x)\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)+$$

$$+\frac{1}{2}A_1\left(-p'(x)\right)^{-1/2}\left(\left(E_2-E_1\right)\frac{p(x)}{\left(-p'(x)\right)}\right)\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)=$$

$$=\frac{1}{2}\left(\left(-p'(x)\right)^{-3/2}p''(x)+\left(E_2-E_1\right)\left(-p'(x)\right)^{-3/2}p(x)\right)A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)=$$

$$=\frac{1}{2}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-3/2}A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)$$

That is

$$\psi_1'(x)=\frac{1}{2}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-3/2}A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right) \quad (16)$$

Using (16), the second derivative of $\psi_1(x)$ is

$$\psi_1''(x)=\frac{1}{2}\left(p'''(x)+\left(E_2-E_1\right)p'(x)\right)\left(-p'(x)\right)^{-3/2}A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)+$$

$$+\frac{1}{2}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-\frac{3}{2}\left(-p'(x)\right)^{-5/2}\left(-p''(x)\right)\right)A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)+$$

$$+\frac{1}{2}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-3/2}\left(-\frac{E_2-E_1}{2}\frac{p(x)}{p'(x)}\right)A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)=$$

$$=\frac{1}{2}\left(p'''(x)+\left(E_2-E_1\right)p'(x)\right)\left(-p'(x)\right)^{-3/2}A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)+$$

$$+\frac{3}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-5/2}p''(x)A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)+$$

$$+\frac{1}{2}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-3/2}\left(\frac{E_2-E_1}{2}\frac{p(x)}{\left(-p'(x)\right)}\right)A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)=$$

$$=\frac{1}{2}\left(p'''(x)+\left(E_2-E_1\right)p'(x)\right)\left(-p'(x)\right)^{-3/2}A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)+$$

$$+\frac{3}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-5/2}p''(x)A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)+$$

$$+\frac{E_2-E_1}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-5/2}p(x)A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)$$

Besides, from (15) we have

$$A_1\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)=\left(-p'(x)\right)^{1/2}\psi_1(x)$$

Substituting into the expression of the second derivative, we obtain

$$\psi_1''(x)=\frac{1}{2}\left(p'''(x)+\left(E_2-E_1\right)p'(x)\right)\left(-p'(x)\right)^{-3/2}\left(-p'(x)\right)^{1/2}\psi_1(x)+$$

$$+\frac{3}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-5/2}p''(x)\left(-p'(x)\right)^{1/2}\psi_1(x)+$$

$$+\frac{E_2-E_1}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-5/2}p(x)\left(-p'(x)\right)^{1/2}\psi_1(x)=$$

$$=\frac{1}{2}\left(p'''(x)+\left(E_2-E_1\right)p'(x)\right)\left(-p'(x)\right)^{-1}\psi_1(x)+$$

$$+\frac{3}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-2}p''(x)\psi_1(x)+$$

$$+\frac{E_2-E_1}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\left(-p'(x)\right)^{-2}p(x)\psi_1(x)$$

Thus

$$\frac{\psi_1''(x)}{\psi_1(x)}=\frac{1}{2}\left(p'''(x)+\left(E_2-E_1\right)p'(x)\right)\left(-p'(x)\right)^{-1}+\frac{3}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\underbrace{\left(-p'(x)\right)^{-2}}_{\left(p'(x)\right)^{-2}}p''(x)+$$

$$+\frac{E_2-E_1}{4}\left(p''(x)+\left(E_2-E_1\right)p(x)\right)\underbrace{\left(-p'(x)\right)^{-2}}_{\left(p'(x)\right)^{-2}}p(x)=$$

$$=\frac{1}{2}\left(-\frac{p'''(x)}{p'(x)}-\left(E_2-E_1\right)\right)+\frac{3}{4}\left(\left(\frac{p''(x)}{p'(x)}\right)^2+\left(E_2-E_1\right)\frac{p(x)p''(x)}{p'^2(x)}\right)+$$

$$+\frac{E_2-E_1}{4}\left(\frac{p(x)p''(x)}{p'^2(x)}+\left(E_2-E_1\right)\left(\frac{p(x)}{p'(x)}\right)^2\right)=-\frac{p'''(x)}{2p'(x)}-\frac{E_2-E_1}{2}+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2+$$

$$+\frac{3\left(E_2-E_1\right)}{4}\frac{p(x)p''(x)}{p'^2(x)}+\frac{E_2-E_1}{4}\frac{p(x)p''(x)}{p'^2(x)}+\frac{\left(E_2-E_1\right)^2}{4}\left(\frac{p(x)}{p'(x)}\right)^2=$$

$$=-\frac{p'''(x)}{2p'(x)}-\frac{E_2-E_1}{2}+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2+\frac{\left(E_2-E_1\right)p(x)p''(x)}{p'^2(x)}+\frac{\left(E_2-E_1\right)^2}{4}\left(\frac{p(x)}{p'(x)}\right)^2$$

Thus

$$\frac{\psi_1''(x)}{\psi_1(x)} = \left(\frac{E_2 - E_1}{2}\right)^2 \left(\frac{p(x)}{p'(x)}\right)^2 + \frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2 - \frac{p'''(x)}{2p'(x)} + \frac{(E_2 - E_1)p(x)p''(x)}{p'^2(x)} - \frac{E_2 - E_1}{2} \quad (17)$$

Besides, from (6), i.e. from the energy eigenvalue equation for $\psi_1(x)$, we obtain

$$\frac{\psi_1''(x)}{\psi_1(x)} + E_1 - V(x) = 0 \Rightarrow V(x) = \frac{\psi_1''(x)}{\psi_1(x)} + E_1$$

Substituting $\frac{\psi_1''(x)}{\psi_1(x)}$ from (17), the potential is then written as

$$V(x) = \left(\frac{E_2 - E_1}{2}\right)^2 \left(\frac{p(x)}{p'(x)}\right)^2 + \frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2 - \frac{p'''(x)}{2p'(x)} + \frac{(E_2 - E_1)p(x)p''(x)}{p'^2(x)} + E_1 - \frac{E_2 - E_1}{2} \quad (18)$$

To determine the potential uniquely, we need its value at a reference point. The reference value of the potential may well be a limiting value at infinity.
Using the reference value, we determine the potential uniquely and also, we obtain an equation for the two energies E_1 and E_2.
In the region(s) of the domain $x \geq 0$ where the derivative $p'(x)$ is positive, i.e. $p'(x) > 0$, (14) is written as

$$\psi_1(x) = A_1 \left(p'(x)\right)^{-1/2} \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) \quad (19)$$

Using (19), the first derivative of $\psi_1(x)$ is

$$\psi_1'(x) = -\frac{1}{2}A_1\left(p'(x)\right)^{-3/2} p''(x)\exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) +$$
$$+A_1\left(p'(x)\right)^{-1/2}\left(-\frac{E_2 - E_1}{2}\frac{p(x)}{p'(x)}\right)\exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) =$$
$$= -\frac{1}{2}\left(p''(x)\left(p'(x)\right)^{-3/2} + (E_2 - E_1)p(x)\left(p'(x)\right)^{-3/2}\right)A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right)$$

Thus

$$\psi_1'(x) = -\frac{1}{2}\left(p''(x) + (E_2 - E_1)p(x)\right)\left(p'(x)\right)^{-3/2} A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) \quad (20)$$

Using (20), the second derivative of $\psi_1(x)$ is

$$\psi_1''(x) = -\frac{1}{2}\left(p'''(x) + (E_2 - E_1)p'(x)\right)\left(p'(x)\right)^{-3/2} A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) -$$

$$-\frac{1}{2}\left(p''(x) + (E_2 - E_1)p(x)\right)\left(-\frac{3}{2}\left(p'(x)\right)^{-5/2} p''(x)\right) A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) -$$

$$-\frac{1}{2}\left(p''(x) + (E_2 - E_1)p(x)\right)\left(p'(x)\right)^{-3/2}\left(-\frac{E_2 - E_1}{2}\frac{p(x)}{p'(x)}\right) A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) =$$

$$= -\frac{1}{2}\left(p'''(x) + (E_2 - E_1)p'(x)\right)\left(p'(x)\right)^{-3/2} A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) +$$

$$+\frac{3}{4}\left(p''(x) + (E_2 - E_1)p(x)\right)p''(x)\left(p'(x)\right)^{-5/2} A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) +$$

$$+\frac{E_2 - E_1}{4}\left(p''(x) + (E_2 - E_1)p(x)\right)p(x)\left(p'(x)\right)^{-5/2} A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right)$$

Besides, from (19) we have

$$A_1 \exp\left(-\frac{E_2 - E_1}{2}\int_x dy \frac{p(y)}{p'(y)}\right) = \left(p'(x)\right)^{1/2}\psi_1(x)$$

Substituting into the expression of the second derivative, we obtain

$$\psi_1''(x) = -\frac{1}{2}\left(p'''(x) + (E_2 - E_1)p'(x)\right)\left(p'(x)\right)^{-1}\psi_1(x) +$$

$$+\frac{3}{4}\left(p''(x) + (E_2 - E_1)p(x)\right)p''(x)\left(p'(x)\right)^{-2}\psi_1(x) +$$

$$+\frac{E_2 - E_1}{4}\left(p''(x) + (E_2 - E_1)p(x)\right)p(x)\left(p'(x)\right)^{-2}\psi_1(x)$$

Thus

$$\frac{\psi_1''(x)}{\psi_1(x)} = -\frac{p'''(x)}{2p'(x)} - \frac{E_2 - E_1}{2} + \frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2 + \frac{3(E_2 - E_1)p(x)p''(x)}{4p'^2(x)} + \frac{(E_2 - E_1)p(x)p''(x)}{4p'^2(x)} +$$

$$+\frac{(E_2 - E_1)^2}{4}\left(\frac{p(x)}{p'(x)}\right)^2 = \left(\frac{E_2 - E_1}{2}\right)^2\left(\frac{p(x)}{p'(x)}\right)^2 + \frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2 - \frac{p'''(x)}{2p'(x)} + \frac{(E_2 - E_1)p(x)p''(x)}{p'^2(x)} -$$

$$-\frac{E_2 - E_1}{2}$$

That is

$$\frac{\psi_1''(x)}{\psi_1(x)} = \left(\frac{E_2 - E_1}{2}\right)^2\left(\frac{p(x)}{p'(x)}\right)^2 + \frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2 - \frac{p'''(x)}{2p'(x)} + \frac{(E_2 - E_1)p(x)p''(x)}{p'^2(x)} - \frac{E_2 - E_1}{2} \quad (21)$$

Comparing (17) and (21), we see that they are the same expressions.
In the previous case, we saw that, using again (6), the potential is

$$V(x)=\frac{\psi_1''(x)}{\psi_1(x)}+E_1$$

Substituting $\frac{\psi_1''(x)}{\psi_1(x)}$ from (21), the potential is then written as

$$V(x)=\left(\frac{E_2-E_1}{2}\right)^2\left(\frac{p(x)}{p'(x)}\right)^2+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+\frac{(E_2-E_1)p(x)p''(x)}{p'^2(x)}+E_1-\frac{E_2-E_1}{2}$$

which is the same expression as (18).
At the zeros of $p'(x)$, i.e. at the points where $p'(x)=0$, the wave function $\psi_1(x)$, and then $\psi_2(x)$ too, is not differentiable.
However, since the expressions of the potential in the regions where $p'(x)$ is negative and positive are the same, and assuming that $p'(x)$ is continuous, so that the wave functions are continuous, which is necessary for the probability density to be continuous too, if x_0 is a zero of $p'(x)$, the left-side and right-side limits of the potential at x_0 are the same, and we can define the value of the potential at x_0 as the value of these two limits. If x_0 is a simple zero of $p'(x)$, then the limits are infinity, and thus x_0 is a singular point of the potential.
Therefore, the simple zeros of $p'(x)$ are singular points of the potential.
To give a summary of what we've done so far, the wave functions (14) and (5), provided that they are square integrable and finite for every x, describe, respectively, the ground state, of energy E_1, and a bound excited state, of energy $E_2>E_1$, of the potential

$$V(x)=\left(\frac{E_2-E_1}{2}\right)^2\left(\frac{p(x)}{p'(x)}\right)^2+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+\frac{(E_2-E_1)p(x)p''(x)}{p'^2(x)}+E_1-\frac{E_2-E_1}{2} \quad (22)$$

The potential is uniquely determined if we know its value at a reference point, which may well be the infinity.
In the previous section, we showed that if the potential is symmetric, $p(x)$ has definite parity.
Using (22), it is easily shown the opposite, i.e. if $p(x)$ has definite parity, then the potential is symmetric.
Indeed, if $p(x)$ has definite parity, then $p'(x)$ has opposite parity from $p(x)$, and thus $\frac{p(x)}{p'(x)}$ is of odd parity, and then $\left(\frac{p(x)}{p'(x)}\right)^2$ is of even parity.

Also, $p''(x)$ has the same parity as $p(x)$, and thus has opposite parity from $p'(x)$, and then $\frac{p''(x)}{p'(x)}$ is of odd parity, and thus $\left(\frac{p''(x)}{p'(x)}\right)^2$ is of even parity.

