Describing a fluid in three-dimensional circular motion with one independent variable by rectangular coordinate

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Abstract: Describe a fluid in three-dimensional circular motion with one independent variable by rectangular coordinate and concludes on the breakdown of Euler and Navier-Stokes equations.

In [1] we showed that the three-dimensional Euler ($\nu = 0$) and Navier-Stokes equations in rectangular coordinates need to be adopted as

(1) $$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3}\nu \nabla_i (\nabla \cdot u) + f_i,$$

for $i = 1,2,3$, where $\alpha_j = \frac{dx_j}{dt}$ is the velocity in Lagrangian description and u_i and the partial derivatives of u_i are in Eulerian description, as well as the scalar pressure p and density of external force f_i. The coefficient of viscosity is ν and by ease we prefer to use the mass density $\rho = 1$ (otherwise substitute p by p/ρ and ν by ν/ρ).

An alternative equation is

(2) $$\frac{\partial p_i}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3}\nu \nabla_i (\nabla \cdot u) + f_i,$$

thus making the pressure a vector: $p = (p_1, p_2, p_3)$. In both equations is valid

(3) $$\frac{Du_i}{Dt} = \frac{Du_i^E}{Dt} = \frac{Du_i^L}{Dt} = \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j}\right)|_L,$$

where the upper letter E refers to Eulerian velocity and L to Lagrangian velocity. The symbol $|_L$ means the respective calculation in Lagrangian description, substituting each x_i as a function of time, initial value and eventually some parameters.

A condition indicated by us in [1] were

(4) $$\begin{cases} \frac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because we have, by definition,

(5) $$u_i = \frac{dx_i}{dt},$$

in Lagrangian description, and for this reason the velocity u_i, *a priori*, is not dependent of others variables x_j, with $x_j \neq x_i$. More than a rigorous mathematical proof, this is a practical approach, which simplifies the original system.

It is very easy to accept the first equation of (4) when there is no link between the spatial coordinates during the movement of the fluid over time, but in a circular motion, for example, it seems to be no longer valid.

Let a circular motion of radius R, centered at $(x_C,\ y_C)$ and with constant angular velocity $\omega > 0$ described by the equations:

$$(6)\qquad \begin{cases} x = x_C + R\cos(\theta_0 + \omega t) \\ y = y_C + R\sin(\theta_0 + \omega t) \end{cases}$$

and consequently

$$(7)\qquad (x - x_C)^2 + (y - y_C)^2 = R^2.$$

Then the velocity components are

$$(8)\qquad \begin{cases} \alpha_1 = u_1^L = \dot{x} = -\omega R\sin(\theta_0 + \omega t) = -\omega(y - y_C) = u_1^E \\ \alpha_2 = u_2^L = \dot{y} = +\omega R\cos(\theta_0 + \omega t) = +\omega(x - x_C) = u_2^E \end{cases}$$

and the acceleration components are

$$(9)\qquad \begin{cases} \frac{Du_1^L}{Dt} = \ddot{x} = -\omega^2 R\cos(\theta_0 + \omega t) = -\omega^2(x - x_C) = \frac{Du_1^E}{Dt} \\ \frac{Du_2^L}{Dt} = \ddot{y} = -\omega^2 R\sin(\theta_0 + \omega t) = -\omega^2(y - y_C) = \frac{Du_2^E}{Dt} \end{cases}$$

Supposing that the particles of fluid obey the motion described by (6) to (9), we have

$$(10)\qquad \begin{cases} \frac{\partial u_1}{\partial y} = -\omega, \quad \frac{\partial u_1}{\partial x} = 0 \\ \frac{\partial u_2}{\partial x} = +\omega, \quad \frac{\partial u_2}{\partial y} = 0 \end{cases}$$

apparently in disagree with (4) if $\omega \neq 0$. But, as x is a function of y and reciprocally, in this circular motion according (7), again (4) turns valid, for any signal of x and y. For to complete a three-dimensional description, we define $z = z_0$, without dependence of time, and $u_3 = 0$.

This is a motion of velocity without potential, because $\frac{\partial u_i}{\partial x_j} \neq \frac{\partial u_j}{\partial x_i}$ for some $i \neq j$, but if $f = (f_1, f_2, f_3)$ has potential we have $\frac{\partial s_i}{\partial x_j} = \frac{\partial s_j}{\partial x_i}$ for all $i, j = 1,2,3$, with

(11) $$S_i = -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} + \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$$

then the system (1) has solution.