Also, $p''(x)$ has opposite parity from both $p'(x)$ and $p'''(x)$, and thus $p'(x)$ and $p'''(x)$ have the same parity, and then $\frac{p'''(x)}{p'(x)}$ is of even parity.

Also, since $\frac{p(x)}{p'(x)}$ and $\frac{p''(x)}{p'(x)}$ are both of odd parity, $\frac{p(x)p''(x)}{p'^2(x)}$ is of even parity.

The constant $E_1 - \frac{E_2 - E_1}{2}$, as any constant, is an even-parity function.

Therefore, using (22), the potential, as a sum of even-parity functions, is of even parity (symmetric).

The case where the function p(x) is a definite-parity polynomial

We'll now assume that $p(x)$ is a polynomial of even or odd parity.

Since we can always incorporate the leading coefficient of $p(x)$ – which by definition is non-zero – into the normalization constant A_2 of the wave function $\psi_2(x)$, we can assume that $p(x)$ is monic, i.e. its leading coefficient is 1.

In this case, as seen from (22), the singularities of the potential are the zeros of $p'(x)$ – if $p'(x)$ has any zeros – and they are poles of second order.

We'll examine separately the case where $p(x)$ is of odd parity and the case where it is of even parity.

I. p(x) is of odd parity and satisfies an integrability condition

If $p(x)$ is an odd-parity polynomial, it will be of odd degree and it will have the form

$$p(x) = \sum_{m=0}^{n} p_{2m+1} x^{2m+1} \quad (23)$$

where $p_{2n+1} = 1$ and $n = 0,1,...$

Since $p(x)$ is of odd parity, it has a zero at 0.

The first derivative of $p(x)$ is

$$p'(x) = \sum_{m=0}^{n} (2m+1) p_{2m+1} x^{2m} \quad (24)$$

Using (23) and (24), we have

$$\frac{p(x)}{p'(x)}=\frac{\sum_{m=0}^{n}p_{2m+1}x^{2m+1}}{\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}=\frac{x\sum_{m=0}^{n}p_{2m+1}x^{2m}}{(2n+1)\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}=\frac{x}{2n+1}\frac{\sum_{m=0}^{n}p_{2m+1}x^{2m}}{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}=$$

$$=\frac{x}{2n+1}\frac{\sum_{m=0}^{n}\left(p_{2m+1}x^{2m}+\left(\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m}-\left(\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m}\right)}{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}=$$

$$=\frac{x}{2n+1}\frac{\sum_{m=0}^{n}\left(\left(1+\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m}-\left(\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m}\right)}{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}=$$

$$=\frac{x}{2n+1}\frac{\sum_{m=0}^{n}\left(\frac{2m+1}{2n+1}p_{2m+1}x^{2m}-\left(\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m}\right)}{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}=$$

$$=\frac{x}{2n+1}\frac{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}-\sum_{m=0}^{n}\left(\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m}}{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}=$$

$$=\frac{x}{2n+1}\left(\frac{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}-\frac{\sum_{m=0}^{n}\left(\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m}}{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}\right)=\frac{x}{2n+1}\left(1-\frac{\sum_{m=0}^{n}\left(\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m}}{\sum_{m=0}^{n}\frac{2m+1}{2n+1}p_{2m+1}x^{2m}}\right)=$$

$$=\frac{x}{2n+1}-\frac{\sum_{m=0}^{n}\left(\frac{2m+1}{2n+1}-1\right)p_{2m+1}x^{2m+1}}{\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}=\frac{x}{2n+1}-\frac{\sum_{m=0}^{n}\left(\frac{2m+1-(2n+1)}{2n+1}\right)p_{2m+1}x^{2m+1}}{\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}=$$

$$=\frac{x}{2n+1}-\frac{\sum_{m=0}^{n}\frac{2(m-n)}{2n+1}p_{2m+1}x^{2m+1}}{\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}=\frac{x}{2n+1}-\frac{\sum_{m=0}^{n}2(m-n)p_{2m+1}x^{2m+1}}{(2n+1)\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}=$$

$$=\frac{x}{2n+1}+\frac{\sum_{m=0}^{n}2(n-m)p_{2m+1}x^{2m+1}}{(2n+1)\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}$$

That is

$$\frac{p(x)}{p'(x)}=\frac{x}{2n+1}+\frac{\sum_{m=0}^{n}2(n-m)p_{2m+1}x^{2m+1}}{(2n+1)\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}$$

Since the coefficient $2(n-m)p_{2m+1}$ vanishes for $m=n$,

$$\sum_{m=0}^{n}2(n-m)p_{2m+1}x^{2m+1}=\sum_{m=0}^{n-1}2(n-m)p_{2m+1}x^{2m+1},$$

and thus

$$\frac{p(x)}{p'(x)}=\frac{x}{2n+1}+\frac{\sum_{m=0}^{n-1}2(n-m)p_{2m+1}x^{2m+1}}{(2n+1)\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}} \quad (25)$$

where $n=1,2,...$

If $n=0$, then $\sum_{m=0}^{n}2(n-m)p_{2m+1}x^{2m+1}=0$, and $\frac{p(x)}{p'(x)}=\frac{x}{2*0+1}=x$, as expected since then $p(x)=x$.

In this case, where $p(x)=x$ and thus $p'(x)=1$, we obtain the wave functions of the ground-state and of the first excited state, and the respective energies, of a harmonic oscillator. Thus, this case can be considered as trivial.

In (25), we observe that the numerator $\sum_{m=0}^{n-1}2(n-m)p_{2m+1}x^{2m+1}$ is of degree $2n-1$, while the denominator $\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}$ is of degree $2n$, and thus, at long distances, i.e. for $|x|\to\infty$, the fraction $\frac{\sum_{m=0}^{n-1}2(n-m)p_{2m+1}x^{2m+1}}{(2n+1)\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}$ goes to zero, and then $\frac{p(x)}{p'(x)}\simeq\frac{x}{2n+1}$. Then, the integration of $\frac{x}{2n+1}$ gives $\frac{x^2}{2(2n+1)}$, which appear in the exponential factor $\exp\left(-\frac{E_2-E_1}{2}\int_x dy\frac{p(y)}{p'(y)}\right)$, and thus, in order for the wave functions to be square integrable, the factor $-\frac{E_2-E_1}{2}$ must be negative, which means that $E_2>E_1$.

As noted, in the fraction $\frac{\sum_{m=0}^{n-1}2(n-m)p_{2m+1}x^{2m+1}}{\sum_{m=0}^{n}(2m+1)p_{2m+1}x^{2m}}$, the degree of denominator is equal to the degree of numerator plus 1.

Then, to proceed, **we'll consider the case where the series in the numerator is proportional to the derivative of the series in the denominator**, i.e.

$$\sum_{m=0}^{n-1} 2(n-m)\,p_{2m+1}x^{2m+1} = a\left(\sum_{m=0}^{n}(2m+1)\,p_{2m+1}x^{2m}\right)' \quad (26)$$

where a is a non-zero real number.

The relation (26) is a condition that allows us to calculate the integral $\int_x dy \frac{p(y)}{p'(y)}$.

From (26), we have

$$\sum_{m=0}^{n-1} 2(n-m)\,p_{2m+1}x^{2m+1} = a\sum_{m=1}^{n} 2m(2m+1)\,p_{2m+1}x^{2m-1} \Rightarrow$$

$$\Rightarrow \sum_{m=0}^{n-1}(n-m)\,p_{2m+1}x^{2m+1} = a\sum_{m=1}^{n} m(2m+1)\,p_{2m+1}x^{2m-1} \quad (27)$$

Changing the summation index to $m' = m+1$, the series $\sum_{m=0}^{n-1}(n-m)\,p_{2m+1}x^{2m+1}$ is written as

$$\sum_{m'=1}^{n}\left(n-(m'-1)\right)p_{2(m'-1)+1}x^{2(m'-1)+1} = \sum_{m'=1}^{n}(n-m'+1)\,p_{2m'-1}x^{2m'-1}$$

Renaming the summation index of the last series to m, we obtain

$$\sum_{m=0}^{n-1}(n-m)\,p_{2m+1}x^{2m+1} = \sum_{m=1}^{n}(n-m+1)\,p_{2m-1}x^{2m-1}$$

Using the last relation, (27) becomes

$$\sum_{m=1}^{n}(n-m+1)\,p_{2m-1}x^{2m-1} = a\sum_{m=1}^{n} m(2m+1)\,p_{2m+1}x^{2m-1}$$

Equating the coefficients of the same degree terms in x, we obtain

$$(n-m+1)\,p_{2m-1} = am(2m+1)\,p_{2m+1}$$

or

$$p_{2m-1} = \frac{am(2m+1)}{n-m+1}\,p_{2m+1} \quad (28)$$

where $m = 1,2,...,n \geq 1$, and $p_{2n+1} = 1$.

From the previous relation, we calculate the coefficients of the polynomial (23).

Besides, using (24), the condition (26) is written as

$$\sum_{m=0}^{n-1} 2(n-m)\,p_{2m+1}x^{2m+1} = ap''(x) \quad (29)$$

By means of (24) and (29), (25) is written as

$$\frac{p(x)}{p'(x)} = \frac{x}{2n+1} + \frac{ap''(x)}{(2n+1)\,p'(x)} \quad (30)$$

where a is a non-zero real number and $n \geq 1$.
This is the differential equation the $(2n+1)$-degree, odd-parity polynomial $p(x)$ must satisfy in order for the condition (26) to hold.
Using (30), the integral $\int_x dy \frac{p(y)}{p'(y)}$ is easily calculated.
We have

$$\int_x dy \frac{p(y)}{p'(y)} = \int_x dy \left(\frac{y}{2n+1} + \frac{ap''(y)}{(2n+1)p'(y)} \right) = \frac{x^2}{2(2n+1)} + \frac{a}{2n+1} \int_x dy \frac{p''(y)}{p'(y)} =$$

$$= \frac{x^2}{2(2n+1)} + \frac{a}{2n+1} \ln |p'(x)|$$

That is

$$\int_x dy \frac{p(y)}{p'(y)} = \frac{x^2}{2(2n+1)} + \frac{a}{2n+1} \ln |p'(x)| \quad (31)$$

We again remind that the integral $\int_x dy \frac{p(y)}{p'(y)}$ is calculated without adding an integration constant.
Substituting (31) into (14), the wave function $\psi_1(x)$ is

$$\psi_1(x) = A_1 |p'(x)|^{-1/2} \exp\left(-\frac{E_2 - E_1}{2} \left(\frac{x^2}{2(2n+1)} + \frac{a}{2n+1} \ln |p'(x)| \right) \right) =$$

$$= A_1 |p'(x)|^{-1/2} \exp\left(-\frac{(E_2 - E_1)x^2}{4(2n+1)} - \frac{a(E_2 - E_1)}{2(2n+1)} \ln |p'(x)| \right) =$$

$$\underset{a \neq 0}{=} A_1 |p'(x)|^{-1/2} \exp\left(-\frac{(E_2 - E_1)x^2}{4(2n+1)} + \ln |p'(x)|^{-a(E_2 - E_1)/2(2n+1)} \right) =$$

$$= A_1 |p'(x)|^{-1/2} |p'(x)|^{-a(E_2 - E_1)/2(2n+1)} \exp\left(-\frac{(E_2 - E_1)x^2}{4(2n+1)} \right) =$$

$$= A_1 |p'(x)|^{-1/2 - a(E_2 - E_1)/2(2n+1)} \exp\left(-\frac{(E_2 - E_1)x^2}{4(2n+1)} \right) =$$

$$= A_1 |p'(x)|^{-1/2(1 + a(E_2 - E_1)/2n+1)} \exp\left(-\frac{(E_2 - E_1)x^2}{4(2n+1)} \right)$$

That is

$$\psi_1(x) = A_1 |p'(x)|^{-1/2(1 + a(E_2 - E_1)/2n+1)} \exp\left(-\frac{(E_2 - E_1)x^2}{4(2n+1)} \right) \quad (32)$$

Then, from (5), i.e.

$$\psi_2(x) = A_2 p(x) \psi_1(x),$$

we calculate and the wave function $\psi_2(x)$.
Since $p(x)$ and $p'(x)$ are polynomials, in order for the two wave functions to be square integrable, it is necessary – but not sufficient if $p'(x)$ has zeros – that $E_2 > E_1$.
For convenience, we set

$$\rho = -\frac{1}{2}\left(\frac{a(E_2 - E_1)}{2n+1} + 1\right) \quad (33)$$

Then, we have

$$-2\rho = \frac{a(E_2 - E_1)}{2n+1} + 1 \Rightarrow \frac{a(E_2 - E_1)}{2n+1} = -(2\rho + 1) \Rightarrow$$

$$\Rightarrow \frac{E_2 - E_1}{2n+1} = -\frac{2\rho+1}{a} \quad (34)$$

Since $E_2 - E_1 > 0$, then $\frac{2\rho+1}{a} < 0$, and thus the signs of a and $2\rho + 1$ must be opposite.
By means of (33) and (34), (32) is written as

$$\psi_1(x) = A_1 \left|p'(x)\right|^\rho \exp\left(-\frac{\left(-\frac{2\rho+1}{a}\right)x^2}{4}\right) = A_1 \left|p'(x)\right|^\rho \exp\left(\frac{(2\rho+1)x^2}{4a}\right)$$

That is

$$\psi_1(x) = A_1 \left|p'(x)\right|^\rho \exp\left(\frac{(2\rho+1)x^2}{4a}\right) \quad (35)$$

and

$$\psi_2(x) = A_2 p(x) \psi_1(x)$$

As explained, the two wave functions must be finite for every x. Thus, if $p'(x)$ has zeros, the parameter ρ must be non-negative, i.e. $\rho \geq 0$.
The previous condition along with the square-integrability condition $\frac{2\rho+1}{a} < 0$ are the two conditions the wave functions $\psi_1(x)$ and $\psi_2(x)$ must fulfill.
Now, substituting the expression of $\frac{p(x)}{p'(x)}$ from the differential equation (30) into the expression of the potential (22), we obtain

$$V(x) = \left(\frac{E_2 - E_1}{2}\right)^2 \left(\frac{x}{2n+1} + \frac{ap''(x)}{(2n+1)p'(x)}\right)^2 + \frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2 - \frac{p'''(x)}{2p'(x)} +$$

$$+(E_2 - E_1)\left(\frac{x}{2n+1} + \frac{ap''(x)}{(2n+1)p'(x)}\right)\frac{p''(x)}{p'(x)} + E_1 - \frac{E_2 - E_1}{2} =$$

$$=\left(\frac{E_2-E_1}{2(2n+1)}\right)^2\left(x+a\frac{p''(x)}{p'(x)}\right)^2+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+\frac{E_2-E_1}{2n+1}\left(x+a\frac{p''(x)}{p'(x)}\right)\frac{p''(x)}{p'(x)}+$$

$$+E_1-\frac{E_2-E_1}{2}=\left(\frac{E_2-E_1}{2(2n+1)}\right)^2\left(x^2+2ax\frac{p''(x)}{p'(x)}+a^2\left(\frac{p''(x)}{p'(x)}\right)^2\right)+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+\frac{(E_2-E_1)x}{2n+1}\frac{p''(x)}{p'(x)}+\frac{a(E_2-E_1)}{2n+1}\left(\frac{p''(x)}{p'(x)}\right)^2+E_1-\frac{E_2-E_1}{2}=$$

$$=\left(\frac{E_2-E_1}{2(2n+1)}\right)^2x^2+\left(\frac{E_2-E_1}{2n+1}\right)^2\frac{ax}{2}\frac{p''(x)}{p'(x)}+\left(\frac{a(E_2-E_1)}{2(2n+1)}\right)^2\left(\frac{p''(x)}{p'(x)}\right)^2+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+\frac{(E_2-E_1)x}{2n+1}\frac{p''(x)}{p'(x)}+\frac{a(E_2-E_1)}{2n+1}\left(\frac{p''(x)}{p'(x)}\right)^2+E_1-\frac{E_2-E_1}{2}=$$

$$=\left(\frac{E_2-E_1}{2(2n+1)}\right)^2x^2+\frac{E_2-E_1}{2n+1}\left(\frac{a(E_2-E_1)}{2(2n+1)}+1\right)\frac{xp''(x)}{p'(x)}+\left(\left(\frac{a(E_2-E_1)}{2(2n+1)}\right)^2+\frac{a(E_2-E_1)}{2n+1}+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-$$

$$-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}$$

That is

$$V(x)=\left(\frac{E_2-E_1}{2(2n+1)}\right)^2x^2+\frac{E_2-E_1}{2n+1}\left(\frac{a(E_2-E_1)}{2(2n+1)}+1\right)\frac{xp''(x)}{p'(x)}+$$

$$+\left(\left(\frac{a(E_2-E_1)}{2(2n+1)}\right)^2+\frac{a(E_2-E_1)}{2n+1}+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}$$

Substituting (34) into the last equation, we obtain

$$V(x)=\left(\frac{-\frac{2\rho+1}{a}}{2}\right)^2x^2-\frac{2\rho+1}{a}\left(-\frac{2\rho+1}{2}+1\right)\frac{xp''(x)}{p'(x)}+$$

$$+\left(\left(-\frac{2\rho+1}{2}\right)^2-(2\rho+1)+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}=$$

$$=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{2\rho+1}{a}\left(\frac{2\rho+1}{2}-1\right)\frac{xp''(x)}{p'(x)}+\left(\left(\rho+\frac{1}{2}\right)^2-2\rho-1+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+E_1-\frac{E_2-E_1}{2}=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{2\rho+1}{a}\left(\frac{2\rho+1-2}{2}\right)\frac{xp''(x)}{p'(x)}+\left(\rho^2+\rho+\frac{1}{4}-2\rho-1+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-$$

$$-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\frac{xp''(x)}{p'(x)}+\left(\rho^2-\rho\right)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+E_1-\frac{E_2-E_1}{2}=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\frac{xp''(x)}{p'(x)}+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}$$

That is

$$V(x)=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\frac{xp''(x)}{p'(x)}+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2} \quad (36)$$

We see that the potential (36) consists of the harmonic oscillator potential $\frac{(2\rho+1)^2}{4a^2}x^2$ and a rational part with even parity.

Defining the degree of a fraction $\frac{f(x)}{g(x)}$ as the degree of $f(x)$ minus the degree of $g(x)$, we have

$$\deg\left(\frac{xp''(x)}{p'(x)}\right)=0$$

$$\deg\left(\frac{p''(x)}{p'(x)}\right)^2=2\deg\left(\frac{p''(x)}{p'(x)}\right)=2(-1)=-2$$

$$\deg\left(\frac{p'''(x)}{p'(x)}\right)=-2$$

Then

$$\lim_{|x|\to\infty}\left(\frac{p''(x)}{p'(x)}\right)^2=0$$

$$\lim_{|x|\to\infty}\frac{p'''(x)}{p'(x)}=0$$

$$\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}= \text{ a non-zero (finite) constant}$$

Since the polynomial $p(x)$ is of odd parity, then $p'(x)$ is of even parity and $p''(x)$, and thus, since x is of odd parity, $xp''(x)$ is of even parity. Then, $\frac{xp''(x)}{p'(x)}$ is of even parity, and thus its limits at plus and minus infinity are equal.

Therefore, at long distances, the potential (36) is approximately the harmonic oscillator potential $\frac{(2\rho+1)^2}{4a^2}x^2$ plus the constant

$$\frac{(2\rho-1)(2\rho+1)}{2a}\left(\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}\right)+E_1-\frac{E_2-E_1}{2}.$$

Choosing the infinity as the reference point of the potential, we set the previous constant equal to zero, i.e.

$$\frac{(2\rho-1)(2\rho+1)}{2a}\left(\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}\right)+E_1-\frac{E_2-E_1}{2}=0 \quad (37)$$

The potential then becomes a rational extension of the harmonic oscillator $\frac{(2\rho+1)^2}{4a^2}x^2$.

Since $xp''(x)$ is an even-parity polynomial of degree $2n$ and $p'(x)$ is also an even-parity polynomial of the same degree, the limit $\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}$ is equal to the ratio of the leading coefficients of the numerator and denominator, i.e. of $xp''(x)$ and $p'(x)$.

From (24), the leading coefficient of $p'(x)$ is $(2n+1)\,p_{2n+1}=2n+1$.

Using (24), the second derivative of $p(x)$ is

$$p''(x)=\sum_{m=1}^{n}2m(2m+1)\,p_{2m+1}x^{2m-1}$$

Thus

$$xp''(x)=\sum_{m=1}^{n}2m(2m+1)\,p_{2m+1}x^{2m}$$

Then, the leading coefficient of $xp''(x)$ is $2n(2n+1)\,p_{2n+1}=2n(2n+1)$.

Thus

$$\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}=\frac{2n(2n+1)}{2n+1}=2n$$

Then, the condition (37) takes the form

$$\frac{(2\rho-1)(2\rho+1)}{2a}2n+E_1-\frac{E_2-E_1}{2}=0$$

Thus

$$E_1-\frac{E_2-E_1}{2}=-\frac{n(2\rho-1)(2\rho+1)}{a} \quad (38)$$

Substituting (38) into the expression of the potential (36), we obtain

$$V(x)=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\frac{xp''(x)}{p'(x)}+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}-\frac{n(2\rho-1)(2\rho+1)}{a}=$$

$$=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\left(\frac{xp''(x)}{p'(x)}-2n\right)+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}=$$

$$=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\left(\frac{xp''(x)-2np'(x)}{p'(x)}\right)+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}$$

That is

$$V(x)=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\left(\frac{xp''(x)-2np'(x)}{p'(x)}\right)+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)} \quad (39)$$

Using (34) and (38), we can express the two energies E_1 and E_2 in terms of the parameters a and ρ.
From (34), we obtain

$$E_2-E_1=-\frac{(2n+1)(2\rho+1)}{a} \quad (40)$$

By means of (40), (38) becomes

$$E_1-\frac{-\frac{(2n+1)(2\rho+1)}{a}}{2}=-\frac{n(2\rho-1)(2\rho+1)}{a}\Rightarrow E_1+\frac{(2n+1)(2\rho+1)}{2a}=-\frac{n(2\rho-1)(2\rho+1)}{a}\Rightarrow$$

$$\Rightarrow E_1=-\frac{2n(2\rho-1)(2\rho+1)}{2a}-\frac{(2n+1)(2\rho+1)}{2a}=-\frac{\left(2n(2\rho-1)+(2n+1)\right)(2\rho+1)}{2a}=$$

$$=-\frac{(4n\rho-2n+2n+1)(2\rho+1)}{2a}=-\frac{(4n\rho+1)(2\rho+1)}{2a}$$

That is

$$E_1=-\frac{(4n\rho+1)(2\rho+1)}{2a} \quad (41)$$

By means of (41), (40) becomes

$$E_2+\frac{(4n\rho+1)(2\rho+1)}{2a}=-\frac{(2n+1)(2\rho+1)}{a}\Rightarrow E_2=-\frac{2(2n+1)(2\rho+1)}{2a}-\frac{(4n\rho+1)(2\rho+1)}{2a}=$$

$$=-\frac{\left(2(2n+1)+(4n\rho+1)\right)(2\rho+1)}{2a}=-\frac{(4n+2+4n\rho+1)(2\rho+1)}{2a}=-\frac{\left(4n(\rho+1)+3\right)(2\rho+1)}{2a}$$

That is

$$E_2=-\frac{\left(4n(\rho+1)+3\right)(2\rho+1)}{2a} \quad (42)$$

As explained, $\frac{2\rho+1}{a}<0$, so that the two wave functions are square integrable. Thus $-\frac{2\rho+1}{2a}>0$, and then E_1 has the same sign as $4n\rho+1$, while E_2 has the same sign as $4n(\rho+1)+3$.

Also, since $4n(\rho+1)+3=4n\rho+1+\underbrace{4n+2}_{>0}>4n\rho+1$, then $E_2>E_1$, as it should.

For $\rho=0$, the square-integrability condition $\frac{2\rho+1}{a}<0$ becomes $\frac{1}{a}<0$, and thus $a<0$.