A calculation for the scalar pressure of this motion is

(12) $$\begin{aligned} p &= \int_L (S_1, S_2, S_3) \cdot dl = \int_L \left(-\frac{Du}{Dt} + f\right) \cdot dl \\ &= \omega^2 \left[\left(\frac{x^2}{2} - x_C x\right)\Big|_{x_0}^{x} + \left(\frac{y^2}{2} - y_C y\right)\Big|_{y_0}^{y}\right] + U - U_0 + \theta(t) \\ &= \omega^2 \left[\left(\frac{x^2}{2} - x_C x\right) - \left(\frac{{x_0}^2}{2} - x_C x_0\right) + \left(\frac{y^2}{2} - y_C y\right) - \left(\frac{{y_0}^2}{2} - y_C y_0\right)\right] + \\ &\quad U - U_0 + \theta(t), \end{aligned}$$

where $f = \nabla U$, $U_0 = U(x_0, y_0, z_0, t)$ and L is any smooth path linking a point (x_0, y_0, z_0) to (x, y, z). We can ignore the use of x_0, y_0, z_0 and U_0, and use only the free function for time, $\theta(t)$, which on the other hand can include the terms in x_0, y_0 and z_0, and nevertheless this solution shows us that the pressure is not uniquely well determined, therefore we get to the negative answer to Smale's 15th problem, according already seen in [2] and [3], even if we assign the velocity value on some surface that we wish and even if $\theta(t)$ and U does not depend explicitly on the variable time t. In this motion the pressure is dependent, besides of x, y and U, without any problematic question, and x_C, y_C and ω, specific parameters of the movement conditions of a particle, of $\theta(t)$, U_0 and more three parameters, x_0, y_0 and z_0, then there is not uniqueness of solution.

Another calculation for pressure is possible due to fact that we can describe the acceleration $\frac{Du}{Dt}$ of a particle of fluid as a function only of time, $\frac{D\alpha}{Dt}$, without the variables x, y, z, and then

(13) $$\begin{aligned} p &= -\frac{D\alpha}{Dt} \cdot \int_L dl + U - U_0 + \theta(t) \\ &= +\omega^2 R[\cos(\theta_0 + \omega t)(x - x_0) + \sin(\theta_0 + \omega t)(y - y_0)] \\ &\quad + U - U_0 + \theta(t), \end{aligned}$$

with

(14) $$\begin{cases} \frac{\partial p}{\partial x} = +\omega^2 R \cos(\theta_0 + \omega t) + f_1 = +\omega^2 (x - x_C) + f_1 \\ \frac{\partial p}{\partial y} = +\omega^2 R \sin(\theta_0 + \omega t) + f_2 = +\omega^2 (y - y_C) + f_2 \\ \frac{\partial p}{\partial z} = f_3 \end{cases}$$

in fact derivatives such as can be obtained from (12).

Note that in order to continue using the traditional form of the Euler and Navier-Stokes equations we will have non-linear equations, which can make it difficult to obtain the solutions and bring all the difficulties that we know. To make sense to use the velocity in Eulerian description rather than the Lagrangian description in α_j it is necessary that, for all $t \geq 0$,

(15) $$u^E(x(t), y(t), z(t), t) = \alpha(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = u^L(t),$$

omitting the use of possible parameters of motion, then nothing more natural than the definitive substitution of the terms $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j}$, as well as $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}$ in the traditional form, by $\frac{Du_i^L}{Dt}$ or $\frac{D\alpha_i}{Dt}$. This brings a great and important simplification in the equations, and to return to having the position as reference it is enough to use the conversion or definition adopted for $x(t), y(t)$ and $z(t)$, including the possible additional parameters, for example, substituting initial positions in function of position and time, etc.

Thus, more appropriate Euler ($\nu = 0$) and Navier-Stokes equations with scalar pressure are, in index notation,

(16) $$\frac{\partial p}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i.$$

If $\nu = 0$ and f is not conservative then there is no solution for Euler equations, as well as if u is conservative and f is not conservative there is no solution for Navier-Stokes equations, which now it is very clear to see from (16) and it is according [4]. More specifically, if u^0, the initial velocity, is conservative (irrotational or potential flow) and f is not conservative then there is no solution for Navier-Stokes equations, because it is impossible to obtain the pressure. This then solve [5] for the cases (C) and (D), the breakdown of solutions, for both u^0 and f belonging to Schwartz Space in case (C), and smooth functions with period 1 in the three orthogonal directions e_1, e_2, e_3 in case (D). As u^0 need obey to the incompressibility condition, $\nabla \cdot u^0 = 0$, with $\nabla \times u^0 = 0$ and $u^0 = \nabla \varphi^0$, where φ^0 is the potential of u^0, we have $\nabla^2 u^0 = 0$ and $\nabla^2 \varphi^0 = 0$, i.e., u^0 and φ^0 are harmonic functions, unlimited functions except the constants, including zero. As u^0 need be limited, we choose $u^0 = 0$ for case (C) (where it is necessary that $\int_{\mathbb{R}^3} |u^0|^2 dx\, dy\, dz$ is finite) and any constant for case (D), of spatially periodic solutions. In case (D) the external force need belonging to Schwartz Space with relation to time.

Note that the application of a non conservative force in fluid is naturally possible and there will always be some movement, even starting from rest. So that this is not a paradoxical situation it seems certain that the pressure in this case cannot be scalar, but rather vector, and thus the equation returns to solution in all cases (assuming all derivatives are possible, etc.). It is as indicated in (2), or substituting p by p_i in (16).

References

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