Besides, for $\rho=0$, the wave function $\psi_1(x)$ becomes $\psi_1(x)=A_1\exp\left(\frac{x^2}{4a}\right)$, with $a<0$, which is the ground-state wave function of a harmonic oscillator. Then, for $\rho=0$, the potential (39) must become a harmonic oscillator potential.
Let us verify it.
For $\rho=0$, (39) becomes

$$V(x)=\frac{x^2}{4a^2}-\frac{1}{2a}\left(\frac{xp''(x)-2np'(x)}{p'(x)}\right)-\frac{p'''(x)}{2p'(x)}=\frac{x^2}{4a^2}-\frac{1}{2a}\left(\frac{xp''(x)}{p'(x)}-2n\right)-\frac{p'''(x)}{2p'(x)}=$$

$$=\frac{x^2}{4a^2}-\frac{1}{2a}\left(\frac{xp''(x)}{p'(x)}-2n+\frac{ap'''(x)}{p'(x)}\right)$$

That is

$$V(x)=\frac{x^2}{4a^2}-\frac{1}{2a}\left(\frac{xp''(x)}{p'(x)}-2n+\frac{ap'''(x)}{p'(x)}\right)\quad(43)$$

Using (30), we'll show that the expression in parentheses vanishes.
Differentiating both members of (30) with respect to x, we obtain

$$\frac{p'(x)}{p'(x)}-\frac{p(x)p''(x)}{p'^2(x)}=\frac{1}{2n+1}+\frac{a}{2n+1}\left(\frac{p'''(x)}{p'(x)}-\frac{p''(x)p''(x)}{p'^2(x)}\right)\Rightarrow$$

$$\Rightarrow 1-\frac{p(x)}{p'(x)}\frac{p''(x)}{p'(x)}=\frac{1}{2n+1}+\frac{a}{2n+1}\left(\frac{p'''(x)}{p'(x)}-\left(\frac{p''(x)}{p'(x)}\right)^2\right)$$

Using (30), the previous equation becomes

$$1-\left(\frac{x}{2n+1}+\frac{ap''(x)}{(2n+1)p'(x)}\right)\frac{p''(x)}{p'(x)}=\frac{1}{2n+1}+\frac{a}{2n+1}\left(\frac{p'''(x)}{p'(x)}-\left(\frac{p''(x)}{p'(x)}\right)^2\right)\Rightarrow$$

$$\Rightarrow 1-\frac{x}{2n+1}\frac{p''(x)}{p'(x)}-\frac{ap''(x)}{(2n+1)p'(x)}\frac{p''(x)}{p'(x)}=\frac{1}{2n+1}+\frac{a}{2n+1}\frac{p'''(x)}{p'(x)}-\frac{a}{2n+1}\left(\frac{p''(x)}{p'(x)}\right)^2\Rightarrow$$

$$\Rightarrow 1-\frac{x}{2n+1}\frac{p''(x)}{p'(x)}-\frac{a}{2n+1}\left(\frac{p''(x)}{p'(x)}\right)^2=\frac{1}{2n+1}+\frac{a}{2n+1}\frac{p'''(x)}{p'(x)}-\frac{a}{2n+1}\left(\frac{p''(x)}{p'(x)}\right)^2\Rightarrow$$

$$\Rightarrow 1-\frac{x}{2n+1}\frac{p''(x)}{p'(x)}=\frac{1}{2n+1}+\frac{a}{2n+1}\frac{p'''(x)}{p'(x)}\Rightarrow 2n+1-\frac{xp''(x)}{p'(x)}=1+\frac{ap'''(x)}{p'(x)}\Rightarrow$$

$$\Rightarrow 0=\frac{xp''(x)}{p'(x)}-2n+\frac{ap'''(x)}{p'(x)}$$

That is

$$\frac{xp''(x)}{p'(x)}-2n+\frac{ap'''(x)}{p'(x)}=0$$

Thus, for $\rho=0$, the potential becomes the harmonic oscillator potential $V(x)=\frac{x^2}{4a^2}$.
Besides, for $\rho=0$, (41) and (42) give, respectively,

$E_1=-\frac{1}{2a}$ and $E_2=-\frac{4n+3}{2a}$.

E_1 is the ground-state energy of the harmonic oscillator $\frac{x^2}{4a^2}$, and since $a<0$, it is positive, as it should.

E_2 is an odd-parity excited-state energy of the harmonic oscillator $\frac{x^2}{4a^2}$, and it is greater than E_1, as it should.

We remind that we work with dimensionless variables.

If $a=-\frac{1}{2}$, then $V(x)=x^2$, and $E_1=1$ and $E_2=4n+3$, while

if $a=-1$, then $V(x)=\frac{x^2}{4}$, and $E_1=\frac{1}{2}$ and $E_2=\frac{4n+3}{2}=(2n+1)+\frac{1}{2}$,

which are, respectively, the ground-state energy and the $(2n+1)$-th excited-state energy of a harmonic oscillator with $\hbar\omega\equiv 1$.

The odd-parity polynomials p(x) for α<0 and for α>0

The differential equation (30) is written as

$$(2n+1)p(x)=xp'(x)+ap''(x)\Rightarrow ap''(x)+xp'(x)-(2n+1)p(x)=0\Rightarrow$$
$$\Rightarrow -2ap''(x)-2xp'(x)+2(2n+1)p(x)=0 \quad (44)$$

i. $a<0$

Then $a=-|a|$, and (44) becomes

$$2|a|p''(x)-2xp'(x)+2(2n+1)p(x)=0 \quad (45)$$

Setting $\tilde{x}=\frac{1}{\sqrt{2|a|}}x$, we have

$$\frac{d}{dx}=\frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}}=\frac{1}{\sqrt{2|a|}}\frac{d}{d\tilde{x}}$$

and

$$\frac{d^2}{dx^2}=\frac{d}{dx}\frac{d}{dx}=\left(\frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}}\right)\left(\frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}}\right)=\left(\frac{1}{\sqrt{2|a|}}\frac{d}{d\tilde{x}}\right)\left(\frac{1}{\sqrt{2|a|}}\frac{d}{d\tilde{x}}\right)=\frac{1}{2|a|}\frac{d^2}{d\tilde{x}^2}$$

Also

$x=\sqrt{2|a|}\tilde{x}$ (46)

Thus, with respect to $\tilde{x}$, (45) is written as

$$2|a|\frac{1}{2|a|}p''(\tilde{x})-2\sqrt{2|a|}\tilde{x}\frac{1}{\sqrt{2|a|}}p'(\tilde{x})+2(2n+1)p(\tilde{x})=0\Rightarrow$$

$$\Rightarrow p''(\tilde{x})-2\tilde{x}p'(\tilde{x})+2(2n+1)p(\tilde{x})=0$$

where now the primes denote differentiation with respect to $\tilde{x}$.
We see that $p(\tilde{x})$ satisfies the Hermite differential equation, i.e.

$$y''(\tilde{x})-2\tilde{x}y'(\tilde{x})+2\lambda y(\tilde{x})=0,$$

for $\lambda=2n+1$.
Thus, since $p(\tilde{x})$ is a polynomial, then $p(\tilde{x})=bH_{2n+1}(\tilde{x})$, where $H_{2n+1}(\tilde{x})$ is the Hermite polynomial of degree $2n+1$, and b is a real non-zero constant.
Obviously, the polynomials $p(\tilde{x})$ and $H_{2n+1}(\tilde{x})$ have the same zeros.
The Hermite polynomials are orthogonal, and a Hermite polynomial of degree n has n zeros.
Thus, $H_{2n+1}(\tilde{x})$ has $2n+1$ zeros, and then $p(\tilde{x})$ has also $2n+1$ zeros.
Since the relation (46) is a linear, one-to-one relation, $p(x)$ has also $2n+1$ zeros.
Using that $\tilde{x}=\frac{1}{\sqrt{2|a|}}x$, the relation $p(\tilde{x})=bH_{2n+1}(\tilde{x})$ is written as

$p(x)=bH_{2n+1}\left(\frac{1}{\sqrt{2|a|}}x\right)$. Then, the constant b is calculated by comparing the leading coefficient of $p(x)$, which is 1, with the leading coefficient of

$H_{2n+1}\left(\frac{1}{\sqrt{2|a|}}x\right)$.

We thus showed that if $a<0$, the odd-parity polynomial $p(x)$ has $2n+1$ zeros.
Then, from Rolle's theorem, the derivative $p'(x)$ has at least $2n$ zeros, and because it is a polynomial of degree $2n$, it has exactly $2n$ zeros.

Therefore, if α<0, p(x) has 2n+1 zeros and p´(x) has 2n zeros.

ii. $a>0$
We showed that, in any case, the coefficients of $p(x)$ are given by (28), i.e.

$p_{2m-1} = \frac{am(2m+1)}{n-m+1} p_{2m+1}$,

where $m = 1, 2, ..., n \geq 1$, and $p_{2n+1} = 1$.

Since $n - m \geq 0$, the fraction $\frac{am(2m+1)}{n-m+1}$ has the same sign as a. Thus, if $a > 0$, then $\frac{am(2m+1)}{n-m+1} > 0$, and since the polynomial $p(x)$ is monic, i.e. $p_{2n+1} = 1 > 0$, all its coefficients are positive.

Then $p(x) > 0$ for $x > 0$. Since $p(x)$ is of odd parity, $p(x) < 0$ for $x < 0$, and, also, $p(0) = 0$.

Thus, if $a > 0$, $p(x)$ has only one zero, at 0.

Since all coefficients of $p(x)$ are positive, the coefficients of $p'(x)$ are positive too, because each coefficient of $p'(x)$ is the product of a respective (positive) coefficient of $p(x)$ and a positive integer.

Now, since $p(x)$ is of odd-parity, $p'(x)$ is of even parity.

Thus, $p'(x)$ is an even-parity polynomial with positive coefficients.

Then, $p'(x) > 0$ for $x > 0$, and since $p'(-x) = p'(x)$, $p'(x) > 0$ for $x < 0$.

Also, $p'(0) = p_1 > 0$.

Thus, the polynomial $p'(x)$ has no zeros.

Therefore, if α>0, p(x) has only one zero, at 0, and p΄(x) has no zeros.

Summary of the case where p(x) is an odd-parity polynomial and satisfies the integrability condition (26)

To summarize the case where $p(x)$ is an odd-parity polynomial of degree $2n+1$, with $n \geq 1$, we have

α<0

Then $p(x)$ has $2n+1$ zeros and $p'(x)$ has $2n$ zeros.

Since $p'(x)$ has zeros, the condition that the wave functions must be finite for every x gives $\rho \geq 0$.

Then, the square integrability condition $\frac{2\rho+1}{a} < 0$ is satisfied.

In this case, the two wave functions have $2n$ common – or special – zeros, which are the simple zeros of $p'(x)$, and the wave function $\psi_2(x)$ has $2n+1$ more zeros, which are the simple zeros of $p(x)$.

At each zero of $p'(x)$, the potential has a pole of second order, and thus the potential has $2n$ second-order poles.

The potential is then a singular rational extension of the harmonic oscillator $\frac{(2\rho+1)^2}{4a^2}x^2$.

The wave function $\psi_1(x)$ is the ground-state wave function, with energy E_1, while the wave function $\psi_2(x)$ is the $(2n+1)$-th excited-state wave function, with energy E_2.

Between the two energy levels E_1 and E_2, there are $2n$ energy levels, as many as the poles of the potential.

The two energies are given by

$$E_1 = -\frac{(4n\rho+1)(2\rho+1)}{2a} \text{ and } E_2 = -\frac{(4n(\rho+1)+3)(2\rho+1)}{2a}$$

The energy E_1 has the same sign as $4n\rho+1$, and thus, since $\rho \geq 0$, it is positive.

That is, the ground-state energy, and thus all excited-state energies too, are positive.

For $\rho = 0$, the potential becomes the harmonic oscillator potential $V(x) = \frac{x^2}{4a^2}$, and the two wave functions $\psi_1(x)$ and $\psi_2(x)$ then become, respectively, the ground-state wave function and an odd-parity excited-state wave function of the previous harmonic oscillator.

α>0

Then $p(x)$ has only one zero, at 0, and $p'(x)$ has no zeros.

Since $p'(x)$ has no zeros, the two wave functions are finite for every x for every value of the parameter ρ.

The square integrability condition $\frac{2\rho+1}{a} < 0$ gives $\rho < -\frac{1}{2}$.

In this case, $\psi_1(x)$ has no zeros, and it is the ground-state wave function, with energy E_1, while $\psi_2(x)$ has one zero, at 0, and it is the first-excited-state wave function, with energy E_2.

The potential has no singularities, it is a smooth function, and it is a regular rational extension of the harmonic oscillator $\frac{(2\rho+1)^2}{4a^2}x^2$.

In this case, between the two energy levels E_1 and E_2 there are no energy levels.

As in the case where $a < 0$, the two energies are given by

$$E_1 = -\frac{(4n\rho+1)(2\rho+1)}{2a} \text{ and } E_2 = -\frac{(4n(\rho+1)+3)(2\rho+1)}{2a}$$

The energy E_1 has the same sign as $4n\rho+1$, while the energy E_2 has the same sign as $4n(\rho+1)+3$, respectively.

Since $\rho < -\frac{1}{2}$, and $n \geq 1$, then $n\rho < -\frac{1}{2} \Rightarrow 4n\rho < -2 \Rightarrow 4n\rho+1 < -1 < 0$.

Thus $E_1 < 0$, i.e. the ground-state energy is always negative.
For the first-excited-state energy, we have the cases
If $4n(\rho+1)+3<0 \Rightarrow 4n(\rho+1)<-3 \Rightarrow \rho+1<-\frac{3}{4n} \Rightarrow \rho<-1-\frac{3}{4n}$, then E_2 is also negative, i.e. the first-excited-state energy is negative.
If $4n(\rho+1)+3=0 \Rightarrow \rho=-1-\frac{3}{4n}$, then the first-excited-state energy is zero.
If $4n(\rho+1)+3>0 \Rightarrow \rho>-1-\frac{3}{4n}$, i.e. if $-\frac{1}{2}>\rho>-1-\frac{3}{4n}$, then the first-excited-state energy becomes positive.
Observe that for $n \to \infty$, i.e. for big odd-parity polynomials $p(x)$, the critical value of ρ, the value for which the first-excited-state energy vanishes, tends to -1.
If $-\frac{1}{2}>\rho>-1$, the first-excited-state energy is positive for every $n \geq 1$, and then the first two energies, i.e. the ground-state energy and the first-excited-state energy, have different signs.

II. p(x) is of even parity and satisfies an integrability condition

If $p(x)$ is an even-parity polynomial, it will be of even degree and it will have the form

$$p(x)=\sum_{m=0}^{n} p_{2m} x^{2m} \quad (47)$$

where $p_{2n}=1$ and $n=1,2,\ldots$
The case $n=0$ $\left(p(x)=1\right)$ is excluded, since then $p'(x)=0$.
Besides, if $p(x)=1$, then $\psi_2(x)=A_2\psi_1(x)$, i.e. the two wave functions are linearly dependent and thus they describe the same eigenstate, i.e. $E_2=E_1$.
Observe also that since $p(x)$ is of even parity, $p'(x)$ is of odd parity, and then is has a zero at 0, which is then a singular point – a second order pole – of the potential. In other words, in this case the potential is always singular.
From (47), the first derivative of $p(x)$ is

$$p'(x)=\sum_{m=1}^{n} 2m p_{2m} x^{2m-1} \quad (48)$$

Then, we have

$$\frac{p(x)}{p'(x)}=\frac{\sum_{m=0}^{n}p_{2m}x^{2m}}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=\frac{\sum_{m=1}^{n}p_{2m}x^{2m}+p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=\frac{x\sum_{m=1}^{n}p_{2m}x^{2m-1}}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=$$

$$=\frac{x\sum_{m=1}^{n}p_{2m}x^{2m-1}}{2n\sum_{m=1}^{n}\frac{2m}{2n}p_{2m}x^{2m-1}}+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=\frac{x\sum_{m=1}^{n}\left(p_{2m}x^{2m-1}+\left(\frac{m}{n}-1\right)p_{2m}x^{2m-1}-\left(\frac{m}{n}-1\right)p_{2m}x^{2m-1}\right)}{2n\sum_{m=1}^{n}\frac{m}{n}p_{2m}x^{2m-1}}+$$

$$+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=\frac{x}{2n}\frac{\sum_{m=1}^{n}\left(\left(1+\frac{m}{n}-1\right)p_{2m}x^{2m-1}-\left(\frac{m}{n}-1\right)p_{2m}x^{2m-1}\right)}{\sum_{m=1}^{n}\frac{m}{n}p_{2m}x^{2m-1}}+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=$$

$$=\frac{x}{2n}\frac{\sum_{m=1}^{n}\left(\frac{m}{n}p_{2m}x^{2m-1}-\left(\frac{m}{n}-1\right)p_{2m}x^{2m-1}\right)}{\sum_{m=1}^{n}\frac{m}{n}p_{2m}x^{2m-1}}+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=$$

$$=\frac{x}{2n}\left(\frac{\sum_{m=1}^{n}\frac{m}{n}p_{2m}x^{2m-1}}{\sum_{m=1}^{n}\frac{m}{n}p_{2m}x^{2m-1}}-\frac{\sum_{m=1}^{n}\left(\frac{m}{n}-1\right)p_{2m}x^{2m-1}}{\sum_{m=1}^{n}\frac{m}{n}p_{2m}x^{2m-1}}\right)+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=$$

$$=\frac{x}{2n}\left(1-\frac{\sum_{m=1}^{n}\left(\frac{m}{n}-1\right)p_{2m}x^{2m-1}}{\sum_{m=1}^{n}\frac{m}{n}p_{2m}x^{2m-1}}\right)+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=\frac{x}{2n}-\frac{\sum_{m=1}^{n}\left(\frac{m}{n}-1\right)p_{2m}x^{2m}}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=$$

$$=\frac{x}{2n}-\frac{\sum_{m=1}^{n}\frac{m-n}{n}p_{2m}x^{2m}}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=\frac{x}{2n}-\frac{\sum_{m=1}^{n}(m-n)p_{2m}x^{2m}}{n\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=$$

$$=\frac{x}{2n}+\frac{\sum_{m=1}^{n}(n-m)p_{2m}x^{2m}}{n\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}+\frac{p_0}{\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}=\frac{x}{2n}+\frac{\sum_{m=1}^{n}(n-m)p_{2m}x^{2m}+np_0}{n\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}$$

That is

$$\frac{p(x)}{p'(x)}=\frac{x}{2n}+\frac{\sum_{m=1}^{n}(n-m)p_{2m}x^{2m}+np_0}{n\sum_{m=1}^{n}2mp_{2m}x^{2m-1}}$$

But

$$\sum_{m=1}^{n}(n-m)p_{2m}x^{2m}+np_0=\sum_{m=0}^{n}(n-m)p_{2m}x^{2m}=\sum_{m=0}^{n-1}(n-m)p_{2m}x^{2m}\ ,$$

because the coefficient $(n-m)p_{2m}$ vanishes for $m=n$.

Then, the previous expression of $\frac{p(x)}{p'(x)}$ is written as

$$\frac{p(x)}{p'(x)}=\frac{x}{2n}+\frac{\sum_{m=0}^{n-1}(n-m)p_{2m}x^{2m}}{n\sum_{m=1}^{n}2mp_{2m}x^{2m-1}} \quad (49)$$

where $n=1,2,...$

In (49), the degree of denominator $\sum_{m=1}^{n}2mp_{2m}x^{2m-1}$ is $2n-1$, while the degree of numerator $\sum_{m=0}^{n-1}(n-m)p_{2m}x^{2m}$ is $2(n-1)=2n-2$.

As in the case of odd-parity polynomials $p(x)$, the degree of denominator is equal to the degree of numerator plus one, and then **we'll consider the case where the series in the numerator is proportional to the derivative of the series in the denominator**, i.e.

$$\sum_{m=0}^{n-1}(n-m)p_{2m}x^{2m}=a\left(\sum_{m=1}^{n}2mp_{2m}x^{2m-1}\right)' \quad (50)$$

where a is a non-zero real number.

As in the case of odd-parity polynomials $p(x)$, the relation (50) is a condition that allows us to calculate the integral $\int_x dy\frac{p(y)}{p'(y)}$.

The condition (50) gives

$$\sum_{m=0}^{n-1}(n-m)p_{2m}x^{2m}=a\sum_{m=1}^{n}2m(2m-1)p_{2m}x^{2m-2}=a\sum_{m=1}^{n}2m(2m-1)p_{2m}x^{2(m-1)}$$

That is

$$\sum_{m=0}^{n-1}(n-m)p_{2m}x^{2m}=a\sum_{m=1}^{n}2m(2m-1)p_{2m}x^{2(m-1)} \quad (51)$$

Changing the summation index of the series in the right-hand side of (51) to $m'=m-1$, we have

$$\sum_{m=1}^{n}2m(2m-1)p_{2m}x^{2(m-1)}=\sum_{m'=0}^{n-1}2(m'+1)(2(m'+1)-1)p_{2(m'+1)}x^{2m'}=$$

$$=\sum_{m'=0}^{n-1}2(m'+1)(2m'+1)p_{2(m'+1)}x^{2m'}=\sum_{m'=0}^{n-1}(2m'+1)(2m'+2)p_{2m'+2}x^{2m'}$$

Renaming the summation index of the last series to m, we end up to

$$\sum_{m=1}^{n}2m(2m-1)p_{2m}x^{2(m-1)}=\sum_{m=0}^{n-1}(2m+1)(2m+2)p_{2m+2}x^{2m}$$

Substituting the last relation into (51), we obtain

$$\sum_{m=0}^{n-1}(n-m)\,p_{2m}x^{2m}=a\sum_{m=0}^{n-1}(2m+1)(2m+2)\,p_{2m+2}x^{2m}$$

Equating the coefficients of the same degree in x, we obtain

$$(n-m)\,p_{2m}=a(2m+1)(2m+2)\,p_{2m+2}$$

or

$$p_{2m}=\frac{a(2m+1)(2m+2)}{n-m}\,p_{2m+2} \quad (52)$$

where $m=0,1,\ldots n-1$ $(n\geq 1)$, and $p_{2n}=1$.

From the previous relation, we calculate the coefficients of the polynomial (47).

By means of (48), the condition (50) is written as

$$\sum_{m=0}^{n-1}(n-m)\,p_{2m}x^{2m}=ap''(x)$$

Using again (48), and the previous relation, (49) is written as

$$\frac{p(x)}{p'(x)}=\frac{x}{2n}+\frac{ap''(x)}{np'(x)} \quad (53)$$

where a is a non-zero real number and $n\geq 1$.

This is the differential equation the $2n$-degree, even-parity polynomial $p(x)$ must satisfy in order for the condition (50) to hold.

Using (53), the integral $\int_x dy\frac{p(y)}{p'(y)}$ becomes

$$\int_x dy\frac{p(y)}{p'(y)}=\int_x dy\left(\frac{y}{2n}+\frac{ap''(y)}{np'(y)}\right)=\frac{x^2}{4n}+\frac{a}{n}\int_x dy\frac{p''(y)}{p'(y)}=\frac{x^2}{4n}+\frac{a}{n}\ln\left|p'(x)\right|$$

That is

$$\int_x dy\frac{p(y)}{p'(y)}=\frac{x^2}{4n}+\frac{a}{n}\ln\left|p'(x)\right| \quad (54)$$

We once again remind that we omit the integration constant in the calculation of the integral $\int_x dy\frac{p(y)}{p'(y)}$.

By means of (54), the wave function $\psi_1(x)$, which is given by (14), is written as

$$\psi_1(x)=A_1\left|p'(x)\right|^{-1/2}\exp\left(-\frac{E_2-E_1}{2}\left(\frac{x^2}{4n}+\frac{a}{n}\ln\left|p'(x)\right|\right)\right)=$$

$$=A_1\left|p'(x)\right|^{-1/2}\exp\left(-\frac{(E_2-E_1)x^2}{8n}-\frac{a(E_2-E_1)}{2n}\ln\left|p'(x)\right|\right)=$$

$$\overset{a\neq 0}{=} A_1 \left|p'(x)\right|^{-1/2} \exp\left(-\frac{(E_2-E_1)x^2}{8n} + \ln\left|p'(x)\right|^{-a(E_2-E_1)/2n}\right) =$$

$$= A_1 \left|p'(x)\right|^{-1/2} \left|p'(x)\right|^{-a(E_2-E_1)/2n} \exp\left(-\frac{(E_2-E_1)x^2}{8n}\right) =$$

$$= A_1 \left|p'(x)\right|^{-1/2-a(E_2-E_1)/2n} \exp\left(-\frac{(E_2-E_1)x^2}{8n}\right) =$$

$$= A_1 \left|p'(x)\right|^{-1/2(1+a(E_2-E_1)/n)} \exp\left(-\frac{(E_2-E_1)x^2}{8n}\right)$$

That is

$$\psi_1(x) = A_1 \left|p'(x)\right|^{-1/2(1+a(E_2-E_1)/n)} \exp\left(-\frac{(E_2-E_1)x^2}{8n}\right) \quad (55)$$

and $\psi_2(x) = A_2 p(x)\psi_1(x)$.

In order for the two wave functions to be square integrable, it is necessary – but not sufficient as $p'(x)$ has zero(s) – that $E_2 > E_1$.

As in the case of odd-parity polynomials $p(x)$, we set the exponent $-\frac{1}{2}\left(\frac{a(E_2-E_1)}{n}+1\right)$ as a new parameter ρ, i.e.

$$\rho = -\frac{1}{2}\left(\frac{a(E_2-E_1)}{n}+1\right) \quad (56)$$

Making use of (56), we obtain

$$-2\rho = \frac{a(E_2-E_1)}{n}+1 \Rightarrow \frac{E_2-E_1}{n} = -\frac{2\rho+1}{a} \quad (57)$$

Since $E_2 > E_1$, then $\frac{2\rho+1}{a} < 0$.

Then, in terms of the parameters a and ρ, the wave function (55) is written as

$$\psi_1(x) = A_1 \left|p'(x)\right|^{\rho} \exp\left(-\frac{\left(-\frac{2\rho+1}{a}\right)x^2}{8}\right) = A_1 \left|p'(x)\right|^{\rho} \exp\left(\frac{(2\rho+1)x^2}{8a}\right)$$

That is

$$\psi_1(x) = A_1 \left|p'(x)\right|^{\rho} \exp\left(\frac{(2\rho+1)x^2}{8a}\right) \quad (58)$$

and $\psi_2(x) = A_2 p(x)\psi_1(x)$.

As noted, since $p(x)$ is of even parity, $p'(x)$ is of odd parity, and thus it has a zero at 0. Then, in order for the two wave functions to be finite at 0, ρ must be non-negative, i.e. $\rho \geq 0$. Then $2\rho+1>0$ and the square-integrability condition $\frac{2\rho+1}{a}<0$ gives $a<0$.

Thus, if p(x) is an even-parity polynomial that satisfies the condition (50), α can be only negative, while ρ is non-negative, i.e. zero or positive.

We'll now substitute the expression of $\frac{p(x)}{p'(x)}$ from the differential equation (53) into the potential (22), to derive an expression of the potential in the case where $p(x)$ is an even-parity polynomial that satisfies the condition (50).
We have

$$V(x)=\left(\frac{E_2-E_1}{2}\right)^2\left(\frac{x}{2n}+\frac{ap''(x)}{np'(x)}\right)^2+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+(E_2-E_1)\left(\frac{x}{2n}+\frac{ap''(x)}{np'(x)}\right)\frac{p''(x)}{p'(x)}+E_1-\frac{E_2-E_1}{2}=$$

$$=\left(\frac{E_2-E_1}{2}\right)^2\left(\frac{x^2}{(2n)^2}+\left(\frac{a}{n}\right)^2\left(\frac{p''(x)}{p'(x)}\right)^2+\frac{ax}{n^2}\frac{p''(x)}{p'(x)}\right)+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+\frac{E_2-E_1}{2n}\frac{xp''(x)}{p'(x)}+\frac{a(E_2-E_1)}{n}\left(\frac{p''(x)}{p'(x)}\right)^2+E_1-\frac{E_2-E_1}{2}=$$

$$=\left(\frac{E_2-E_1}{4n}\right)^2x^2+\left(\frac{a(E_2-E_1)}{2n}\right)^2\left(\frac{p''(x)}{p'(x)}\right)^2+\left(\frac{E_2-E_1}{2n}\right)^2\frac{axp''(x)}{p'(x)}+\frac{3}{4}\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+\frac{E_2-E_1}{2n}\frac{xp''(x)}{p'(x)}+\frac{a(E_2-E_1)}{n}\left(\frac{p''(x)}{p'(x)}\right)^2+E_1-\frac{E_2-E_1}{2}=$$

$$=\left(\frac{E_2-E_1}{4n}\right)^2x^2+\left(\left(\frac{a(E_2-E_1)}{2n}\right)^2+\frac{a(E_2-E_1)}{n}+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2+\frac{E_2-E_1}{2n}\left(\frac{a(E_2-E_1)}{2n}+1\right)\frac{xp''(x)}{p'(x)}-$$

$$-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}$$

That is

$$V(x)=\left(\frac{E_2-E_1}{4n}\right)^2x^2+\frac{E_2-E_1}{2n}\left(\frac{a(E_2-E_1)}{2n}+1\right)\frac{xp''(x)}{p'(x)}+$$

$$+\left(\left(\frac{a(E_2-E_1)}{2n}\right)^2+\frac{a(E_2-E_1)}{n}+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}$$

Substituting (57) into the last equation, we obtain

$$V(x)=\left(\frac{-\frac{2\rho+1}{a}}{4}\right)^2 x^2-\frac{2\rho+1}{2a}\left(-\frac{2\rho+1}{2}+1\right)\frac{xp''(x)}{p'(x)}+$$

$$+\left(\left(-\frac{2\rho+1}{2}\right)^2-(2\rho+1)+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}=$$

$$=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{2\rho+1}{2a}\left(\frac{2\rho+1}{2}-1\right)\frac{xp''(x)}{p'(x)}+\left(\left(\rho+\frac{1}{2}\right)^2-2\rho-1+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+E_1-\frac{E_2-E_1}{2}=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{2\rho+1}{2a}\left(\frac{2\rho+1-2}{2}\right)\frac{xp''(x)}{p'(x)}+\left(\rho^2+\rho+\frac{1}{4}-2\rho-1+\frac{3}{4}\right)\left(\frac{p''(x)}{p'(x)}\right)^2-$$

$$-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\frac{xp''(x)}{p'(x)}+\left(\rho^2-\rho\right)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+$$

$$+E_1-\frac{E_2-E_1}{2}=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\frac{xp''(x)}{p'(x)}+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2}$$

That is

$$V(x)=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\frac{xp''(x)}{p'(x)}+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}+E_1-\frac{E_2-E_1}{2} \quad (59)$$

As in the case of odd-parity polynomials $p(x)$, the potential consists of a harmonic oscillator potential and a rational part with even parity, and also, we have

$$\deg\left(\frac{xp''(x)}{p'(x)}\right)=0$$

$$\deg\left(\frac{p''(x)}{p'(x)}\right)^2=2\deg\left(\frac{p''(x)}{p'(x)}\right)=2(-1)=-2$$

$$\deg\left(\frac{p'''(x)}{p'(x)}\right)=-2$$

Then

$$\lim_{|x|\to\infty}\left(\frac{p''(x)}{p'(x)}\right)^2=0$$

$$\lim_{|x|\to\infty}\frac{p'''(x)}{p'(x)}=0$$

$$\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)} = \text{ a non-zero (finite) constant}$$

The polynomial $p(x)$ is of even parity, and thus $p'(x)$ is of odd parity and $p''(x)$ is of even parity, and then, since x is of odd parity, $xp''(x)$ is of odd parity. Since $xp''(x)$ and $p'(x)$ have the same parity, $\frac{xp''(x)}{p'(x)}$ is of even parity, and thus its limits at plus and minus infinity are equal.

Then, at long distances, the potential is approximately the harmonic oscillator potential $\frac{(2\rho+1)^2}{16a^2}x^2$ plus the constant

$$\frac{(2\rho-1)(2\rho+1)}{4a}\left(\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}\right)+E_1-\frac{E_2-E_1}{2}.$$

As in the case of odd-parity polynomials $p(x)$, we choose the infinity as the reference point of the potential and we set the previous constant equal to zero, i.e.

$$\frac{(2\rho-1)(2\rho+1)}{4a}\left(\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}\right)+E_1-\frac{E_2-E_1}{2}=0 \quad (60)$$

The potential then becomes a rational extension of the harmonic oscillator $\frac{(2\rho+1)^2}{16a^2}x^2$.

Since the derivative $p'(x)$ has simple zero(s), the potential has singularities, and thus it is a singular rational extension of the harmonic oscillator.

Since $xp''(x)$ is an odd-parity polynomial of degree $2n-1$ and $p'(x)$ is also an odd-parity polynomial of the same degree, the limit $\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}$ is equal to the ratio of the leading coefficients of the two polynomials.

From (48), we see that the leading coefficient of $p'(x)$ is $2np_{2n}=2n$.

Besides, using (48), we have

$$p''(x)=\sum_{m=1}^{n}2m(2m-1)p_{2m}x^{2m-2} \Rightarrow xp''(x)=\sum_{m=1}^{n}2m(2m-1)p_{2m}x^{2m-1}$$

Thus, the leading coefficient of $xp''(x)$ is $2n(2n-1)p_{2n}=2n(2n-1)$.

Then, the limit $\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}$ is

$$\lim_{|x|\to\infty}\frac{xp''(x)}{p'(x)}=\frac{2n(2n-1)}{2n}=2n-1$$

Substituting into the condition (60), we obtain

$$\frac{(2\rho-1)(2\rho+1)}{4a}(2n-1)+E_1-\frac{E_2-E_1}{2}=0\Rightarrow$$
$$\Rightarrow E_1-\frac{E_2-E_1}{2}=-\frac{(2n-1)(2\rho-1)(2\rho+1)}{4a} \quad (61)$$

Substituting the previous equation into the potential (59), we obtain

$$V(x)=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\frac{xp''(x)}{p'(x)}+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}-$$
$$-\frac{(2n-1)(2\rho-1)(2\rho+1)}{4a}=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\left(\frac{xp''(x)}{p'(x)}-(2n-1)\right)+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-$$
$$-\frac{p'''(x)}{2p'(x)}=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\left(\frac{xp''(x)-(2n-1)p'(x)}{p'(x)}\right)+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)}$$

That is

$$V(x)=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\left(\frac{xp''(x)-(2n-1)p'(x)}{p'(x)}\right)+\rho(\rho-1)\left(\frac{p''(x)}{p'(x)}\right)^2-\frac{p'''(x)}{2p'(x)} \quad (62)$$

As in the case of odd-parity polynomials $p(x)$, we'll use the relations (57) and (61) to express the two energies E_1 and E_2 in terms of the parameters a and ρ.
From (57), we obtain

$$E_2-E_1=-\frac{n(2\rho+1)}{a} \quad (63)$$

Substituting into (61) yields

$$E_1-\frac{-\frac{n(2\rho+1)}{a}}{2}=-\frac{(2n-1)(2\rho-1)(2\rho+1)}{4a}\Rightarrow E_1+\frac{n(2\rho+1)}{2a}=-\frac{(2n-1)(2\rho-1)(2\rho+1)}{4a}\Rightarrow$$
$$\Rightarrow E_1=-\frac{(2n-1)(2\rho-1)(2\rho+1)}{4a}-\frac{2n(2\rho+1)}{4a}=-\frac{((2n-1)(2\rho-1)+2n)(2\rho+1)}{4a}=$$
$$=-\frac{(4n\rho-2n-2\rho+1+2n)(2\rho+1)}{4a}=-\frac{(4n\rho-2\rho+1)(2\rho+1)}{4a}=-\frac{(2\rho(2n-1)+1)(2\rho+1)}{4a}$$

That is

$$E_1=-\frac{(2\rho(2n-1)+1)(2\rho+1)}{4a} \quad (64)$$

Substituting (64) into (63) yields

$$E_2+\frac{(2\rho(2n-1)+1)(2\rho+1)}{4a}=-\frac{n(2\rho+1)}{a}\Rightarrow E_2=-\frac{4n(2\rho+1)}{4a}-\frac{(2\rho(2n-1)+1)(2\rho+1)}{4a}=$$
$$=-\frac{(4n+2\rho(2n-1)+1)(2\rho+1)}{4a}$$

That is

$$E_2=-\frac{(2\rho(2n-1)+4n+1)(2\rho+1)}{4a} \quad (65)$$

As explained, in the case of even-parity polynomials $p(x)$ satisfying the condition (50), $a<0$ and $\rho\geq 0$.

Thus $-\frac{2\rho+1}{4a}>0$ and also, since $n\geq 1$, then $2\rho(2n-1)+1>0$, and thus $2\rho(2n-1)+4n+1>2\rho(2n-1)+1>0$.

Then, from (64) and (65), we see that both energies are positive, and $E_2>E_1$, as it should.

For $\rho=0$, (58) gives $\psi_1(x)=A_1\exp\left(\frac{x^2}{8a}\right)$, with $a<0$, which is the ground-state wave function of a harmonic oscillator.

Then, for $\rho=0$, the potential (62) must become a harmonic oscillator potential.

Let us verify it.

For $\rho=0$, (62) becomes

$$V(x)=\frac{x^2}{16a^2}-\frac{1}{4a}\left(\frac{xp''(x)-(2n-1)p'(x)}{p'(x)}\right)-\frac{p'''(x)}{2p'(x)}=$$
$$=\frac{x^2}{16a^2}-\frac{1}{4a}\left(\frac{xp''(x)}{p'(x)}-(2n-1)\right)-\frac{p'''(x)}{2p'(x)}=\frac{x^2}{16a^2}-\frac{1}{4a}\left(\frac{xp''(x)}{p'(x)}-(2n-1)+\frac{2ap'''(x)}{p'(x)}\right)$$

That is

$$V(x)=\frac{x^2}{16a^2}-\frac{1}{4a}\left(\frac{xp''(x)}{p'(x)}-(2n-1)+\frac{2ap'''(x)}{p'(x)}\right)$$

Using the differential equation (53), we'll show that the expression in parentheses vanishes.

Differentiating both members of (53) with respect to x, we obtain

$$\frac{p'(x)}{p'(x)}-\frac{p(x)p''(x)}{p'^2(x)}=\frac{1}{2n}+\frac{a}{n}\left(\frac{p'''(x)}{p'(x)}-\frac{p''(x)p''(x)}{p'^2(x)}\right)\Rightarrow$$
$$\Rightarrow 1-\frac{p(x)}{p'(x)}\frac{p''(x)}{p'(x)}=\frac{1}{2n}+\frac{a}{n}\left(\frac{p'''(x)}{p'(x)}-\left(\frac{p''(x)}{p'(x)}\right)^2\right)$$

Using again (53), the previous equation is written as

$$1-\left(\frac{x}{2n}+\frac{ap''(x)}{np'(x)}\right)\frac{p''(x)}{p'(x)}=\frac{1}{2n}+\frac{a}{n}\left(\frac{p'''(x)}{p'(x)}-\left(\frac{p''(x)}{p'(x)}\right)^2\right)\Rightarrow$$

$$\Rightarrow 1-\frac{xp''(x)}{2np'(x)}-\frac{a}{n}\left(\frac{p''(x)}{p'(x)}\right)^2=\frac{1}{2n}+\frac{ap'''(x)}{np'(x)}-\frac{a}{n}\left(\frac{p''(x)}{p'(x)}\right)^2\Rightarrow$$

$$\Rightarrow 1-\frac{xp''(x)}{2np'(x)}=\frac{1}{2n}+\frac{ap'''(x)}{np'(x)}\Rightarrow 2n-\frac{xp''(x)}{p'(x)}=1+\frac{2ap'''(x)}{p'(x)}\Rightarrow$$

$$\Rightarrow 0=\frac{xp''(x)}{p'(x)}+1-2n+\frac{2ap'''(x)}{p'(x)}=\frac{xp''(x)}{p'(x)}-(2n-1)+\frac{2ap'''(x)}{p'(x)}$$

That is

$$\frac{xp''(x)}{p'(x)}-(2n-1)+\frac{2ap'''(x)}{p'(x)}=0$$

Thus, for $\rho=0$, the potential becomes the harmonic oscillator potential

$$V(x)=\frac{x^2}{16a^2}.$$

Besides, for $\rho=0$, (64) and (65) give, respectively,

$$E_1=-\frac{1}{4a} \text{ and } E_2=-\frac{4n+1}{4a}$$

Since a is negative, the ground-state energy E_1 of the harmonic oscillator $\frac{x^2}{16a^2}$ is positive, and thus all excited-state energies are also positive, as expected.

The energy E_2 is an even-parity excited-state energy of the previous harmonic oscillator.

If $a=-\frac{1}{2}$, the two energies become $E_1=\frac{1}{2}$ and $E_2=\frac{4n+1}{2}=2n+\frac{1}{2}$, i.e. we obtain, respectively, the ground-state energy and the $2n$-th excited-state energy of a harmonic oscillator with $\hbar\omega\equiv 1$.

The even-parity polynomials p(x) for α<0

The differential equation (53) is written as

$$2np(x)=xp'(x)+2ap''(x)\Rightarrow 2ap''(x)+xp'(x)-2np(x)=0\Rightarrow$$
$$\Rightarrow -4ap''(x)-2xp'(x)+4np(x)=0$$

As explained, in this case, where $p(x)$ is of even parity and satisfies the condition (50), a can be only negative, and $\rho\geq 0$.

Since $a<0$, $a=-|a|$, and the previous differential equation is written as

$$4|a|p''(x)-2xp'(x)+4np(x)=0 \quad (66)$$

Setting $\tilde{x} = \frac{1}{\sqrt{4|a|}} x$, we have

$$\frac{d}{dx} = \frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}} = \frac{1}{\sqrt{4|a|}}\frac{d}{d\tilde{x}}$$

and

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \left(\frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}}\right)\left(\frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}}\right) = \frac{1}{4|a|}\frac{d^2}{d\tilde{x}^2}$$

Also

$$x = \sqrt{4|a|}\tilde{x} \quad (67)$$

Thus, (66) becomes

$$4|a|\frac{1}{4|a|}p''(\tilde{x}) - 2\sqrt{4|a|}\tilde{x}\frac{1}{\sqrt{4|a|}}p'(\tilde{x}) + 4np(\tilde{x}) = 0 \Rightarrow$$

$$\Rightarrow p''(\tilde{x}) - 2\tilde{x}p'(\tilde{x}) + 4np(\tilde{x}) = 0$$

where, now, the primes denote differentiation with respect to $\tilde{x}$.
We see that $p(\tilde{x})$ satisfies the Hermite differential equation, i.e.

$$y''(\tilde{x}) - 2\tilde{x}y'(\tilde{x}) + 2\lambda y(\tilde{x}) = 0,$$

for $\lambda = 2n$.
Thus, since $p(\tilde{x})$ is a polynomial, $p(\tilde{x}) = cH_{2n}(\tilde{x})$, where $H_{2n}(\tilde{x})$ is the Hermite polynomial of degree $2n$ and c a real non-zero constant. $p(\tilde{x})$ and $H_{2n}(\tilde{x})$ have the same zeros, and since $H_{2n}(\tilde{x})$ has $2n$ zeros, $p(\tilde{x})$ has also $2n$ zeros. Then, since the relation (67) is a linear, one-to-one relation, $p(x)$ has $2n$ zeros too.

Using that $\tilde{x} = \frac{1}{\sqrt{4|a|}} x$, we obtain $p(x) = cH_{2n}\left(\frac{1}{\sqrt{4|a|}} x\right)$, and we calculate the constant c by comparing the leading coefficient of $p(x)$, which is 1, with the leading coefficient of $H_{2n}\left(\frac{1}{\sqrt{4|a|}} x\right)$.

We thus showed that if $a < 0$, the even-parity polynomial $p(x)$ has $2n$ zeros.
Then, from Rolle's theorem, the derivative $p'(x)$ has at least $2n-1$ zeros, and as it is a polynomial of degree $2n-1$, it has exactly $2n-1$ zeros $(n \geq 1)$.

Therefore, if α<0, p(x) has 2n zeros and p´(x) has 2n-1 zeros (n>0).

Summary of the case where p(x) is an even-parity polynomial and satisfies the integrability condition (50)

In this case, the two wave functions have $2n-1$ common – or special – zeros, which are the simple zeros of $p'(x)$, while $\psi_2(x)$ has $2n$ more zeros, which are the simple zeros of $p(x)$.
At each zero of $p'(x)$, the potential has a pole of second order, and thus it has $2n-1$ poles of second order. The potential is then a singular rational extension of the harmonic oscillator $\frac{(2\rho+1)^2}{16a^2}x^2$.
The wave function $\psi_1(x)$ is the ground-state wave function, with energy E_1, while the wave function $\psi_2(x)$ is the $2n$-th excited-state wave function, with energy E_2.
Between the two energy levels E_1 and E_2, there are $2n-1$ energy levels, as many as the poles of the potential.
The two energies are given by

$$E_1 = -\frac{(2\rho(2n-1)+1)(2\rho+1)}{4a} \text{ and } E_2 = -\frac{(2\rho(2n-1)+4n+1)(2\rho+1)}{4a},$$

and they are both positive.
As the ground-state energy E_1 is positive, all energies of the potential are positive.

Examples

1.

$p(x)=1$
As mentioned, this case is excluded, since then the two wave functions are linearly dependent, and thus they describe the same eigenstate.

2.

$p(x)=x$
Then $p'(x)=1$ and $\int_x dy\frac{p(y)}{p'(y)}=\frac{x^2}{2}$
Thus, from (14) and (5), the two wave functions are respectively written as

$$\psi_1(x)=A_1\exp\left(-\frac{(E_2-E_1)x^2}{4}\right)$$

$$\psi_2(x)=A_2x\exp\left(-\frac{(E_2-E_1)x^2}{4}\right)$$

with $E_2>E_1$, so that the wave functions are square integrable.
These are, respectively, the ground-state wave function and the first-excited-state wave function of a harmonic oscillator.

Indeed, from (22), using that $p''(x) = p'''(x) = 0$, we obtain

$$V(x) = \left(\frac{E_2 - E_1}{2}\right)^2 x^2 + E_1 - \frac{E_2 - E_1}{2},$$

which is a harmonic oscillator potential plus a constant.
Setting $E_1 - \frac{E_2 - E_1}{2} = 0$, the potential becomes

$$V(x) = \left(\frac{E_2 - E_1}{2}\right)^2 x^2$$

Observe that in order to calculate uniquely the two energies, we need one more equation. This equation can come from the "strength" of the harmonic oscillator, i.e. how big is the factor $\left(\frac{E_2 - E_1}{2}\right)^2$.

If $\xi = \left(\frac{E_2 - E_1}{2}\right)^2$, then using this equation and the condition $E_1 - \frac{E_2 - E_1}{2} = 0$, we calculate the two energies, i.e. the ground-state energy and the first-excited-state energy of the harmonic oscillator.

3.

$p(x) = x^2 + p_0 \quad (n = 1)$

This is the first non-trivial case.
Since $p(x)$ is of even parity, $a < 0$ and $\rho \geq 0$.
Using the recursion relation (52), we calculate the coefficient p_0, which is

$$p_0 = \frac{a*1*2}{1-0} \underbrace{p_2}_{1} = 2a$$

Then

$$p(x) = x^2 + 2a \quad (68)$$

Since $a < 0$, the binomial (68) has two zeros, at $\pm\sqrt{2|a|}$.
Using (68), the derivatives of $p(x)$ are

$$p'(x) = 2x,\ p''(x) = 2,\ p'''(x) = 0$$

Then, the wave function (58) is written as

$$\psi_1(x) = A_1 |2x|^\rho \exp\left(\frac{(2\rho+1)x^2}{8a}\right) = A_1 2^\rho |x|^\rho \exp\left(\frac{(2\rho+1)x^2}{8a}\right)$$

Incorporating the constant 2^ρ into the normalization constant A_1, we write $\psi_1(x)$ as

$$\psi_1(x) = A_1 |x|^\rho \exp\left(\frac{(2\rho+1)x^2}{8a}\right) \quad (69)$$

Also, using (68), the wave function (5) is written as

$$\psi_2(x) = A_2\left(x^2+2a\right)\psi_1(x) \quad (70)$$

Substituting the derivatives of $p(x)$ and $n=1$ into (62), we obtain the expression of the potential, which is then

$$V(x) = \frac{(2\rho+1)^2}{16a^2}x^2 + \frac{(2\rho-1)(2\rho+1)}{4a}\left(\frac{\overbrace{2x-(2-1)(2x)}^{2x-2x=0}}{2x}\right) + \rho(\rho-1)\left(\frac{2}{2x}\right)^2 =$$

$$= \frac{(2\rho+1)^2}{16a^2}x^2 + \frac{\rho(\rho-1)}{x^2}$$

That is

$$V(x) = \frac{(2\rho+1)^2}{16a^2}x^2 + \frac{\rho(\rho-1)}{x^2} \quad (71)$$

with $a<0$ and $\rho \geq 0$.

For $\rho > 1$, the potential (71) is the potential of an isotonic oscillator **[5]**, and it is exactly solvable.

We observe that the potential (71) has a pole of second order, at 0, as expected, since the derivative $p'(x)$ has a pole at 0.

The potential (71) is then a singular rational extension of the harmonic oscillator $\frac{(2\rho+1)^2}{16a^2}x^2$.

Substituting $n=1$ into the general expressions (64) and (65), the energies of the two eigenstates are

$$E_1 = -\frac{\left(2\rho(2-1)+1\right)(2\rho+1)}{4a} = -\frac{(2\rho+1)^2}{4a}$$

$$E_2 = -\frac{\left(2\rho(2-1)+4+1\right)(2\rho+1)}{4a} = -\frac{(2\rho+5)(2\rho+1)}{4a}$$

That is

$$E_1 = -\frac{(2\rho+1)^2}{4a} \quad (72)$$

$$E_2 = -\frac{(2\rho+5)(2\rho+1)}{4a} \quad (73)$$

We see that both energies are positive.

The two wave functions have a common zero at 0, which is the zero of $p'(x)$, and the wave function $\psi_2(x)$ has two more zeros, which are the zeros of $p(x)$.

Therefore, $\psi_1(x)$ is the ground-state wave function, with energy given by (72), while $\psi_2(x)$ is the second-excited-state wave function, with energy given by (73), of the potential (71).
We also observe that the potential (71) becomes a harmonic oscillator potential for two values of ρ, and particularly for $\rho=0$ and for $\rho=1$.
However, if $\rho=1$, both wave functions (69) and (70) are not differentiable at 0, due to the presence of the term $|x|$, and this is unacceptable for a smooth polynomial potential, as is the harmonic oscillator potential.
On the contrary, if $\rho=0$, the absolute value of x vanishes, and both wave functions become smooth.
Thus, the value $\rho=0$ is the right one to obtain the harmonic oscillator potential.
For $\rho=0$, the potential (71) becomes $V(x)=\frac{x^2}{16a^2}$, in accordance with what we've found in the general case of even-parity polynomials $p(x)$ satisfying the condition (50).

4.

$p(x)=x^3+p_1x \ \ (n=1)$

Using the recursion relation (28), we calculate the coefficient p_1, which is

$$p_1=\frac{a*1*3}{1-1+1}\underbrace{p_3}_{1}=3a$$

Then

$p(x)=x^3+3ax$ (74)

The derivatives of $p(x)$ are then

$$p'(x)=3x^2+3a=3\left(x^2+a\right)$$

$$p''(x)=6x$$

$$p'''(x)=6$$

Then, using (35), the wave function $\psi_1(x)$ is

$$\psi_1(x)=A_1\left|3\left(x^2+a\right)\right|^{\rho}\exp\left(\frac{(2\rho+1)x^2}{4a}\right)=A_1 3^{\rho}\left|x^2+a\right|^{\rho}\exp\left(\frac{(2\rho+1)x^2}{4a}\right)$$

Incorporating the constant 3^{ρ} into the normalization constant A_1, we write $\psi_1(x)$ as

$$\psi_1(x)=A_1\left|x^2+a\right|^{\rho}\exp\left(\frac{(2\rho+1)x^2}{4a}\right) \quad (75)$$

Using (74), the wave function (5) is written as

$\psi_2(x) = A_2 x(x^2+3a)\psi_1(x)$ (76)

As showed for odd-parity polynomials $p(x)$ satisfying the condition (26), the domains of a and ρ are $a>0$ and $\rho<-\frac{1}{2}$, or $a<0$ and $\rho\geq 0$.

Substituting the derivatives of $p(x)$ and $n=1$ into (39), we obtain the potential, which is then

$$V(x)=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\left(\frac{x6x-2*3(x^2+a)}{3(x^2+a)}\right)+\rho(\rho-1)\left(\frac{6x}{3(x^2+a)}\right)^2-\frac{6}{2*3(x^2+a)}=$$

$$=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{2a}\left(\frac{-6a}{3(x^2+a)}\right)+\rho(\rho-1)\left(\frac{2x}{x^2+a}\right)^2-\frac{1}{x^2+a}=$$

$$=\frac{(2\rho+1)^2}{4a^2}x^2-\frac{(2\rho-1)(2\rho+1)}{x^2+a}+\frac{4\rho(\rho-1)x^2}{(x^2+a)^2}-\frac{1}{x^2+a}=$$

$$=\frac{(2\rho+1)^2}{4a^2}x^2-\frac{(2\rho-1)(2\rho+1)+1}{x^2+a}+\frac{4\rho(\rho-1)x^2}{(x^2+a)^2}=\frac{(2\rho+1)^2}{4a^2}x^2-\frac{4\rho^2}{x^2+a}+\frac{4\rho(\rho-1)x^2}{(x^2+a)^2}=$$

$$=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{4\rho(\rho-1)x^2-4\rho^2(x^2+a)}{(x^2+a)^2}=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{4\rho^2x^2-4\rho x^2-4\rho^2x^2-4a\rho^2}{(x^2+a)^2}=$$

$$=\frac{(2\rho+1)^2}{4a^2}x^2+\frac{-4\rho x^2-4a\rho^2}{(x^2+a)^2}=\frac{(2\rho+1)^2}{4a^2}x^2-\frac{4\rho(x^2+a\rho)}{(x^2+a)^2}$$

That is

$$V(x)=\frac{(2\rho+1)^2}{4a^2}x^2-\frac{4\rho(x^2+a\rho)}{(x^2+a)^2} \quad (77)$$

If $a>0$, the potential (77) has no singularities, it is differentiable everywhere and its derivatives are also differentiable, i.e. the potential is a smooth function of x.

The potential is then a regular rational extension of $\frac{(2\rho+1)^2}{4a^2}x^2$.

In this case, where $a>0$, $\rho<-\frac{1}{2}$. If $\rho=-1$, the potential (77) becomes

$$V(x)=\frac{(-1)^2}{4a^2}x^2-\frac{4(-1)(x^2-a)}{(x^2+a)^2}=\frac{x^2}{4a^2}+\frac{4(x^2-a)}{(x^2+a)^2}$$

That is

$$V(x)=\frac{x^2}{4a^2}+\frac{4(x^2-a)}{(x^2+a)^2} \quad (78)$$

Since $a>0$, the potential (78) is a generalized isotonic oscillator **[5, 6]**.
If $a<0$, the potential has two singularities, two poles of second order at $\pm\sqrt{|a|}$.

For $\rho=0$ $(a<0)$, the potential (77) becomes $V(x)=\frac{x^2}{4a^2}$, and the wave functions (75) and (76) are, respectively, the ground-state wave function and the third-excited-state wave function (as it has three zeros, at $0,\pm\sqrt{3|a|}$) of the harmonic oscillator $\frac{x^2}{4a^2}$.

Substituting $n=1$ into (41) and (42), we obtain the two energies, which are

$$E_1=-\frac{(4\rho+1)(2\rho+1)}{2a} \quad (79)$$

$$E_2=-\frac{(4\rho+7)(2\rho+1)}{2a} \quad (80)$$

If $a>0$ and $\rho<-\frac{1}{2}$, $\psi_1(x)$ has no zeros, while $\psi_2(x)$ has one zero, at 0.

Thus, $\psi_1(x)$ is the ground-state wave function and $\psi_2(x)$ is the first-excited-state wave function of the potential (77), with energies given by (79) and (80), respectively.

As $4\rho+1<-\frac{4}{2}+1=-1<0$, then from (79) we see that $E_1<0$, i.e. the ground-state energy is negative.

From (80), we see that the first-excited-state energy E_2 is negative if $\rho<-\frac{7}{4}$, it is zero if $\rho=-\frac{7}{4}$, and it is positive if $-\frac{7}{4}<\rho<-\frac{1}{2}$.

If $a<0$ and $\rho\geq 0$, the two wave functions have two common zeros, at $\pm\sqrt{|a|}$, and $\psi_2(x)$ has three more zeros, at $0,\pm\sqrt{3|a|}$, which are the zeros of $p(x)$.

Thus, $\psi_1(x)$ is the ground-state wave function and $\psi_2(x)$ is now the third-excited-state wave function, with energies given by (79) and (80), respectively. Both energies are now positive.

In this case, the potential (77) has two second-order poles at $\pm\sqrt{|a|}$, and thus it is a singular rational extension of $\frac{(2\rho+1)^2}{4a^2}x^2$.

We see that the presence of the two poles in the potential raises the ground-state energy, so that it is always positive, and it also results in the presence of two energy levels between E_1 and E_2.

5.

$p(x)=x^4+p_2x^2+p_0$ $(n=2)$

As showed for even-parity polynomials $p(x)$ satisfying the condition (50), $a<0$ and $\rho\geq 0$.

Using the recursion relation (52), we calculate the coefficients p_0 and p_2.
We have

$$p_2 = \frac{a(2+1)(2+2)}{2-1}\underbrace{p_4}_{1} = 12a$$

$$p_0 = \frac{2a}{2-0}p_2 = a12a = 12a^2$$

Thus

$$p(x) = x^4 + 12ax^2 + 12a^2 \quad (81)$$

The derivatives of $p(x)$ are then

$$p'(x) = 4x^3 + 24ax = 4x(x^2 + 6a)$$

$$p''(x) = 12x^2 + 24a = 12(x^2 + 2a)$$

$$p'''(x) = 24x$$

Using (58), the wave function $\psi_1(x)$ then becomes

$$\psi_1(x) = A_1\left|4x(x^2+6a)\right|^{\rho}\exp\left(\frac{(2\rho+1)x^2}{8a}\right) = A_1 4^{\rho}\left|x(x^2+6a)\right|^{\rho}\exp\left(\frac{(2\rho+1)x^2}{8a}\right)$$

Incorporating the constant 4^{ρ} into the normalization constant A_1, $\psi_1(x)$ is written as

$$\psi_1(x) = A_1\left|x(x^2+6a)\right|^{\rho}\exp\left(\frac{(2\rho+1)x^2}{8a}\right) \quad (82)$$

Also, using (81), the wave function $\psi_2(x)$ is written as, from (5),

$$\psi_2(x) = A_2\left(x^4 + 12ax^2 + 12a^2\right)\psi_1(x) \quad (83)$$

As shown for the even-parity polynomials $p(x)$ satisfying the condition (50), a can be only negative, and then, for this case, $p(x)$ has four zeros and $p'(x)$ has three zeros.
Then, the wave functions (82) and (83) have three common zeros, which are the three zeros of $p'(x)$, while $\psi_2(x)$ has four more zeros, which are the zeros of $p(x)$.
Thus, $\psi_1(x)$ is the ground-state wave function, while $\psi_2(x)$ is the fourth-excited-state wave function of the potential we'll calculate now.
Substituting the derivatives of $p(x)$ and $n=2$ into (62), we obtain the expression of the potential, which is then

$$V(x)=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\left(\frac{x12\left(x^2+2a\right)-(2*2-1)4x\left(x^2+6a\right)}{4x\left(x^2+6a\right)}\right)+$$

$$+\rho(\rho-1)\left(\frac{12\left(x^2+2a\right)}{4x\left(x^2+6a\right)}\right)^2-\frac{24x}{2*4x\left(x^2+6a\right)}=\frac{(2\rho+1)^2}{16a^2}x^2+$$

$$+\frac{(2\rho-1)(2\rho+1)}{4a}\left(\frac{12x\left(x^2+2a\right)-12x\left(x^2+6a\right)}{4x\left(x^2+6a\right)}\right)+\rho(\rho-1)\left(\frac{3\left(x^2+2a\right)}{x\left(x^2+6a\right)}\right)^2-\frac{3}{x^2+6a}=$$

$$=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\left(\frac{3\left(x^2+2a\right)-3\left(x^2+6a\right)}{x^2+6a}\right)+\frac{9\rho(\rho-1)\left(x^2+2a\right)^2}{x^2\left(x^2+6a\right)^2}-\frac{3}{x^2+6a}=$$

$$=\frac{(2\rho+1)^2}{16a^2}x^2+\frac{(2\rho-1)(2\rho+1)}{4a}\left(\frac{-12a}{x^2+6a}\right)+\frac{9\rho(\rho-1)\left(x^2+2a\right)^2}{x^2\left(x^2+6a\right)^2}-\frac{3}{x^2+6a}=$$

$$=\frac{(2\rho+1)^2}{16a^2}x^2-\frac{3(2\rho-1)(2\rho+1)}{x^2+6a}+\frac{9\rho(\rho-1)\left(x^2+2a\right)^2}{x^2\left(x^2+6a\right)^2}-\frac{3}{x^2+6a}$$

That is

$$V(x)=\frac{(2\rho+1)^2}{16a^2}x^2-\frac{3(2\rho-1)(2\rho+1)}{x^2+6a}+\frac{9\rho(\rho-1)\left(x^2+2a\right)^2}{x^2\left(x^2+6a\right)^2}-\frac{3}{x^2+6a}\quad(84)$$

But

$$-\frac{3(2\rho-1)(2\rho+1)}{x^2+6a}+\frac{9\rho(\rho-1)\left(x^2+2a\right)^2}{x^2\left(x^2+6a\right)^2}-\frac{3}{x^2+6a}=$$

$$=-\frac{3\left(4\rho^2-1\right)}{x^2+6a}-\frac{3}{x^2+6a}+\frac{9\rho(\rho-1)\left(x^2+2a\right)^2}{x^2\left(x^2+6a\right)^2}=-\frac{12\rho^2-3+3}{x^2+6a}+\frac{9\rho(\rho-1)\left(x^2+2a\right)^2}{x^2\left(x^2+6a\right)^2}=$$

$$=-\frac{12\rho^2}{x^2+6a}+\frac{9\rho(\rho-1)\left(x^2+2a\right)^2}{x^2\left(x^2+6a\right)^2}=\frac{9\rho(\rho-1)\left(x^2+2a\right)^2-12\rho^2x^2\left(x^2+6a\right)}{x^2\left(x^2+6a\right)^2}=$$

$$=\frac{3\left(\left(3\rho^2-3\rho\right)\left(x^4+4ax^2+4a^2\right)-4\rho^2x^4-24a\rho^2x^2\right)}{x^2\left(x^2+6a\right)^2}=$$

$$=\frac{3\left(3\rho^2x^4+12a\rho^2x^2+12a^2\rho^2-3\rho x^4-12a\rho x^2-12a^2\rho-4\rho^2x^4-24a\rho^2x^2\right)}{x^2\left(x^2+6a\right)^2}=$$

$$=\frac{3\left(-\rho^2x^4-12a\rho^2x^2+12a^2\rho^2-3\rho x^4-12a\rho x^2-12a^2\rho\right)}{x^2\left(x^2+6a\right)^2}=$$

$$= \frac{3\left(-\left(\rho^2+3\rho\right)x^4-12a\left(\rho^2+\rho\right)x^2+12a^2\left(\rho^2-\rho\right)\right)}{x^2\left(x^2+6a\right)^2} =$$

$$= \frac{-3\left(\rho^2+3\rho\right)x^4-36a\left(\rho^2+\rho\right)x^2+36a^2\left(\rho^2-\rho\right)}{x^2\left(x^2+6a\right)^2} =$$

$$= \frac{-3\rho\left(\rho+3\right)x^4-36a\rho\left(\rho+1\right)x^2+36a^2\rho\left(\rho-1\right)}{x^2\left(x^2+6a\right)^2} =$$

$$= -\frac{3\rho\left(\left(\rho+3\right)x^4+12a\left(\rho+1\right)x^2-12a^2\left(\rho-1\right)\right)}{x^2\left(x^2+6a\right)^2}$$

Then, the potential (84) is written as

$$V\left(x\right) = \frac{\left(2\rho+1\right)^2}{16a^2}x^2 - \frac{3\rho\left(\left(\rho+3\right)x^4+12a\left(\rho+1\right)x^2-12a^2\left(\rho-1\right)\right)}{x^2\left(x^2+6a\right)^2} \quad (85)$$

The potential (85) has three poles of second order, at $0, \pm\sqrt{6|a|}$, which are the zeros of $p'\left(x\right)$.

The potential is then a singular rational extension of the harmonic oscillator $\frac{\left(2\rho+1\right)^2}{16a^2}x^2$.

As noted $\rho \geq 0$. For $\rho = 0$, the potential (85) gives the harmonic oscillator potential $\frac{x^2}{16a^2}$, in accordance with what we've found in the general case of even-parity polynomials $p\left(x\right)$ satisfying the condition (50). Then, the wave function (82) becomes $\psi_1\left(x\right) = A_1 \exp\left(\frac{x^2}{8a}\right)$ and it is the ground-state wave function of the harmonic oscillator $\frac{x^2}{16a^2}$ $\left(a<0\right)$, while the wave function (83) is then the fourth-excited-state wave function of the previous oscillator.

Substituting $n=2$ into the general expressions (64) and (65), the energies of the two eigenstates are

$$E_1 = -\frac{\left(6\rho+1\right)\left(2\rho+1\right)}{4a} \quad (86)$$

$$E_2 = -\frac{\left(6\rho+9\right)\left(2\rho+1\right)}{4a} \quad (87)$$

Since $\rho \geq 0$ and $a<0$, both energies (86) and (87) are positive, and $E_2 > E_1$, as expected.

The energy (86) is the ground-state energy, while the energy (87) is the fourth-excited-state energy of the potential (85).

The presence of three poles in the potential results in the presence of three energy levels between E_1 and E_2.

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